Applied Mathematics and
Computer Science
1 (1) (2016), 1–8

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/amcs

Some inequalities for log φ-convex functions

Muhammad Aslam Noor[a], Muhammad Uzair Awan[b], Khalida Inayat Noor[c]

[a]*Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*
[b]*Mathematics Department, Government College University, Faisalabad, Pakistan*
[c]*Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*

Abstract. In this paper, we derive several Hermite-Hadamard type integral inequalities for log φ-convex functions. Our results represent refinement and improvement of the previously known results. Several special cases are discussed.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

named after C. Hermite and J. Hadamard. Inequality (1.1) can be considered as necessary and sufficient condition for a function to be convex. For useful details on Hermite-Hadamard type of integral inequalities (see [1,2,3,4,6,7,10,11,12,14,15,16]).

In recent years concept of convexity has been generalized and extended in several directions using novel and innovative ideas (see [2,4,5,8,10,12]). A significant generalization of classical convexity was the introduction of φ-convexity by Noor [10]. Noor [10] investigated various basic properties for the class of φ-convex function. Noor [8] extended Hermite-Hadamard type integral inequalities for φ-convex function. It is worth to mention here that φ-convex functions are nonconvex functions. In this paper, we consider the class of log φ-convex functions which was also introduced by Noor [10]. We derive several Hermite-Hadamard type inequalities for log φ-convex functions. Our results generalize several known results. The ideas and techniques used in the paper are interesting and may stimulate further research in this area.

2010 *Mathematics Subject Classification*. 26D15; 26A51.
Keywords. Convex functions; φ-convex functions; log φ-convex functions; Hermite-Hadamard.
Received: 2 May 2016; Accepted: 20 May 2016
Communicated by Dragan S. Djordjevic
Email addresses: noormaslam@hotmail.com (Muhammad Aslam Noor), awan.uzair@gmail.com (Muhammad Uzair Awan), khalidanoor@hotmail.com (Khalida Inayat Noor)

2. Preliminaries

In this section, we recall some basic results. Let $\mathbb{R}^n$ be a finite dimensional euclidian space, whose inner product and norm is denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K_φ be a nonempty closed set in $\mathbb{R}^n$ and suppose $f, \varphi : K \to \mathbb{R}$ be continuous functions, where $0 \leq \varphi \leq \frac{\pi}{2}$.

Definition 2.1 ([10]). *Let $u \in K_\varphi$. Then the set K_φ is said to be φ-convex, if*

$$u + te^{i\varphi}(v-u) \in K_\varphi, \quad \forall\, u, v \in K_\varphi, \;\; t \in [0,1].$$

For $\varphi = 0$, the set K_φ reduces to the classical convex set K. That is,

$$u + t(v-u) \in K, \quad \forall u, v \in K, \; t \in [0,1].$$

Remark 2.2 ([10]). *Definition 2.1 says that there is a path starting from a point u which is contained in K_φ. We do not require that the point v should be one of the end point of the path. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in K_\varphi$, then $e^{i\varphi}(v-u) = v-u$, if and only if $\varphi = 0$. Then the φ-convex K_φ becomes the convex set K. It is clear that every convex set is a φ-convex set, but the converse is not necessarily true.*

Definition 2.3 ([10]). *A function f on the φ-convex set K_φ is said to be φ-convex with respect to φ, if*

$$f(u + te^{i\varphi}(v-u)) \leq (1-t)f(u) + tf(v)\,, \quad \forall\, u, v \in K_\varphi, \; t \in [0,1].$$

For $\varphi = 0$, φ-convex function reduces to convex functions. This implies that every convex function is φ-convex function. However the converse is not true, see [14].

Definition 2.4 ([10]). *A function f on the φ-convex set K_φ is said to be* log *φ-convex with respect to φ, if*

$$f(u + te^{i\varphi}(v-u)) \leq [f(u)]^{1-t}[f(v)]^t\,, \quad \forall\, u, v \in K_\varphi, \; t \in [0,1].$$

Remark 2.5. *From Definition 2.4, we have*

$$\log f(u + te^{i\varphi}(v-u)) \leq (1-t)\log(f(u)) + t\log(f(v))\,, \quad \forall\, u, v \in K_\varphi, \; t \in [0,1].$$

Using Remark 2.5, we have if f is differentiable log *φ-convex function.*

$$f(v) \geq f(u) \exp\left\langle \frac{f'_\varphi(u)}{f(u)}, v-u \right\rangle, \quad \forall u, v \in K_\varphi,$$

where $f'_\varphi(.)$ is the φ-derivative of f, see [10].

3. Main Results

In this section, we prove our main results.

Theorem 3.1. *Let $f, g : I = [a, a + e^{i\varphi}(b-a)] \to (0, \infty)$ be* log *φ-convex functions. If $\alpha + \beta = 1$, then*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx$$
$$\leq \alpha \frac{f(a)+f(b)}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a))\right]^{\frac{1-\alpha}{\alpha}} + \beta \frac{g(a)+g(b)}{2}\left[L_{(\frac{1}{\beta}-1)}(g(b), g(a))\right]^{\frac{1-\beta}{\beta}}.$$

Proof. Let f and g be log φ-convex functions. Using inequality

$$xy \leq \alpha x^{\frac{1}{\alpha}} + \beta y^{\frac{1}{\beta}}, \quad \alpha, \beta > 0, \alpha + \beta = 1,$$

we have

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(x)dx = \int_0^1 f(a + te^{i\varphi}(b-a))g(a + te^{i\varphi}(b-a))$$

$$\leq \int_0^1 \left\{ \alpha(f(a + te^{i\varphi}(b-a)))^{\frac{1}{\alpha}} + \beta(g(a + te^{i\varphi}(b-a)))^{\frac{1}{\beta}} \right\} dt$$

$$\leq \int_0^1 \left\{ \alpha[(f(a))^{1-t}(f(b))^t]^{\frac{1}{\alpha}} + \beta[(g(a))^{1-t}(g(b))^t]^{\frac{1}{\beta}} \right\} dt$$

$$= \alpha(f(a))^{\frac{1}{\alpha}} \int_0^1 \left(\frac{f(b)}{f(a)}\right)^{\frac{t}{\alpha}} dt + \beta(g(a))^{\frac{1}{\beta}} \int_0^1 \left(\frac{g(b)}{g(a)}\right)^{\frac{t}{\beta}} dt$$

$$= \alpha^2(f(a))^{\frac{1}{\alpha}} \int_0^{\frac{1}{\alpha}} \left(\frac{f(b)}{f(a)}\right)^{u} du + \beta^2(g(a))^{\frac{1}{\beta}} \int_0^{\frac{1}{\beta}} \left(\frac{g(b)}{g(a)}\right)^{v} dv$$

$$= \alpha^2 \frac{(f(b))^{\frac{1}{\alpha}} - (f(a))^{\frac{1}{\alpha}}}{\log f(b) - \log f(a)} + \beta^2 \frac{(g(b))^{\frac{1}{\beta}} - (g(a))^{\frac{1}{\beta}}}{\log g(b) - \log g(a)}$$

$$= \alpha^2 \frac{(f(b))^{\frac{1}{\alpha}} - (f(a))^{\frac{1}{\alpha}}}{f(b) - f(a)} L(f(b), f(a)) + \beta^2 \frac{(g(b))^{\frac{1}{\alpha}} - (g(a))^{\frac{1}{\beta}}}{g(b) - g(a)} L(g(b), g(a))$$

$$= \alpha \left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a)) \right]^{\frac{\alpha}{1-\alpha}} L(f(b), f(a)) + \beta \left[L_{(\frac{1}{\beta}-1)}(g(b), g(a)) \right]^{\frac{\beta}{1-\beta}} L(g(b), g(a))$$

$$\leq \alpha \frac{f(a) + f(b)}{2} \left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a)) \right]^{\frac{\alpha}{1-\alpha}} + \beta \frac{g(a) + g(b)}{2} \left[L_{(\frac{1}{\beta}-1)}(g(b), g(a)) \right]^{\frac{\beta}{1-\beta}}.$$

This completes the proof. □

Theorem 3.2. *Let $f, g : I = [a, a + e^{i\varphi}(b-a)] \to (0, \infty)$ be* log *φ-convex functions with $a < a + e^{i\varphi}(b-a)$. Then, we have*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(2a + e^{i\varphi}(b-a) - x)dx$$

$$\leq \alpha \frac{f(a) + f(b)}{2} \left[L_{(\frac{1}{\alpha}-1)}(f(b), f(a)) \right]^{\frac{\alpha}{1-\alpha}} + \beta \frac{g(a) + g(b)}{2} \left[L_{(\frac{1}{\beta}-1)}(g(b), g(a)) \right]^{\frac{\beta}{1-\beta}}.$$

Proof. The proof directly follows from the proof of Theorem 3.1. □

Theorem 3.3. *Let $f_1, f_2, \ldots, f_n$ be* log *φ-convex functions. Then for $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, we have*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \leq \sum_{i=1}^{n} \left\{ \alpha_i \frac{f_i(a) + f_i(b)}{2} \left[L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{\alpha_i}{1-\alpha_i}} \right\}. \tag{3.1}$$

Proof. Since $f_1, f_2, \ldots, f_n$ be log φ-convex functions and using inequality

$$f_1.f_2 \ldots f_n \leq \alpha_1(f_1)^{\frac{1}{\alpha_1}} + \alpha_2(f_2)^{\frac{1}{\alpha_1}} + \ldots + \alpha_n(f_n)^{\frac{1}{\alpha_n}}, \quad \alpha_1, \alpha_2, \ldots, \alpha_n > 0, \sum_{i=1}^{n} \alpha_i = 1,$$

we have

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \leq \int_0^1 \left\{ \sum_{i=1}^{n} \alpha_i (f_i(a + te^{i\varphi}(b-a)))^{\frac{1}{\alpha_i}} \right\} dt$$

$$\leq \int_0^1 \left\{ \sum_{i=1}^{n} \alpha_i [(f_i(a))^{1-t}(f_i(b))^t]^{\frac{1}{\alpha_i}} \right\} dt = \sum_{i=1}^{n} \alpha_i (f_i(a))^{\frac{1}{\alpha_i}} \int_0^1 \left(\frac{f_i(b)}{f_i(a)} \right)^{\frac{t}{\alpha_i}} dt$$

$$= \sum_{i=1}^{n} (\alpha_i)^2 (f_i(a))^{\frac{1}{\alpha_i}} \int_0^{\frac{1}{\alpha_i}} \left(\frac{f_i(b)}{f_i(a)} \right)^u du = \sum_{i=1}^{n} (\alpha_i)^2 \frac{(f_i(b))^{\frac{1}{\alpha_i}} - (f_i(a))^{\frac{1}{\alpha_i}}}{\log f_i(b) - \log f_i(a)}$$

$$= \sum_{i=1}^{n} (\alpha_i)^2 \frac{(f_i(b))^{\frac{1}{\alpha_i}} - (f_i(a))^{\frac{1}{\alpha_i}}}{f_i(b) - f_i(a)} L(f_i(b), f_i(a)) = \sum_{i=1}^{n} \alpha_i \left[L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(b), f_i(a))$$

$$\leq \sum_{i=1}^{n} \alpha_i \frac{f_i(a) + f_i(b)}{2} \left[L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{1-\alpha_i}{\alpha_i}}.$$

This completes the proof. □

Remark 3.4. *If we suppose,* $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$ *in inequality* (3.1). *Then*

$$\frac{n}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \leq \sum_{i=1}^{n} \left\{ \frac{f_i(a) + f_i(b)}{2} [L_{n-1}(f_i(b), f_i(a))]^{n-1} \right\}.$$

Theorem 3.5. *Let* $f_1, f_2, \ldots, f_n$ *be* log φ*-concave functions. Then, for* $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ *and* $\sum_{i=1}^{n} \alpha_i = 1$, *we have*

$$\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^{n} f_i(x)dx \geq \sum_{i=1}^{n} \left\{ \alpha_i \frac{f_i(a) + f_i(b)}{2} \left[L_{(\frac{1}{\alpha_i}-1)}(f_i(b), f_i(a)) \right]^{\frac{\alpha_i}{1-\alpha_i}} \right\}.$$

Theorem 3.6. *Let f and g be increasing and* log φ*-convex functions on* $I = [a, a + e^{i\varphi}(b-a)]$. *Then*

$$f\left(\frac{2a + e^{i\varphi}(b-a)}{2} \right) L[g(a), g(b)] + g\left(\frac{2a + e^{i\varphi}(b-a)}{2} \right) L[f(a), f(b)]$$

$$\leq \frac{1}{b-a} \int_a^b f(x) w(2a + e^{i\varphi}(b-a) - x)dx + L[f(a)g(a), f(b)g(b)].$$

Proof. Let f and g be log φ-convex functions. Then we have

$$f(a + te^{i\varphi}(b-a)) \leq [f(a)]^{1-t}[f(b)]^t$$
$$g(a + (1-t)e^{i\varphi}(b-a)) \leq [g(a)]^t [g(b)]^{1-t}.$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2 < x_3 < x_4$, we have

$$\begin{aligned}
&f(a + te^{i\varphi}(b-a))[g(a)]^t[g(b)]^{1-t} + g(a + (1-t)e^{i\varphi}(b-a))[f(a)]^{1-t}[f(b)]^t \\
&\leq f(a + te^{i\varphi}(b-a))w(a + (1-t)e^{i\varphi}(b-a)) + [f(a)]^{1-t}[f(b)]^t[g(a)]^t[g(b)]^{1-t}.
\end{aligned}$$

Integrating above inequalities with respect to t on $[0, 1]$, we have

$$\begin{aligned}
&\int_0^1 f(a + te^{i\varphi}(b-a))[g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(a + (1-t)e^{i\varphi}(b-a))[f(a)]^{1-t}[f(b)]^t dt \\
&\leq \int_0^1 f(a + te^{i\varphi}(b-a))g(a + (1-t)e^{i\varphi}(b-a))dt + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^t[g(b)]^{1-t}dt.
\end{aligned}$$

Now, since f and g are increasing, then, we have

$$\begin{aligned}
&\int_0^1 f(a + te^{i\varphi}(b-a))dt \int_0^1 [g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(a + (1-t)e^{i\varphi}(b-a))dt \int_0^1 [f(a)]^{1-t}[f(b)]^t dt \\
&\leq \int_0^1 f(a + te^{i\varphi}(b-a))g(a + (1-t)e^{i\varphi}(b-a))dt + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^t[g(b)]^{1-t}dt.
\end{aligned}$$

Now after simple integration, we have

$$\begin{aligned}
&\frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dxL[g(a), g(b)] \\
&+ \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} g(2a + e^{i\varphi}(b-a) - x)dxL[f(a), f(b)] \\
&\leq \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)g(2a + e^{i\varphi}(b-a) - x)dx + L[f(a)g(a), f(b)g(b)].
\end{aligned}$$

Now, using the left hand side of Hermite-Hadamard's inequality for $\log \varphi$-convex functions, we have

$$\begin{aligned}
&f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)L[g(a), g(b)] + g\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right)L[f(a), f(b)] \\
&\leq \frac{1}{b-a} \int_a^b f(x)w(2a + e^{i\varphi}(b-a) - x)dx + L[f(a)g(a), f(b)g(b)].
\end{aligned}$$

The desired result. □

Theorem 3.7. *Let $f_1, f_2, \dots, f_n$ be differentiable* log *φ-convex functions on I^0(interior of I). Then, we have*

$$\left.\begin{array}{l} \int\limits_a^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \geq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_2(v)f_3(v)\dots f_n(v)\exp\left(\Delta_1\right)dv \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_3(v)\dots f_n(v)\exp\left(\Delta_2\right)dv \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_2(v)\dots f_{n-1}(v)\exp\left(\Delta_n\right)dv \end{array}\right\}, \tag{3.2}$$

where

$$\Delta_i = \left\langle \frac{f'_{\varphi_i}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_i(\frac{2a+e^{i\varphi}(b-a)}{2})}, v - \frac{2a+e^{i\varphi}(b-a)}{2} \right\rangle$$

Proof. Since $f_1, f_2 \dots, f_n$ be differentiable log φ-convex functions, so we have

$$f_1(v) \geq f_1(u)\exp\left[\left\langle \frac{f'_{\varphi_1}(u)}{f_1(u)}, v-u \right\rangle\right], \tag{3.3}$$

$$f_2(v) \geq f_2(u)\exp\left[\left\langle \frac{f'_{\varphi_2}(u)}{f_2(u)}, v-u \right\rangle\right], \tag{3.4}$$

$$\vdots$$

$$f_n(v) \geq f_n(u)\exp\left[\left\langle \frac{f'_{\varphi_n}(u)}{f_n(u)}, v-u \right\rangle\right], \tag{3.5}$$

Multiplying (3.3) by $\alpha_1 f_2(v)f_3(v)\dots f_n(v)$, (3.4) by $\alpha_2 f_1(v)f_3(v)\dots f_n(v)$ and (3.5) by $\alpha_n f_1(v)f_2(v)\dots f_{n-1}(v)$ respectively and then adding the resultant, we have

$$\left.\begin{array}{l} \sum\limits_{i=1}^{n} f_i(v) \\ \geq \alpha_1 f_1(u)f_2(v)f_3(v)\dots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_1}(u)}{f_1(u)}, v-u \right\rangle\right] \\ \quad +\alpha_2 f_2(u)f_1(v)f_3(v)\dots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_2}(u)}{f_2(u)}, v-u \right\rangle\right] \\ \quad \vdots \\ \quad +\alpha_n f_n(u)f_1(v)f_2(v)\dots f_{n-1}(v)\exp\left[\left\langle \frac{f'_{\varphi_n}(u)}{f_n(u)}, v-u \right\rangle\right] \end{array}\right\}. \tag{3.6}$$

Putting $u = \frac{2a+e^{i\varphi}(b-a)}{2}$ in (3.6), we have

$$\left.\begin{array}{l} \sum\limits_{i=1}^{n} f_i(v) \\ \geq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) f_2(v)f_3(v)\dots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_1}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_1(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle\right] \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) f_1(v)f_3(v)\dots f_n(v)\exp\left[\left\langle \frac{f'_{\varphi_2}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_2(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle\right] \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) f_1(v)f_2(v)\dots f_{n-1}(v)\exp\left[\left\langle \frac{f'_{\varphi_n}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_n(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle\right] \end{array}\right\}.$$

Integrating both sides of above inequality on $[a, a+e^{i\varphi}(b-a)]$, we have

$$\left.\begin{array}{l} \int\limits_a^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \geq \alpha_1 f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_2(v)f_3(v)\dots f_n(v)\exp(\Delta_1)\,dv \\ \quad +\alpha_2 f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_3(v)\dots f_n(v)\exp(\Delta_2)\,dv \\ \quad \vdots \\ \quad +\alpha_n f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_2(v)\dots f_{n-1}(v)\exp(\Delta_n)\,dv \end{array}\right\},$$

This completes the proof. □

Remark 3.8. *If we suppose,* $\alpha_1 = \alpha_2 = \dots = \alpha_n = \frac{1}{n}$ *in inequality* (3.2). *Then*

$$\left.\begin{array}{l} n\int\limits_a^{a+e^{i\varphi}(b-a)} \sum\limits_{i=1}^{n} f_i(v)dv \\ \geq f_1\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_2(v)f_3(v)\dots f_n(v)\exp(\Delta_1)\,dv \\ \quad +f_2\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_3(v)\dots f_n(v)\exp(\Delta_2)\,dv \\ \quad \vdots \\ \quad +f_n\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) \int\limits_a^{a+e^{i\varphi}(b-a)} f_1(v)f_2(v)\dots f_{n-1}(v)\exp(\Delta_n)\,dv \end{array}\right\},$$

where

$$\Delta_i = \left\langle \frac{f'_{\varphi_i}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_i(\frac{2a+e^{i\varphi}(b-a)}{2})}, v-\frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle$$

Theorem 3.9. *Let $f_1, f_2, \dots, f_n$ be differentiable* log *φ-concave functions on I^0(interior of I). Then, we have*

$$\left.\begin{aligned}
&\int_a^{a+e^{i\varphi}(b-a)} \sum_{i=1}^n f_i(v)dv\\
&\leq \alpha_1 f_1\left(\tfrac{2a+e^{i\varphi}(b-a)}{2}\right) \int_a^{a+e^{i\varphi}(b-a)} f_2(v)f_3(v)\dots f_n(v)\exp\left(\Delta_1\right)dv\\
&\quad+\alpha_2 f_2\left(\tfrac{2a+e^{i\varphi}(b-a)}{2}\right) \int_a^{a+e^{i\varphi}(b-a)} f_1(v)f_3(v)\dots f_n(v)\exp\left(\Delta_2\right)dv\\
&\quad\vdots\\
&\quad+\alpha_n f_n\left(\tfrac{2a+e^{i\varphi}(b-a)}{2}\right) \int_a^{a+e^{i\varphi}(b-a)} f_1(v)f_2(v)\dots f_{n-1}(v)\exp\left(\Delta_n\right)dv
\end{aligned}\right\},$$

where

$$\Delta_i = \left\langle \frac{f'_{\varphi_i}(\frac{2a+e^{i\varphi}(b-a)}{2})}{f_i(\frac{2a+e^{i\varphi}(b-a)}{2})}, v - \frac{2a+e^{i\varphi}(b-a)}{2}\right\rangle$$

Remark 3.10. *If $\varphi = 0$, then* log *φ-convex (concave) functions become* log*-convex (concave) in classical sense, thus our results continue to hold for* log*-convex (concave) functions.*

Acknowledgement. The authors are grateful to editor and anonymous referee for his/her valuable suggestions and comments.

References

[1] B. G. Pachpatte, *A note on integral inequalities involving two log-convex functions*, Math. Inequal. Appl. 7 (2004), 511515.

[2] G. Cristescu, M. A. Noor, M. U. Awan, *Bounds of the second degree cumulative frontier gaps of functions with generalized convexity*, Carpathian Journal of Mathematics. 31(2) (2015), 173-180.

[3] S. S. Dragomir and C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.

[4] S. S. Dragomir and B. Mond, *Integral inequalities of Hadamard's type for* log*-convex functions*, Demonstration Math. 2 (1998), 354-364.

[5] M. A. Noor, *Advanced convex analysis*, Lecture Notes, Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan, 2010.

[6] M. A. Noor, *Extended general variational inequalities*, Applied Mathematics Letters. 22, (2009), 182-186.

[7] M. A. Noor, *On Hadamard integral inequalities involving two log-preinvex functions*, J. Inequal. Pure. Appl. Math. 8 (3) (2007).

[8] M. A. Noor, *On Hermite-Hadamarad integral inequalities for product of two nonconvex functions* J. Adv. Math. Stud. 2 (2009), 53-62.

[9] M. A. Noor, *Some developments in general variational inequalities*, Appl. Math. Comput. 152 (2004), 199-277.

[10] M. A. Noor, *Some new classes of nonconvex functions*, Nonl. Funct. Anal. Appl. 11 (1) (2006), 165-171.

[11] M. A. Noor, K. I. Noor, M. A. Ashraf, M. U. Awan and B. Bashir, *Hermite-Hadamard Inequalities for h_φ-convex Functions*, Nonl. Anal. Forum. 18 (2013), 65-76.

[12] M. A. Noor, K. I. Noor, M. U. Awan, *Hermite-Hadamard inequalities for relative semi-convex functions and applications*, Filomat, 28 (2)(2014), 221-230.

[13] M. A. Noor, M. U. Awan, K. I. Noor, *On some inequalities for relative semi-convex functions*, J. Inequal. Appl. 2013, 2013:332.

[14] M. E. Ozdemir, M. Avci and A. O. Akdemir, *Simpson type inequalities via φ-convexity*, (2012), in Press.

[15] M. Z. Sarikaya, H. Bozkurt and N. Alp, *On Hadamard type integral inequalities for nonconvex functions*, preprint, (2015).

[16] G-S Yang, K-L Tseng and H-T Wang, *A note on integral inequalities of Hadamard type for log-convex and log-concave functions*, Taiwanese J. Math. 16(2) (2012), 479-496.