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Integral Inequalities for Relative Harmonic (s, η)-Convex Functions

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Abstract. We introduce the concept of relative harmonic (s, η)-convex functions as a generalization of harmonic convex functions. Some basic inequalities related to relative harmonic (s, η)-convex functions are proved. We also investigate the famous Jensen, Hermite-Hadamard and Fejer type inequalities for this class of functions. The ideas and techniques of this paper may stimulate further research.

1. Introduction

Convexity plays an important role in pure and applied mathematics. Convex sets and convex functions play a significant role especially in nonlinear programming and optimization theory. This motivates many researchers to study this branch of mathematical analysis. On the other hand it should be noticed that in new problems related to convexity, generalized notions for convex sets and functions are required to reach favorite and applicable results. In recent years, various inequalities for convex functions and their variant forms are being developed using innovative techniques, see [1, 4, 9, 11, 13–15]. Anderson et al. [1] and Iscan [7] introduced and studied harmonic convex functions. In particular, it has been shown that f is a harmonic convex function, if and only if, it satisfies the Hermite-Hadamard inequality.
Recently, Gordji et al.[3] considred a new class of convex functions, which is called φ-convex functions. For some properties of the φ-convex functions, see [3, 4, 16] and the references therein. Motivated and inspired by the research work in this field, we introduce and investigate new class of convex functions, which is called relative harmonic (s, η)-convex functions. This class of convex functions can be viewed as an important generalization of harmonic convex functions. We discuss some basic results of harmonic (s, η)-convex functions. We also derive the Hermite-Hadamard and Fejer type inequalities for this class of functions. Our results include a wide class of known and new inequalities for harmonic convex functions and their variant forms as special cases.

2. Preliminaries

In this section we recall some known and new concept.

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Definition 2.1. [17]. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.2. [7]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be a harmonic convex function, if and only if,*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Now we define some new concepts.

Definition 2.3. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be relative harmonic (s, η)-convex function, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq h[(1-t)^s]f(x) + h(t^s)[f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

Note that for $t = \frac{1}{2}$, we have

$$f\Big(\frac{2xy}{x+y}\Big) \leq h\Big(\frac{1}{2^s}\Big)[2f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I. \tag{1}$$

We would like to mention that Definition 2.13 has a clear geometric interpretation. This definition essentially says that there is a path starting from a point $f(x)$ which is contained in $\mathbb{R}$. We do not require that the point $f(y)$ should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that $f(y)$ should be an end point of the path for every pairs of the point $f(x)$, $f(y) \in \mathbb{R}$, then $\eta(f(y), f(x)) = f(y) - f(x)$, and consequently relative harmonic (s, η)-convex function reduces to relative harmonic s-convex function. Thus it is true that every relative harmonic (s, η)-convex function is also a relative harmonic s-convex function with respect to $\eta(f(y), f(x)) = f(y) - f(x)$, but the converse is not necessarily true.

Now we discuss some special cases of relative harmonic (s, η) convex function.

I. If we take $h(t^s) = t^s$ in Definition 2.3 then it reduces to the Definition of harmonic (s, η)-convex functions.

Definition 2.4. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic (s, η)-convex function in second sense, where $s \in [-1, 1]$, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq (1-t)^s f(x) + t^s[f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

II. If we take $h(t^s) = t^s$ and $s = 1$ in Definition 2.3 then it reduces to the Definition of harmonic η-convex functions.

Definition 2.5. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic η-convex function, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq f(x) + t\eta(f(y), f(x)), \qquad \forall x, y \in I, t \in [0, 1].$$

III. If we take $h(t^s) = \frac{1}{t^s}$ in Definition 2.3 then it reduces to the Definition of Godunova-Levin harmonic (s, η)-convex functions.

Definition 2.6. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin harmonic (s, η)-convex in second sense, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s}[f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

IV. If we take $h(t^s) = \frac{1}{t^s}$ and $s = 1$ in Definition 2.3 then it reduces to the Definition of Godunova-Levin harmonic η-convex functions.

Definition 2.7. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin harmonic η-convex, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq \frac{1}{1-t} f(x) + \frac{1}{t}[f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

V. If we take $h(t^s) = (\frac{\sqrt{t}}{2\sqrt{1-t}})^s$ in Definition 2.3 then it reduces to the Definition of MT-harmonic (s, η)-convex functions.

Definition 2.8. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be MT-harmonic (s, η)-convex function in second sense, where $s \in [-1, 1]$, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq \Big(\frac{\sqrt{1-t}}{2\sqrt{t}}\Big)^s f(x) + \Big(\frac{\sqrt{t}}{2\sqrt{1-t}}\Big)^s [f(x) + \eta(f(y), f(x))].$$

VI. If we take $h(t^s) = (\frac{\sqrt{t}}{2\sqrt{1-t}})^s$ and $s = 1$ in Definition 2.3 then it reduces to the Definition of MT-harmonic η-convex functions.

Definition 2.9. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be MT-harmonic η-convex, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}}[f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

VII. If we take $h(t^s) = t^s(1-t)^s$ in Definition 2.3 then it reduces to the Definition of generalized harmonic (s, η)-convex functions.

Definition 2.10. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized harmonic (s, η)-convex, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq t^s(1-t)^s[2f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

VIII. If we take $h(t^s) = t^s(1-t)^s$ and $s = 1$ in Definition 2.3 then it reduces to the Definition of generalized harmonic η-convex functions.

Definition 2.11. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized harmonic η-convex, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq t(1-t)[2f(x) + \eta(f(y), f(x))], \qquad \forall x, y \in I, t \in (0, 1).$$

IX. If we take $h(t^s) = 1$ in Definition 2.3 then it reduces to the Definition of harmonic (η, P) functions.

Definition 2.12. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic (η, P), if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq 2f(x) + \eta(f(y), f(x)), \qquad \forall x, y \in I.$$

Definition 2.13. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be relative harmonic* log (s, η)*-convex function, if*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq [f(x)]^{h[(1-t)^s]}[f(x) + \eta(f(y), f(x))]^{h(t^s)}, \qquad \forall x, y \in I, t \in [0, 1].$$

It follows that

$$\log f\Big(\frac{xy}{tx+(1-t)y}\Big) \quad \leq \quad h[(1-t)^s]\log f(x) + h(t^s)\log[f(x)+\eta(f(y),f(x))].$$

For appropriate and suitable choices of $h(t^s)$, one can obtain several new classes of relative harmonic log (s,η)-convex functions. This shows that the relative harmonic (s,η)-convex functions is quite general and unifying ones.
We also make use of the following simple and important known fact in obtaining the main results of this paper.

Remark 2.14. *If $I=[a,b]\subseteq \mathbb{R}\setminus\{0\}$ and if we consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$ defined by $g(t)=f\left(\frac{1}{t}\right)$, then f is relative harmonic (s,η)-convex on $[a,b]$, if and only if, g is relative (s,η)-convex in the usual sense on $\left[\frac{1}{b},\frac{1}{a}\right]$.*

Lemma 2.15. *Suppose that $a,b,c\in\mathbb{R}$. Then*

1. $\min\{a,b\}\leq \frac{a+b}{2}$.
2. *if $c\geq 0$, $c.\min\{a,b\}=\min\{ca,cb\}$.*

3. Main results

In this section, we obtain Hermite-Hadamard and Fejer type inequalities for relative harmonic (s,η)-convex functions, where η is a bifunction bounded from above on $f([a,b])\times f([a,b])$, satisfies a Lipschitz condition.

Theorem 3.1. *Let $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be relative harmonic (s,η)-convex function. If $f\in L[a,b]$. Then*

$$\begin{aligned}\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x \quad &\leq \quad \min\Big\{f(a)\int_0^1[h[(1-t)^s]+h(t^s)]\mathrm{d}t\\ &\quad +\eta(f(b),f(a))\int_0^1 h(t^s)\mathrm{d}t,\ f(b)\int_0^1[h[(1-t)^s]+h(t^s)]\mathrm{d}t\\ &\quad +\eta(f(a),f(b))\int_0^1 h(t^s)\mathrm{d}t\Big\}\\ &\leq \quad \frac{f(a)+f(b)}{2}\int_0^1[h[(1-t)^s]+h(t^s)]\mathrm{d}t\\ &\quad +\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1 h(t^s)\mathrm{d}t.\end{aligned}$$

Proof. Let f be relative harmonic (s,η)-convex function. Then

$$f\Big(\frac{ab}{ta+(1-t)b}\Big)\leq h[(1-t)^s]f(a)+h(t^s)[f(a)+\eta(f(b),f(a))],$$

and

$$f\Big(\frac{ab}{(1-t)a+tb}\Big)\leq h[(1-t)^s]f(b)+h(t^s)[f(b)+\eta(f(a),f(b))], \qquad \forall x,y\in I, t\in[0,1].$$

Thus, we have

$$\begin{aligned}f\Big(\frac{ab}{ta+(1-t)b}\Big)+f\Big(\frac{ab}{(1-t)a+tb}\Big) \quad &\leq \quad [f(a)+f(b)][h[(1-t)^s]+h(t^s)]\\ &+ \quad h(t^s)[\eta(f(a),f(b))+\eta(f(b),f(a))],\end{aligned} \tag{2}$$

Integrating over $[0,1]$, we have

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x &\leq \frac{f(a)+f(b)}{2}\int_0^1 [h[(1-t)^s]+h(t^s)]\mathrm{d}t\\ &+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1 h(t^s)\mathrm{d}t,\end{aligned}$$

the required result. □

Theorem 3.2. *Let $f: I=[a,b]\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be relative harmonic (s,η)-convex function. If $f\in L[a,b]$, then*

$$\begin{aligned}f\Big(\frac{2ab}{a+b}\Big) &\leq \min\Big\{h\Big(\frac{1}{2^s}\Big)[2f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)]\\ &\quad ,h\Big(\frac{1}{2^s}\Big)[2f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)]\Big\}\\ &\leq h\Big(\frac{1}{2^s}\Big)\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)+f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]\\ &\quad +\frac{1}{2}h\Big(\frac{1}{2^s}\Big)\Big[\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\\ &\quad +\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big]. \end{aligned}\tag{3}$$

Proof. Let f be relative harmonic (s,η)-convex function. Then taking $x=\frac{ab}{ta+(1-t)b}$ and $y=\frac{ab}{(1-t)a+tb}$ in (4), we have

$$f\Big(\frac{2ab}{a+b}\Big)\leq h\Big(\frac{1}{2^s}\Big)[2f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)],$$

and

$$f\Big(\frac{2ab}{a+b}\Big)\leq h\Big(\frac{1}{2^s}\Big)[2f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)].$$

Adding the above two inequalities, we have

$$\begin{aligned}f\Big(\frac{2ab}{a+b}\Big) &\leq h\Big(\frac{1}{2^s}\Big)\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)+f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]\\ &\quad +\frac{1}{2}h\Big(\frac{1}{2^s}\Big)\Big[\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\\ &\quad +\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big],\end{aligned}$$

the required result. □

Theorem 3.3. *Let $f: I=[a,b]\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be relative harmonic (s,η)-convex function. If $f\in L[a,b]$, then*

$$\frac{1}{2h\Big(\frac{1}{2^s}\Big)}f\Big(\frac{2ab}{a+b}\Big)-\frac{ab}{2(b-a)}\int_a^b\frac{\eta\Big(f(x),f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)}{x^2}\mathrm{d}x \leq \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x \tag{4}$$

Proof. Integrating inequality (3), we obtain the required result. □

We now obtain some Fejer type integral inequalities for relative harmonic (s,η)-convex functions.

Theorem 3.4. *Let $f, g : I = [a, b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be relative harmonic (s, η)-convex functions. If $fg \in L[a, b]$, then*

$$\begin{aligned}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \quad &\leq \quad \frac{1}{2(b-a)^s}[f(a)+f(b)]\int_a^b [a^s(b-x)^s + b^s(x-a)^s]\frac{g(x)}{x^{s+2}}\mathrm{d}x \\ &\quad + \frac{1}{2}\Big[\frac{\eta(f(b), f(a)) + \eta(f(a), f(b))}{(b-a)^s}\Big]\int_a^b b^s(x-a)^s\frac{g(x)}{x^{s+2}}\mathrm{d}x.\end{aligned}$$

where $g : [a, b] \subseteq \mathbb{R}\setminus\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x) = g\Big(\frac{abx}{[a+b]x - ab}\Big), \quad \forall x \in [a, b].$$

Proof. Let f be relative harmonic (s, η)-convex function. Then multiplying inequality (2) with $g\big(\frac{ab}{ta+(1-t)b}\big)$ and integrating over t, we have

$$\begin{aligned}&\int_0^1 \Big[f\Big(\frac{ab}{ta+(1-t)b}\Big) + f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]g\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t \\ \leq\quad & [f(a)+f(b)]\int_0^1 [h[(1-t)^s] + h(t^s)]g\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t \\ & + [\eta(f(a), f(b)) + \eta(f(b), f(a))]\int_0^1 h(t^s)g\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t,\end{aligned}$$

Since g is symmetric, we have

$$\begin{aligned}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \quad &\leq \quad \frac{1}{2(b-a)^s}[f(a)+f(b)]\int_a^b [a^s(b-x)^s + b^s(x-a)^s]\frac{g(x)}{x^{s+2}}\mathrm{d}x \\ &\quad + \frac{1}{2}\Big[\frac{\eta(f(b), f(a)) + \eta(f(a), f(b))}{(b-a)^s}\Big]\int_a^b b^s(x-a)^s\frac{g(x)}{x^{s+2}}\mathrm{d}x,\end{aligned}$$

the required result. □

Theorem 3.5. *Let $f : I = [a, b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be relative harmonic (s, η)-convex function. If $fg \in L[a, b]$, then*

$$\begin{aligned}& f\Big(\frac{2ab}{a+b}\Big)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \\ \leq\quad & \int_a^b \frac{g(x)}{x^2}\min\Big\{f(x) + \frac{1}{2}\eta\Big(f\Big(\frac{abx}{(a+b)x-ab}\Big), f(x)\Big), f\Big(\frac{abx}{(a+b)x-ab}\Big) \\ & + \frac{1}{2}\eta\Big(f(x), f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)\Big\}\mathrm{d}x \\ \leq\quad & \int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x + \frac{1}{2}\int_a^b \frac{g(x)}{x^2}\Big[\eta\Big(f(x), f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)\Big]\mathrm{d}x\end{aligned}$$

where $g : [a, b] \subseteq \mathbb{R}\setminus\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x) = g\Big(\frac{abx}{[a+b]x - ab}\Big), \quad \forall x \in [a, b].$$

Proof. Let f, g be relative harmonic (s,η)-convex functions. Then Multiplying (3) with $g\Big(\frac{ab}{ta+(1-t)b}\Big)$ and integrating over t, we have

$$\begin{aligned}
& f\Big(\frac{2ab}{a+b}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t \\
\leq\ & \int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\min\Big\{h\Big(\frac{1}{2^s}\Big)[2f\Big(\frac{ab}{ta+(1-t)b}\Big)+\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big) \\
& ,f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)], h\Big(\frac{1}{2^s}\Big)2f\Big(\frac{ab}{(1-t)a+tb}\Big)+\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big\}\mathrm{d}t \\
=\ & \min\Big\{2h\Big(\frac{1}{2^s}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)f\Big(\frac{ab}{ta+(1-t)b}\Big)\mathrm{d}t \\
& +h\Big(\frac{1}{2^s}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\eta\Big(\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big) \\
& ,f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big)\Big)\mathrm{d}t, 2h\Big(\frac{1}{2^s}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)f\Big(\frac{ab}{(1-t)a+tb}\Big)\mathrm{d}t \\
& +\frac{1}{2}\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\mathrm{d}t\Big\} \\
\leq\ & h\Big(\frac{1}{2^s}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\Big[f\Big(\frac{ab}{ta+(1-t)b}\Big)+f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big]\mathrm{d}t \\
& +\frac{1}{2}h\Big(\frac{1}{2^s}\Big)\int_0^1 g\Big(\frac{ab}{ta+(1-t)b}\Big)\Big[\eta\Big(f\Big(\frac{ab}{(1-t)a+tb}\Big),f\Big(\frac{ab}{ta+(1-t)b}\Big)\Big) \\
& +\eta\Big(f\Big(\frac{ab}{ta+(1-t)b}\Big),f\Big(\frac{ab}{(1-t)a+tb}\Big)\Big)\Big]\mathrm{d}t
\end{aligned}$$

By the symmetry of g on $[a,b]$, we have

$$\begin{aligned}
& \frac{1}{2h\big(\frac{1}{2^s}\big)}f\Big(\frac{2ab}{a+b}\Big)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \\
\leq\ & \int_a^b \frac{g(x)}{x^2}\min\Big\{f(x)+\frac{1}{2}\eta\Big(f\Big(\frac{abx}{(a+b)x-ab}\Big),f(x)\Big),f\Big(\frac{abx}{(a+b)x-ab}\Big) \\
& +\frac{1}{2}\eta(f(x),f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)\Big\}\mathrm{d}x \\
\leq\ & \int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x+\frac{1}{2}\int_a^b \frac{g(x)}{x^2}\Big[\eta\Big(f(x),f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)\Big]\mathrm{d}x,
\end{aligned}$$

which gives the refinement of the (4). □

Corollary 3.6. *Under the assumptions of Theorem 3.4 and Theorem 3.5, we have*

(i). *If $g(x)=1$, then we have the Hermite-Hadamard type inequality:*

$$\begin{aligned}
& \frac{1}{2h\big(\frac{1}{2^s}\big)}f\Big(\frac{2ab}{a+b}\Big)-\frac{ab}{2(b-a)}\int_a^b \frac{\eta\Big(f(x),f\Big(\frac{abx}{(a+b)x-ab}\Big)\Big)}{x^2}\mathrm{d}x \\
\leq\ & \frac{f(a)+f(b)}{2}\int_0^1 [h[(1-t)^s]+h(t^s)]\mathrm{d}t \\
& +\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{2}\int_0^1 h(t^s)\mathrm{d}t.
\end{aligned}$$

(ii). *If $g(x) = 1$ and M_η as the upper bound of η, then we have the Hermite-Hadamard type inequality:*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2^s}\right)} f\left(\frac{2ab}{a+b}\right) - \frac{M_\eta}{2} &\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2}\int_0^1 [h[(1-t)^s] + h(t^s)]\mathrm{d}t \\ &\quad + M_\eta \int_0^1 h(t^s)\mathrm{d}t.\end{aligned}$$

(iii). *If we consider M_η as the upper bound of η, then*

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2^s}\right)} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x - \frac{M_\eta}{2}\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \\ \leq\ & \int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \leq \frac{1}{2(b-a)^s}[f(a)+f(b)]\int_a^b [a^s(b-x)^s + b^s(x-a)^s]\frac{g(x)}{x^{s+2}}\mathrm{d}x \\ &+ \frac{M_\eta}{2(b-a)^s}\int_a^b b^s(x-a)^s \frac{g(x)}{x^{s+2}}\mathrm{d}x.\end{aligned}$$

(iv). *If we set $g(x) = 1$ and $\eta(f(x), f(y)) = f(x) - f(y)$, then classic form of Hermite-Hadamard-Fejer inequality can be obtained, see* [8].

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