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On bounds for incidence energy of a graph

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Abstract. Let G be a simple connected graph with n vertices and m edges, and let $q_1 \geq q_2 \geq \cdots \geq q_n$ be its signless Laplacian eigenvalues. The incident energy of G is defined as $IE(G) = \sum_{i=1}^{n} \sqrt{q_i}$. Some new bounds for $IE(G)$ are obtained.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with n vertices, m edges and let $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$, be a sequence of its vertex degrees. If vertices v_i and v_j are adjacent, we denote it as $i \sim j$. For the edge $e \in E$ connecting the vertices v_i and v_j, the degree of edge is $d(e) = d_i + d_j - 2$.

The numeric quantity associated with a graph which characterize the topology of graph and is invariant under graph automorphism is called graph invariant or topological index. A large number of topological indices have been derived depending on vertex degrees.

The first Zagreb index is vertex–degree–based graph invariant, introduced by Gutman and Trinajstić in [5], defined as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2.$$

Since

$$M_1 = \sum_{i \sim j} (d_i + d_j) = \sum_{i=1}^{m} (d(e_i) + 2), \tag{1.1}$$

(see [12]), the first Zagreb index can also be considered as an edge-degree-based topological index. Details on the first Zagreb index, as well as some other topological indices can be found in [1, 2, 8–10].

Let $\mathbf{A}$ be the adjacency matrix of G, and $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of its vertex degrees. Signless Laplacian matrix of G is defined as $\mathbf{Q} = \mathbf{D} + \mathbf{A}$ (see [4]). Eigenvalues of matrix $\mathbf{Q}$,

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$q_1 \geq q_2 \geq \cdots \geq q_n > 0$, are signless Laplacian eigenvalues of G. They satisfy the following identities

$$\sum_{i=1}^{n} q_i = \operatorname{tr}(D+A) = 2m \quad \text{and} \quad \sum_{i=1}^{n} q_i^2 = \operatorname{tr}(D+A)^2 = M_1 + 2m, \tag{1.2}$$

where $\operatorname{tr}(B)$ denotes a trace of a square matrix B.

Gutman et all [6] defined incident energy of graph G, $IE(G)$, as

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i}.$$

From (1.2) we have that

$$M_1 = \sum_{i=1}^{n} q_i(q_i - 1),$$

which means that M_1 can also be considered as signless-Laplacian-spectrum-based graph invariant.

In this paper we first analyze some known lower bounds for $IE(G)$ in terms of graph parameters n, m and Δ, reported in the literature. Then we establish new lower and upper bounds for this graph invariant.

2. Preliminaries

In this section we recall some results from the literature for q_1 and $IE(G)$, as well as some analytical inequalities for real number sequences, which will be used subsequently.

Denote by

$$T = \frac{1}{2}\left(\Delta + \delta + \sqrt{(\Delta-\delta)^2 + 4\Delta}\right).$$

Lemma 2.1. [3, 15] *Let G be a connected graph with $n \geq 2$ vertices and Δ be the maximum vertex degree of G. Then*

$$q_1 \geq T \geq 1 + \Delta, \tag{2.1}$$

with either equalities if and only if G is a star graph $K_{1,n-1}$.

Lemma 2.2. [7] *Let G be a graph with n vertices and m edges. Then*

$$IE(G) \geq \sqrt{\frac{(2m)^3}{M_1 + 2m}}, \tag{2.2}$$

with equality if and only if all non-zero signless Laplacian eigenvalues of G are equal.

Lemma 2.3. [7] *Let G be a graph with n vertices and m edges. Then*

$$IE(G) \geq \frac{2m}{\sqrt{n}}, \tag{2.3}$$

with equality in (2.3) *if and only if $G \cong \overline{K_n}$ or $G \cong K_2$.*

Based on the identity

$$M_1 = \sum_{i=1}^{n} d_i^2 \leq n\Delta^2, \tag{2.4}$$

the following result was proven in [13].

Lemma 2.4. [13] *Let G be a connected graph with n vertices, $m > 1$ edges and maximum vertex degree Δ. Then*

$$IE(G) \geq 2m\sqrt{\frac{2m}{n\Delta^2 + 2m}}, \tag{2.5}$$

with equality in (2.5) *if and only if* $G \cong K_2$.

As noted in [13], the lower bounds for $IE(G)$ given by (2.3) and (2.5) are not comparable. Therefore, we have that

$$IE(G) \geq \max\left\{\frac{2m}{\sqrt{n}}, 2m\sqrt{\frac{2m}{n\Delta^2 + 2m}}\right\}, \tag{2.6}$$

with equality if and only if $G \cong K_2$.

Lemma 2.5. [14] *Let G be a simple connected graph with n vertices. Then*

$$q_1 \leq 2\Delta, \tag{2.7}$$

with equality if and only if G is a regular graph.

Lemma 2.6. [11] *Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequences. Then for any real r such that $r \geq 1$ or $r \leq 0$, holds*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i\right)^r. \tag{2.8}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$. If $0 < r < 1$, then the sense of (2.8) *reverses.*

The inequality (2.8) is referred to as Jensen's inequality in the literature.

3. Main results

At the beginning we present a new proof of inequality (2.2) that is simpler than the one given in [7]. Then, we give a comment on inequalities (2.3)–(2.6). In the second part of this section we determine some new bounds for the invariant $IE(G)$.

New proof of inequality (2.2): For $r = 3$, $p_i = a_i = \sqrt{q_i}$, $i = 1, 2, \dots, n$, the inequality (2.8) becomes

$$\left(\sum_{i=1}^{n} \sqrt{q_i}\right)^2 \sum_{i=1}^{n} q_i^2 \geq \left(\sum_{i=1}^{n} q_i\right)^3,$$

that is

$$IE(G)^2(M_1 + 2m) \geq (2m)^3,$$

wherefrom (2.2) immediately follows.

Denote by

$$\Delta_e = \max_{1\leq i\leq m}\{d(e_i) + 2\}.$$

From (1.1) we get

$$M_1 = \sum_{i=1}^{m}(d(e_i) + 2) \leq m\Delta_e \leq 2m\Delta \leq n\Delta^2. \tag{3.1}$$

According to the first two inequalities in (3.1) and (2.2), the following inequalities hold

$$IE(G) \geq 2m\sqrt{\frac{2}{\Delta_e+2}}$$

and

$$IE(G) \geq \frac{2m}{\sqrt{1+\Delta}}.$$

Both of these inequalities are stronger than (2.3) and (2.5), and therefore it follows

$$2m\sqrt{\frac{2}{\Delta_e+2}} \geq \max\left\{\frac{2m}{\sqrt{n}}, 2m\sqrt{\frac{2m}{n\Delta^2+2m}}\right\}$$

and

$$\frac{2m}{\sqrt{1+\Delta}} \geq \max\left\{\frac{2m}{\sqrt{n}}, 2m\sqrt{\frac{2m}{n\Delta^2+2m}}\right\}.$$

In the following theorem we determine a new lower bound for $IE(G)$ in terms of parameters m, Δ and T, and the first Zagreb index, M_1.

Theorem 3.1. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$IE(G) \geq \sqrt{T} + \sqrt{\frac{8(m-\Delta)^3}{M_1+2m-T^2}}. \tag{3.2}$$

Equality holds if and only if $G \cong K_2$.

Proof. For $r = 3$ the inequality (2.8) becomes

$$\left(\sum_{i=2}^{n} p_i\right)^2 \sum_{i=2}^{n} p_i a_i^3 \geq \left(\sum_{i=2}^{n} p_i a_i\right)^3.$$

For $p_i = a_i = \sqrt{q_i}$, $i = 2, 3, \ldots, n$, the above inequality becomes

$$\left(\sum_{i=2}^{n} \sqrt{q_i}\right)^2 \sum_{i=2}^{n} q_i^2 \geq \left(\sum_{i=2}^{n} q_i\right)^3,$$

i.e.

$$(IE(G) - \sqrt{q_1})^2(M_1 + 2m - q_1^2) \geq (2m - q_1)^3,$$

wherefrom we obtain

$$IE(G) \geq \sqrt{q_1} + \sqrt{\frac{(2m-q_1)^3}{M_1+2m-q_1^2}}. \tag{3.3}$$

Since the function

$$f(x) = \sqrt{x} + \sqrt{\frac{(2m-q_1)^3}{M_1+2m-x^2}}$$

is monotone increasing for $0 \leq x < \sqrt{M_1+2m}$, for $x = q_1 \geq T$ from (3.3) we have that

$$IE(G) \geq \sqrt{T} + \sqrt{\frac{(2m-q_1)^3}{M_1+2m-T^2}}.$$

Now, (3.2) follows from the above and (2.7).

Equality in (3.3) holds if and only if $q_2 = q_3 = \cdots = q_n$. Equality in (2.1) holds if and only if $G \cong K_{1,n-n}$. Equality in (2.7) is attained if and only if G is a regular graph. Therefore we conclude that equality in (3.2) holds if and only if $G \cong K_2$. □

In the next theorem we determine a new upper bound for $IE(G)$ in terms of n, m T, and M_1.

Theorem 3.2. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$IE(G) \leq \sqrt{T} + \left((n-1)^3(M_1 + 2m - T^2)\right)^{\frac{1}{4}}. \tag{3.4}$$

Equality holds if and only if $G \cong K_2$.

Proof. For $r = 4$ the inequality (2.8) can be considered as

$$\left(\sum_{i=2}^{n} p_i\right)^3 \sum_{i=2}^{n} p_i a_i^4 \geq \left(\sum_{i=2}^{n} p_i a_i\right)^4.$$

For $p_i = 1$, $a_i = \sqrt{q_i}$, $i = 2, 3, \ldots, n$, this inequality transforms into

$$(n-1)^3 \sum_{i=2}^{n} q_i^2 \geq \left(\sum_{i=2}^{n} \sqrt{q_i}\right)^4,$$

that is

$$IE(G) \leq \sqrt{q_1} + \left((n-1)^3(M_1 + 2m - q_1^2)\right)^{\frac{1}{4}}. \tag{3.5}$$

The function

$$f(x) = \sqrt{x} + \left((n-1)^3(M_1 + 2m - x^2)\right)^{\frac{1}{4}}$$

is monotone decreasing for $\sqrt{\frac{M_1+2m}{n}} \leq x \leq \sqrt{M_1 + 2m}$. According to (1.2) it holds

$$M_1 = \sum_{i=1}^{n} q_i(q_i - 1) \leq (q_1 - 1) \sum_{i=1}^{n} q_i = 2m(q_1 - 1),$$

therefore

$$q_1 \geq \frac{M_1}{2m} + 1.$$

One can easily verify that

$$q_1 \geq T \geq 1 + \Delta \geq \frac{M_1}{2m} + 1 \geq \sqrt{\frac{M_1 + 2m}{n}}.$$

Now from $f(x) = f(q_1) \leq f(T)$ and (3.5) we arrive at (3.4).

Equality in (3.5) holds if and only if $q_2 = q_3 = \cdots = q_n$. Equality in (2.1) holds if and only if $G \cong K_{1,n-1}$. Consequently, equality in (3.4) holds if and only if $G \cong K_2$. □

If G is a bipartite graph, then $q_n = 0$. In a similar way as in case of Theorem 3.2, the following result can be proved.

Theorem 3.3. *Let G be a simple connected bipartite graph with $n \geq 2$ vertices and m edges. Then*

$$IE(G) \leq \sqrt{T} + \left((n-2)^3(M_1 + 2m - T^2)\right)^{\frac{1}{4}}.$$

Equality holds if and only if $G \cong K_{1,n-1}$.

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