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The Existence and Uniqueness of Pseudo-Almost Periodic Solutions of Semilinear Cauchy Inclusions of First Order

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Abstract. The main aim of this paper is to investigate pseudo-almost periodic solutions of abstract semilinear Cauchy inclusions of first order with multivalued linear operators generating infinitely differentiable degenerate semigroups with removable singularites at zero. Our results seem to be new even for a class of almost sectorial single-valued linear operators.

1. Introduction and preliminaries

Almost periodicity of solutions of abstract differential equations in Banach spaces has received much attention so far (for further information on the subject, we refer the reader to the monograph [12] by Y. Hino, T. Naito, N. V. Minh, J. S. Shin and references cited therein). The class of Banach space valued pseudo-almost periodic functions was introduced in the doctoral dissertation of C. Zhang [24] (1992), while the class of weighted pseudo-almost periodic functions was introduced by T. Diagana [8] (2006). From 1992 onwards, a great number of mathematicians has investigated pseudo-almost periodic solutions and weighted pseudo-almost periodic solutions for various classes of abstract Volterra integro-differential equations in Banach spaces (see e.g. [1]-[2], [7], [20] and [25]-[26]).

Let $(X, \|\cdot\|)$ be a Banach space. In this paper, we continue our investigations [14]-[16] by analyzing pseudo-almost periodic solutions of abstract semilinear Cauchy inclusions of first order with multivalued linear operators $\mathcal{A} : D(\mathcal{A}) \subseteq X \to P(X)$ satisfying the condition [10, (P), p. 47] introduced by A. Favini and A. Yagi:

(P) There exist finite constants $c,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Psi_c := \Big\{\lambda \in \mathbb{C} : \Re\lambda \geq -c\big(|\Im\lambda| + 1\big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

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The main goal of Theorem 2.4 is to prove the existence of a unique pseudo-almost periodic mild solution of the following semilinear differential inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1}$$

where $f : \mathbb{R} \times X \to X$ is pseudo-almost periodic. In Corollary 2.5, we state a simple consequence of Theorem 2.4 provided that the usually considered Lipschitz type condition is satisfied. It is very straightforward to incorporate our results in the study of pseudo-almost periodic solutions of abstract higher-order semilinear differential equations in Hölder spaces ([22]-[23], [16]), as well as in the study of pseudo-almost periodic solutions for semilinear Poisson heat equation

$$\begin{cases} \frac{\partial}{\partial x}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)), \quad t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

in L^p-spaces (here, Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$, $X = L^p(\Omega)$ and Δ denotes the Dirichlet Laplacian).

Suppose that the condition (P) holds. Then there exists a degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ and there exists a finite constant $M > 0$ such that $\|T(t)\| \leq Me^{-ct}t^{\beta-1}$, $t > 0$ ([15]). Concerning the Poisson heat equation in the Sobolev space $X = H^{-1}(\Omega)$, it is worth noting that the above conclusions hold with the number $\beta = 1$ and the region $\Psi_c = \mathbb{C} \setminus (-\infty, c)$, so that $(T(t))_{t>0}$ is bounded at zero and strongly continuous on $\overline{D(\mathcal{A})}$ (cf. [10, Example 3.3, pp. 74-75, and Remarks, p. 52]). If this is the case, then we are in a position to investigate pseudo-almost periodic mild solutions of the following semilinear fractional differential inclusion of order $\alpha \in (1,2)$:

$$D_t^\alpha u(t) \in \mathcal{A}u(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in \mathbb{R}, \tag{2}$$

where $f : \mathbb{R} \times X \to X$ is pseudo-almost periodic and $D_t^{\alpha-1}$ is the fractional derivative of a Riemann-Liouville type (cf. [2, Section 3] for more details; the Leray-Schauder alternative can be also employed here). The analysis of existence and uniqueness of pseudo-almost periodic solutions for (2) will be considered in our forthcoming paper [17].

We use the standard notation henceforth. By X and Y we denote two complex Banach spaces; the symbol $L(X) \equiv L(X, X)$ denotes the space of all bounded continuous operators from X into X. Let $I = \mathbb{R}$ or $I = [0, \infty)$. The space consisted of all bounded continuous functions from I into X, equipped with the sup-norm, is one of Banach's and we use the abbreviation $C_b(I : X)$ to designate it.

The class of almost periodic functions was introduced by H. Bohr in 1925 and later generalized by many other authors (cf. [3]-[4], [6], [11] and [19] for more details on the subject). Assume that $f : I \to X$ is continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ iff $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$. The space consisted of all almost periodic functions from the interval I into X will be denoted by $AP(I : X)$.

By $PAP_0(\mathbb{R} : X)$ we denote the space consisting of all bounded continuous functions $\Phi : \mathbb{R} \to X$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Phi(s)\|\, ds = 0;$$

$PAP_0(\mathbb{R} \times Y : X)$ denotes the space consisting of all continuous functions $\Phi : \mathbb{R} \times Y \to X$ such that $\{\Phi(t, y) : t \in \mathbb{R}\}$ is bounded for all $y \in Y$, and

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Phi(s, y)\|\, ds = 0,$$

uniformly in $y \in Y$. A function $f \in C_b(\mathbb{R} : X)$ is said to be pseudo-almost periodic iff it can be written as $f(t) = g(t) + \Phi(t)$, $t \in \mathbb{R}$, where $g \in AP(\mathbb{R} : X)$ and $\Phi \in PAP_0(\mathbb{R} : X)$. The parts $g(\cdot)$ and $\Phi(\cdot)$ are called the almost periodic part of $f(\cdot)$ and the ergodic perturbation of $f(\cdot)$. The vector space consisting of such functions is usually denoted by $PAP(\mathbb{R}:X)$; the sup-norm turns $PAP(\mathbb{R}:X)$ into a closed subspace of Banach space $C_b(\mathbb{R}:X)$.

For the purpose of research of pseudo-almost periodic properties of solutions to semilinear Cauchy inclusions, we need to recall the following well-known definitions and results (see e.g. R. Agarwal, B. de Andrade, C. Cuevas [2]):

Definition 1.1. *Let $1 \leq p < \infty$.*

(i) A function $f : I \times Y \to X$ is called almost periodic iff $f(\cdot,\cdot)$ is bounded, continuous as well as for every $\epsilon > 0$ and every compact $K \subseteq Y$ there exists $l(\epsilon, K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\epsilon, K)$ contains a number τ with the property that $\|f(t+\tau, y) - f(t, y)\| \leq \epsilon$ for all $t \in I$, $y \in K$. The collection of such functions will be denoted by $AP(I \times Y : X)$.

(ii) A function $f : \mathbb{R} \times Y \to X$ is said to be pseudo-almost periodic iff it is continuous and admits a decomposition $f = g + q$, where $g \in AP(\mathbb{R} \times Y : X)$ and $q \in PAP_0(\mathbb{R} \times Y : X)$. Denote by $PAP(\mathbb{R} \times Y : X)$ the vector space consisting of all pseudo-almost periodic functions.

Arguing as in the well known result of H.-X. Li, F.-L. Huang and J.-Y. Li [20, Theorem 2.1], we can prove the following auxiliary lemma (recall that, in contrast to [20], we assume the boundedness of almost periodic component a priori):

Lemma 1.2. *Let $f \in PAP(\mathbb{R} \times Y : X)$ and $h \in PAP(\mathbb{R} : Y)$. Then the mapping $t \mapsto f(t, h(t))$, $t \in \mathbb{R}$ belongs to the space $PAP(\mathbb{R} : X)$ provided that the following conditions hold:*

(i) The set $\{f(t,y) : t \in \mathbb{R},\ y \in B\}$ is bounded for every bounded subset $B \subseteq Y$.

(ii) $f(t,y)$ is uniformly continuous in each bounded subset of Y uniformly in $t \in \mathbb{R}$. That is, for any $\epsilon > 0$ and $B \subseteq Y$ bounded, there exists $\delta > 0$ such that $x,\ y \in B$ and $\|x-y\| \leq \delta$ imply $\|f(t,x) - f(t,y)\| \leq \epsilon$ for all $t \in \mathbb{R}$.

The theory of abstract degenerate differential equations is still an active field of research. We refer the reader to the monographs by R. W. Carroll, R. W. Showalter [5], A. Favini, A. Yagi [10], I. V. Melnikova, A. I. Filinkov [21] and M. Kostić [13] for more details about abstract degenerate differential equations. In the remaining part of this section, we will present a brief overview of definitions from the theory of multivalued linear operators in Banach spaces (cf. [10], [13] and references cited therein).

Let X and Y be Banach spaces. A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following conditions hold:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

In the case that $X = Y$, then we say that $\mathcal{A}$ is an MLO in X.

Let us recall that, if $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, then $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Assuming $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear submanifold of Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. Then the set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted by either $N(\mathcal{A})$ or Kern$(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be simply shown that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$; $\mathcal{A}$ is said to be injective iff $\mathcal{A}^{-1}$ is single-valued.

Assume that $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs. Then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It is evident that $\mathcal{A}+\mathcal{B}$ is likewise an MLO.

Let $\mathcal{A}$ be an MLO in X. Then the resolvent set of $\mathcal{A}$, $\rho(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $X = R(\lambda - \mathcal{A})$;

(ii) $(\lambda - \mathcal{A})^{-1}$ is a single-valued bounded operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ ($\lambda \in \rho(\mathcal{A})$); $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). The basic properties of resolvent sets of single-valued linear operators continue to hold in our framework ([10], [13]).

2. Pseudo-almost periodic solutions of abstract semilinear Cauchy inclusions of first order

We start our work in this section by stating the following simple lemma.

Lemma 2.1. *Suppose that $f : \mathbb{R} \to X$ is almost periodic. Then the function $F(\cdot)$, given by*

$$F(t) := \int_{-\infty}^{t} T(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{3}$$

is well-defined and almost periodic.

Proof. Let $1 < q < \infty$, provided $\beta = 1$, and let $1 < q < 1/1-\beta$, provided $\beta \in (0,1)$. Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$ and $f(\cdot)$ is Stepanov p-almost periodic (cf. [15] for the notion and more details). Therefore, the assertion of lemma is a simple consequence of [15, Proposition 2.11]. $\square$

Now we state the following important lemma:

Lemma 2.2. *Suppose that $f : \mathbb{R} \to X$ is pseudo-almost periodic. Then the function $F(\cdot)$, given by* (3), *is well-defined and pseudo-almost periodic.*

Proof. By definition, there exist two functions $g(\cdot) \in AP(\mathbb{R} : X)$ and $\Phi \in PAP_0(\mathbb{R} : X)$ such that $f(t) = g(t) + \Phi(t)$, $t \in \mathbb{R}$. Owing to Lemma 2.1, we have that the function

$$t \mapsto \int_{-\infty}^{t} T(t-s)g(s)\,ds, \quad t \in \mathbb{R}$$

is almost periodic so that it suffices to show that the function

$$t \mapsto \Psi(t) := \int_{-\infty}^{t} T(t-s)\Phi(s)\,ds, \quad t \in \mathbb{R}$$

belongs to the space $PAP_0(\mathbb{R} : X)$. This can be verified to be true on the basis of information given in the proof of [2, Lemma 2.14]; for the sake of of completeness, we will include all relevant details. Due to the boundedness of function $\Phi(\cdot)$, we have that there exists a finite constant $M' > 0$ such that:

$$\left\| \int_{-\infty}^{t} T(t-s)\Phi(s)\,ds \right\| \leq M' \int_{-\infty}^{t} e^{-c(t-s)} \big(t-s\big)^{\beta-1}\,ds = M'\Gamma(\beta)c^{-\beta}, \quad t \in \mathbb{R},$$

where $\Gamma(\cdot)$ denotes the Gamma function. Since $\int_{-\infty}^{t} T(t-s)\Phi(s)\,ds = \int_0^{\infty} T(s)\Phi(t-s)\,ds$, $t \in \mathbb{R}$, we can apply the dominated convergence theorem in order to see that the function $\Psi(\cdot)$ is continuous on $\mathbb{R}$. Therefore, it remains to be proved that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|\Psi(s)\|\,ds = 0. \tag{4}$$

Towards this end, observe that the Fubini theorem implies:

$$\frac{1}{2r} \int_{-r}^{r} \|\Psi(s)\|\,ds \leq \frac{1}{2r} \int_{-r}^{r} \int_0^{\infty} \|T(v)\| \|\Phi(s-v)\|\,dv\,ds \leq \frac{M}{2r} \int_0^{\infty} e^{-cv} v^{\beta-1} \Phi_r(s)\,ds,$$

where $\Phi_r(s) := 1/(2r)\int_{-r}^{r} \|\Phi(s-v)\|\,dv$, $s \geq 0$. Since $\Phi_r(\cdot)$ is bounded on $[0,\infty)$ and $\lim_{r\to\infty} \Phi_r(s) = 0$, $s \geq 0$, we can apply the dominated convergence theorem again to see that (4) holds true. $\square$

The notion of a mild solution of (2) is introduced in the following definition:

Definition 2.3. *By a mild solution of* (2), *we mean any continuous function* $u(\cdot)$ *such that* $u(t) = (\Lambda u)(t)$, $t \in \mathbb{R}$, *where*

$$t \mapsto (\Lambda u)(t) := \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds,\ t \in \mathbb{R}.$$

Suppose that the inequality

$$\|f(t,x) - f(t,y)\| \leq L_f(t)\|x-y\|, \quad t \in \mathbb{R},\ x,\ y \in X \tag{5}$$

holds with some bounded non-negative function $L_f(\cdot)$. Set, for every $n \in \mathbb{N}$,

$$M_n := M^n \sup_{t\in\mathbb{R}} \int_{-\infty}^{t}\int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} e^{-c(t-x_n)}\bigl(t-x_n\bigr)^{\beta-1} \times \prod_{i=2}^{n} e^{-c(x_i-x_{i-1})}\bigl(x_i-x_{i-1}\bigr)^{\beta-1} \prod_{i=1}^{n} L_f(x_i)\, dx_1\, dx_2 \cdots dx_n.$$

Then a simple calculation shows that

$$\Bigl\|\bigl(\Lambda^n u\bigr) - \bigl(\Lambda^n v\bigr)\Bigr\|_{\infty} \leq M_n \bigl\|u-v\bigr\|_{\infty}, \quad u,\ v \in C_b(\mathbb{R} : X),\ n \in \mathbb{N}. \tag{6}$$

The following result is in a close connection with [16, Theorem 2.10]:

Theorem 2.4. *Suppose that the following conditions hold:*

(i) $f \in PAP(\mathbb{R} \times X : X)$ *is pseudo-almost periodic.*

(ii) The inequality (5) *holds with some bounded non-negative function* $L_f(\cdot)$.

(iii) $\sum_{n=1}^{\infty} M_n < \infty$.

Then there exists a unique pseudo-almost periodic solution of inclusion (2).

Proof. Using Lemma 1.2 and Lemma 2.2, we get that the mapping $\Lambda : PAP(\mathbb{R} : X) \to PAP(\mathbb{R} : X)$ is well-defined. Making use of (6) and Weissinger's fixed point theorem [9, Theorem D.7], we obtain the existence of a unique pseudo-almost periodic mild solution of inclusion (2), as claimed. □

If $L_f \equiv L$ is constant in Theorem 2.4, then it can be simply verified by a direct calculation that $M_n \leq M^n L^n(\Gamma(\beta)c^{-\beta})^n$, $n \in \mathbb{N}$. Hence, we have the following corollary:

Corollary 2.5. *Suppose that the function* $f(\cdot, u(\cdot))$ *is pseudo-almost periodic and* (5) *holds with* $L_f \equiv L \in [0, c^{\beta} M^{-1}\Gamma(\beta)^{-1})$. *Then there exists a unique pseudo-almost periodic solution of inclusion* (2).

Remark 2.6. *(i) In the case that* $\beta = 1$ *and* $L_f \in L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$, *the proof of [2, Theorem 3.2] shows that the condition* $\sum_{n=1}^{\infty} M_n < \infty$ *automatically holds.*

(ii) In [2, Proposition 3.3], the authors have analyzed a version of Corollary 2.5 by assuming that the Stepanov L^1*-norm of function* $L_f \in C_b(\mathbb{R} : X)$ *is sufficiently small. A similar assertion can be proved in our framework; cf. [15] and [16] for more details in this direction.*

We close the paper with the observation that we will consider the pseudo-almost periodic solutions of fractional relaxation semilinear inclusion

$$D_{t,+}^{\alpha} u(t) \in \mathcal{A}u(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where $f : \mathbb{R} \times X \to X$ is pseudo-almost periodic and $D_{t,+}^{\alpha}$ denotes the Weyl-Liouville fractional derivative of order $\alpha \in (0,1)$, in our forthcoming monograph [18].

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