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New L-fuzzy Fixed Point Results of Integral Type

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Abstract. In this paper, we analyze the existence of fixed points of L-fuzzy mappings defined on complete metric spaces satisfying rational contractive inequalities of integral type. Moreover, in support of our main result, a non-trivial example is provided.

1. Introduction

Let (X,d) be a complete metric space. The well-celebrated Banach contraction principle (also called the Banach fixed point theorem) (see[7]) guarantees a unique fixed point if a mapping $T : X \longrightarrow X$ is a contraction, that is, if there exists a real number $\alpha \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \leq \alpha d(x,y).$$

As simple as the Banach fixed point theorem, it is the most applied result in the study of existence and uniqueness of solution of nonlinear problems arising in mathematics and its applications to engineering and life sciences. Shortly, in 1930, Caccioppli [12] published an analogue sort of Banach fixed point theorem. Due to the similarity between the two results, they are sometimes jointly called the Banach-Caccioppoli fixed point theorem. In the case of single-valued mappings, the aforementioned two theorems have been generalized by many researchers in various ways (see, for example, [1, 2, 11, 16]) and the references therein. One may also consult Rhoades [22] for multitude definitions of contractive type mappings. Two obvious intersecting properties of most generalizations of the Banach fixed point theorem is that their proofs are similar and the contractive conditions consist of linear combinations of the distances between two distinct points and their images. The first-two most embraced extensions of Banach-Caccioppoli principle involving rational inequalities were presented by Dass-Gupta [14] and Jaggi [19]. On the other hand, the earliest known fixed point theorem whose statement and proof are significantly different from Banach-Caccioppoli theorem was presented in 1976 by Caristi [13, Theorem 2.1]. Fixed point theorem for mappings satisfying contractive condition of integral type was initiated by Branciari [9]. Given a complete metric space (X,d) with $x, y \in X$, and for some $\lambda \in (0,1)$, Branciari discussed the self mapping T on X satisfying the contractive conditions of the form

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \lambda \int_0^{d(x,y)} \varphi(t)dt,$$

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for any Lebesgue integrable function $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ which is summable on each compact subset of $[0,+\infty)$ and satisfies $\int_0^{\varepsilon} \varphi(t)dt > 0$, for all $\varepsilon > 0$.

Away from single-valued mappings, in 1969, Nadler [21] initiated the study of fixed point theorems for multi-valued mappings. Nadler's contraction principle motivated many researchers and hence, the idea has been refined in different directions (see, for instance, [3, 4, 8, 23]).

On the other hand, a number of practical and theoretic problems in economics, management sciences, engineering, environment sciences, medical sciences, and a large number of other fields involve vagueness and the complexity of modeling uncertain data. Conventional mathematical tools are not usually successful because the imprecisions in these domains may be of various kinds. Dating back to about five decades away, researchers have been proposing a number of theories for handling imprecise environments. One of these is the theory of fuzzy sets introduced by Zadeh[26]. Classically, fuzzy set is characterized by a membership function which assigns to each of its elements a grade of membership ranging between zero and one. A particular generalization of fuzzy sets by replacing the interval $[0,1]$ of range by a complete distributive lattice was presented by Goguen [17] and called L-fuzzy sets. Meanwhile, there are many generalizations of fuzzy set theory, some of these are intuitionistic fuzzy sets [6], soft sets [20] and hesitant fuzzy sets [24]. The arena of applied mathematics witnessed tremendous developments as a result of the introduction of fuzzy sets. The study of fixed points of fuzzy mappings was pioneered by Weiss [25] and Butnairu [10]. Whereas, fixed point theorems for fuzzy set-valued mappings have been investigated by Heilpern [18] who initiated the idea of fuzzy contractions and proved a fixed point theorem parallel to the Banach-Cacciopoli principle in the frame of fuzzy sets.

Recently, Azam et al [5] established a new common fixed point theorem for a pair of intuitionistic fuzzy mappings on a complete metric space in connection with the Hausdorff metric by using a contractive condition involving rational expression as follows:

Theorem 1.1. *[5] Let (X,d) be a complete metric space, S,T be any two intuitionistic fuzzy mappings on X, and for $x \in X$, there exists $(\alpha,\beta)_{Sx}, (\alpha,\beta)_{Tx} \in (0,1] \times [0,1)$ such that $[Sx]_{(\alpha,\beta)_{Sx}}, [Tx]_{(\alpha,\beta)_{Tx}}$ are nonempty closed and bounded subsets of X. If*

$$\begin{aligned} H\left([Sx]_{(\alpha,\beta)_{Sx}}, [Ty]_{(\alpha,\beta)Ty}\right) \leq\; & ad(x,y) + bd(x,[Sx]_{(\alpha,\beta)S(x)}) \\ & + cd(y,[Ty]_{(\alpha,\beta)T(y)}) \\ & + \frac{ed(x,[Sx]_{(\alpha,\beta)S(x)})d(y,[Ty]_{(\alpha,\beta)T(y)})}{1+d(x,y)}, \end{aligned}$$

and

$$c + \frac{ed(x,[Sx]_{(\alpha,\beta)_{S(x)}})}{1+d(x,y)} < 1, \quad b + \frac{ed(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(x,y)} < 1,$$

where a,b,c,e, are nonnegative real numbers with $a+b+c+e<1$, then there exists $z \in X$ such that $z \in [Sz]_{(\alpha,\beta)Sz} \cap [Tz]_{(\alpha,\beta)Tz}$.

In this paper, an integral reformulation of Theorem 1.1 due to Azam et al. [5] in the setting of L-fuzzy mapping is analyzed. Thereafter, in support of our main result, a nontrivial example is provided. Moreover, some associated consequences are deduced.

2. Preliminaries

In this section, some basic concepts that are needed in the sequel are recalled. For these preliminaries, we follow[15, 17, 18, 26]. Let X be a reference set. Recall that an ordinary subset A of X is determined by its characteristic function χ_A, defined by $\chi_A : A \longrightarrow \{0,1\}$:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

The value $\chi_A(x)$ specifies, wether an element belongs to A or not. This idea is used to define fuzzy sets by allowing an element $x \in A$ to assume any possible value in the interval $[0,1]$. Thus, a fuzzy set A in X is a set of ordered pair given as

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where $\mu_A : X \longrightarrow [0,1] = I$ and $\mu_A(x)$ is called the membership function of x or the degree to which $x \in X$ belongs to the fuzzy set A.
A relatively important notion in fuzzy set theory is that of an α-level set. If A is a fuzzy set in X, the (crisp) set of elements in X belonging to A at least of degree $\alpha \in (0,1]$ is called the α-level set, denoted by $[A]_\alpha$. That is,

$$[A]_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}.$$

On the other hand,

$$[A]_\alpha^* = \{x \in X : \mu_A(x) > \alpha\},$$

is called the strong α-level set or strong α-level cut. We denote by I^X, the family of all fuzzy sets in X.

Example 2.1. An organizing committee of an international conference wants to know the facilities needed to play host to the event. An indicated facility by some of the members is accommodation. If

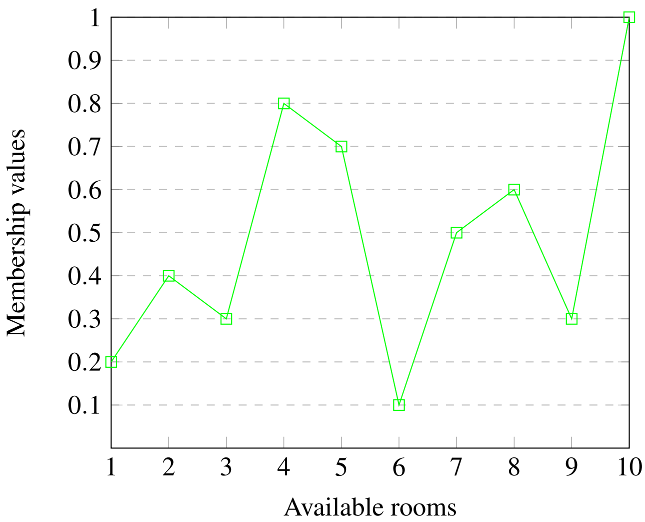
$$X = \{1,2,3,4,5,6,7,8,9,10\}$$

is the set of all available accomodations, then the fuzzy set

$$A = \{\text{few rooms are availble }\}$$

may be seen graphically as in Figure 1.

Figure 1: Graphical representation of the fuzzy set in Example 2.1



Observe that in Figure 1, the α-level set and strong α-level set for $\alpha = 0.5$, are respectively

$$[A]_\alpha = \{4,5,7,8,10\} \quad \text{and} \quad [A]_\alpha^* = \{4,5,8,10\}.$$

Let $CB(X)$ denotes the set of all non-empty closed and bounded subsets of a metric space X. For $A, B \in I^X$, $A \subset B$ implies $A(x) \leq B(x)$ for each $x \in X$. Let $\alpha \in [0,1]$ such that $[A]_\alpha, [B_\alpha] \in CB(X)$. Then, we define

$$p_\alpha(A,B) = \inf_{x \in [A]_\alpha, y \in [B]_\alpha} d(x,y),$$

$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha),$$

$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

Definition 2.1. Let X be an arbitrary set and Y a metric space. A mapping $T : X \longrightarrow I^Y$ is called a fuzzy mapping. In other words, a fuzzy mapping T is a fuzzy subset of $X \times Y$ with membership function $T(x)(y)$. The function value $T(x)(y)$ is the membership value of y in $T(x)$. An element $u \in X$ is called a fuzzy fixed point of T if there exists an $\alpha \in (0,1]$ such that $u \in [Tu]_\alpha$. Similarly, u is said to be a common fuzzy fixed point of S and T if $u \in [Su]_\alpha \cap [Tu]_\alpha$.

Example 2.2. Let $X = [-5,5]$ and $Y = [-4,4]$. Define $T : X \longrightarrow I^Y$ by

$$T(x)(y) = \frac{|x| + |y|}{|x| + |y| + 20}.$$

Then T is a fuzzy mapping. The possible membership values of y in the fuzzy set $T(x)$ is shown in Figure 1.

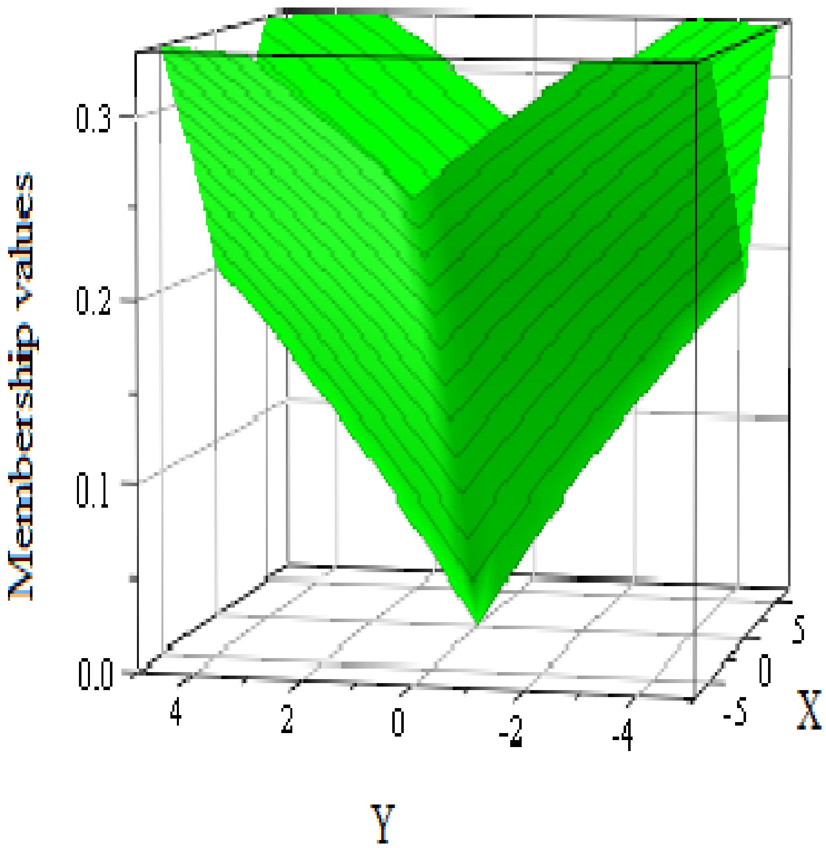


Figure 1: Graphical representation of the fuzzy mapping in Example 2.2

Definition 2.2. A relation $\preceq$ on a set L is called a partial order if it is

(i) Reflexive

(ii) Antisymmetric

(iii) Transitive.

A set L together with a partial ordering $\preceq$ is called a partially ordered set (poset, for short) and is denoted by $(L, \preceq_L)$. Recall that partial orderings are used to give an order to sets that may not have a natural one.

Definition 2.3. Let X be a nonempty set and $(X, \preceq)$ be a partially ordered set. Then any two elements $x, y \in X$ are said to be comparable if either $x \preceq y$ or $y \preceq x$.

Definition 2.4. A partially ordered set $(L, \preceq_L)$ is called

(i) a lattice , if $x \vee y \in L$, $x \wedge y \in L$ for any $x, y \in L$;

(ii) a complete lattice, if $\bigvee A \in L$, $\bigwedge A \in L$ for any $A \subseteq L$;

(iii) distributive lattice if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for any $x, y, z \in L$.

Recall also that a partially ordered set L is called a complete lattice if for every doubleton $\{x, y\}$ in L, either $\sup\{x, y\} = x \bigvee y$ or $\inf\{x, y\} = x \bigwedge y$ exists.

Definition 2.5. Let L be a lattice with top element 1_L and bottom element 0_L and let $x, y \in L$. Then y is called a complement of x , if $x \vee y = 1_L$ and $x \wedge y = 0_L$. If $x \in L$ has a complement , then it is unique. We denote by x^c, the complement of x.

Definition 2.6. Let $(L, \preceq)$ be a partially ordered set.

(a) L is called a Boolean lattice , if

  (i) L is a distributive lattice;

  (ii) L has 0_L and 1_L;

  (iii) each $x \in L$ has the complement x^c.

(b) L is called a complete Boolean lattice if

  (i) L is a complete distributive lattice;

  (ii) L has 0_L and 1_L;

  (iii) each $x \in L$ has a complement x^c.

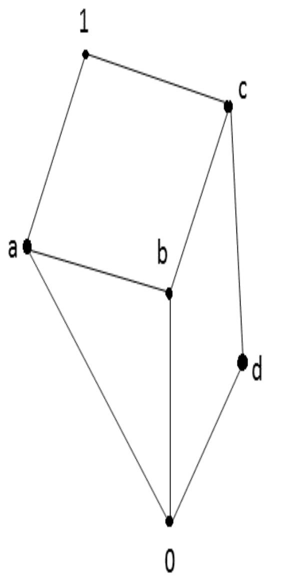


Figure 3: Distributive Lattice

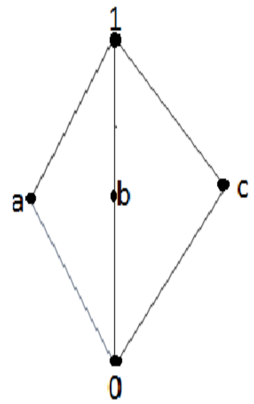


Figure 4: Non-Distributive Lattice

Notice that in Figure 3 and 4, $0 = 0_L$ and $1 = 1_L$, are respectively the bottom and top elements of the lattice.

Definition 2.7. An L-fuzzy set A on a nonempty set X is a function with domain X and whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L respectively.

Remark 2.1. The class of L-fuzzy sets is larger than the class of fuzzy sets as an L-fuzzy set reduces to a fuzzy set if $L = [0,1]$.

Denote the class of all L-fuzzy sets on a nonempty set X by L^X (to mean a function $\quad : X \longrightarrow L$).

Definition 2.8. The α_L-level set of an L-fuzzy set A is denoted by $[A]_{\alpha L}$ and is defined as follows:

$$[A]_{\alpha L} = \{x : \alpha_L \preceq_L A(x), \text{if} \quad \alpha_L \in L \setminus \{0_L\}\},$$

and
$[A]_{0L} = \overline{\{x : 0_L \preceq_L A(x)\}}$, where $\overline{Y}$ is the closure of a crisp set Y.

Definition 2.9. Let X be an arbitrary nonempty set and Y a metric space. A mapping $T : X \longrightarrow L^Y$ is called an L-fuzzy mapping. The function $T(x)(y)$ is the degree of membership of y in $T(x)$. For any two L-fuzzy mappings $S, T : X \longrightarrow L^Y$, a point $u \in X$ is called an L-fuzzy fixed point of S if $u \in [Su]_{\alpha_L}$, where $\alpha_L \in L \setminus \{0_L\}$. A point u is known as a common L-fuzzy fixed point of S and T if $u \in [Su]_{\alpha_L} \cap [Tu]_{\alpha_L}$.

The following Lemma due to Nadler [21] will be required in our presentation.

Lemma 2.1. *[21] Let A and B be nonempty closed and bounded subsets of a metric space X. If $x \in A$, then*

$$d(x,B) \leq H(A,B).$$

Denote by ψ, the class of functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which satisfy the following conditions:

(i) φ is nonnegative, Lebesgue integrable, and

(ii) $\int_0^\tau \varphi(t)dt > 0$, for each $\tau > 0$.

3. Main results

In this section, first, we present our main result (Theorem 3.1) and then obtain some of its consequences.

Theorem 3.1. *Let (X,d) be a complete metric space, $S,T : X \longrightarrow L^X$ be two L-fuzzy mappings, $\varphi \in \psi$ and for $x \in X$, there exists $\alpha_L \in L\setminus\{0_L\}$ such that $[Sx]_{\alpha_L}, [Tx]_{\alpha_L} \in CB(X)$ for all $x \in X$. If*

$$\begin{aligned}\int_0^{H\left([Sx]_{\alpha_L},[Ty]_{\alpha_L}\right)} \varphi(t)dt \quad \leq \quad & p\int_0^{d(x,y)} \varphi(t)dt \\ & +q\int_0^{d(x,[Sx]_{\alpha_L})} \varphi(t)dt \\ & +r\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt \\ & +f\int_0^{\left(\frac{d(x,[Sx]_{\alpha_L})d(y,[Ty]_{\alpha_L})}{1+d(x,y)}\right)} \varphi(t)dt,\end{aligned} \tag{3.1}$$

where p,q,r,f, are nonnegative real numbers with $p+q+r+f<1$, then there exists $\rho \in (X,d)$ such that $\rho \in [S\rho]_{\alpha_L} \cap [T\rho]_{\alpha_L}$,for some $\alpha_L \in L\setminus\{0_L\}$.

Proof. We consider the following three possible cases:

(i) $p+q+f=0$;

(ii) $p+r+f=0$;

(iii) $p+q+f\neq 0$, $p+r+f\neq 0$.

Case (i): $p+q+f=0$. Let $x \in (X,d)$ be arbitrary. Then, for $x \in X$, there exists $\alpha_L \in L\setminus\{0_L\}$ such that $[Sx]_{\alpha_L}$ is a nonempty closed and bounded subset of X. Let $y \in [Sx]_{\alpha_L}$ and $u \in [Ty]_{\alpha_L}$. Then by Lemma 2.1, we have

$$d(y,[Ty]_{\alpha_L}) \leq H\left([Sx]_{\alpha_L},[Ty]_{\alpha_L}\right). \tag{3.2}$$

From ineqs. (3.17) and (3.2), we obtain

$$\begin{aligned}\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt \quad \leq \quad & \int_0^{H([Sx]_{\alpha_L},[Ty]_{\alpha_L})} \varphi(t)dt \\ \leq \quad & p\int_0^{d(x,y)} \varphi(t)dt + q\int_0^{d(x,[Sx]_{\alpha_L})} \varphi(t)dt \\ & +r\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt + f\int_0^{\frac{d(x,[Sx]_{\alpha_L})d(y,[Ty]_{\alpha_L})}{1+d(x,y)}} \varphi(t)dt\end{aligned} \tag{3.3}$$

Using $p+q+f=0$, ineq. (3.3) becomes

$$\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt \leq r\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt,$$

which implies $y \in [Ty]_{\alpha_L}$.

Similarly,

$$\int_0^{d(y,[Sy]_{\alpha_L})} \varphi(t)dt \leq \int_0^{H\left([Sy]_{\alpha_L},[Ty]_{\alpha_L}\right)} \varphi(t)dt \tag{3.4}$$

From (3.17) and (3.4), we have

$$\int_0^{d(y,[Sy]_{\alpha_L})} \varphi(t)dt \leq 0. \tag{3.5}$$

Consequently, $y \in [Sy]_{\alpha_L} \cap [Ty]_{\alpha_L}$.

Case (ii): $p+r+f=0$. For $x \in (X,d)$, by hypothesis, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[Sx]_{\alpha_L}$ is a nonempty closed and bounded subset of X. Let $y \in [Sx]_{\alpha_L}$ and $u \in [Ty]_{\alpha_L}$. Then, by Lemma 2.1,

$$d(u,[Su]_{\alpha_L}) \leq H\left([Su]_{\alpha_L},[Ty]_{\alpha_L}\right). \tag{3.6}$$

From (3.17) and (3.6), we get

$$\begin{aligned}\int_0^{d(u,[Su]_{\alpha_L})} \varphi(t)dt \quad &\leq \quad p\int_0^{d(u,y)} \varphi(t)dt + q\int_0^{d(u,[Su]_{\alpha_L})} \varphi(t)dt \\ &+ r\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt + f\int_0^{\frac{d(u,[Su]_{\alpha_L})d(y,[Ty]_{\alpha_L})}{1+d(u,y)}} \varphi(t)dt\end{aligned}$$

Using $p+r+f=0$, the above ineq. reduces to

$$\int_0^{d(u,[Su]_{\alpha_L})} \varphi(t)dt \leq q\int_0^{d(u,[Su]_{\alpha_L})} \varphi(t)dt,$$

this implies $u \in [Su]_{\alpha_L}$. Similarly, one can show that $u \in [Tu]_{\alpha_L}$. Hence, $u \in [Su]_{\alpha_L} \cap [Tu]_{\alpha_L}$.

Case (iii): $p+q+f \neq 0$, $p+r+f \neq 0$.
Let $\max\left\{\left(\frac{p+q}{1-r-f}\right),\left(\frac{p+r}{1-q-f}\right)\right\} = \xi$.
Observe that $\xi = 0$ implies $p=q=r=0$ and proof follows trivially. Assume that $\xi \neq 0$. Since $p+q+r+f < 1$, then clearly $0 < \xi < 1$.
Let $x_0 \in (X,d)$. Then by hypothesis, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[Sx_0]_{\alpha_L}$ is a nonempty closed and bounded subset of X. Let $x_1 \in [Sx_0]_{\alpha_L}$. Then for this x_1, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[Tx_1]_{\alpha_L}$ is a nonempty closed and bounded subset of X. Hence, by Lemma 2.1, there exists $x_2 \in [Tx_1]_{\alpha_L}$ and $x_3 \in [Sx_2]_{\alpha_L}$ such that

$$d(x_1,x_2) \leq H\left([Sx_0]_{\alpha_L},[Tx_1]_{\alpha_L}\right) + \xi(1-r-f) \tag{3.7}$$

$$d(x_2,x_3) \leq H\left([Sx_2]_{\alpha_L},[Tx_1]_{\alpha_L}\right) + \xi^2(1-q-f) \tag{3.8}$$

Continuing in this fashion, a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X can be generated as:

$$x_{2k+1} = [Sx_{2k}]_{\alpha_L}, k = 0,1,2,\cdots$$

$$x_{2k+2} = [Tx_{2k+1}]_{\alpha_L}, k = 0,1,2,\cdots$$

such that

$$d(x_{2k+1},x_{2k+2}) \leq H\left([Sx_{2k}]_{\alpha_L},[Tx_{2k+1}]_{\alpha_L}\right) + \xi^{2k+1}(1-r-f) \tag{3.9}$$

$$d(x_{2k+2},x_{2k+3}) \leq H\left([Sx_{2k+2}]_{\alpha_L},[Tx_{2k+1}]_{\alpha_L}\right) + \xi^{2k+2}(1-q-f) \tag{3.10}$$

Now, from (3.17) and (3.9), we have

$$\begin{aligned}
\int_0^{d(x_1,x_2)} \varphi(t)dt &\leq p\int_0^{d(x_0,x_1)} \varphi(t)dt + q\int_0^{d(x_0,[Sx_0]_{\alpha_L})} \varphi(t)dt \\
&\quad + r\int_0^{d(x_1,[Tx_1]_{\alpha_L})} \varphi(t)dt \\
&\quad + f\int_0^{\frac{d(x_0,[Sx_0]_{\alpha_L})d(x_1,[Tx_1]_{\alpha_L})}{1+d(x_0,x_1)}} \varphi(t)dt + \xi(1-r-f) \\
&\leq p\int_0^{d(x_0,x_1)} \varphi(t)dt + q\int_0^{d(x_0,x_1)} \varphi(t)dt \\
&\quad + r\int_0^{d(x_1,x_2)} \varphi(t)dt \\
&\quad + f\int_0^{\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}} \varphi(t)dt + \xi(1-r-f) \\
&\leq (p+q)\int_0^{d(x_0,x_1)} \varphi(t)dt + (r+f)\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi(1-r-f).
\end{aligned}$$

This yields

$$\begin{aligned}
\int_0^{d(x_1,x_2)} \varphi(t)dt &\leq \left(\frac{p+q}{1-r-f}\right)\int_0^{d(x_0,x_1)} \varphi(t)dt + \xi \\
&\leq \xi\int_0^{d(x_0,x_1)} \varphi(t)dt + \xi.
\end{aligned} \tag{3.11}$$

Again, from (3.17) and (3.10), we have

$$\begin{aligned}
\int_0^{d(x_2,x_3)} \varphi(t)dt &\leq p\int_0^{d(x_1,x_2)} \varphi(t)dt + q\int_0^{d(x_2,[Sx_2]_{\alpha_L})} \varphi(t)dt \\
&\quad + r\int_0^{d(x_1,[Tx_1]_{\alpha_L})} \varphi(t)dt \\
&\quad + f\int_0^{\frac{d(x_2,[Sx_2]_{\alpha_L})d(x_1,[Tx_1]_{\alpha_L})}{1+d(x_1,x_2)}} \varphi(t)dt + \xi^2(1-q-f) \\
&\leq p\int_0^{d(x_1,x_2)} \varphi(t)dt + q\int_0^{d(x_2,x_3)} \varphi(t)dt \\
&\quad + r\int_0^{d(x_1,x_2)} \varphi(t)dt \\
&\quad + f\int_0^{\frac{d(x_2,x_3)d(x_1,x_2)}{1+d(x_1,x_2)}} \varphi(t)dt + \xi^2(1-q-f) \\
&\leq (p+r)\int_0^{d(x_1,x_2)} \varphi(t)dt + (q+f)\int_0^{d(x_2,x_3)} \varphi(t)dt + \xi^2(1-q-f).
\end{aligned}$$

Simplifying the above inequality, we have

$$\begin{aligned}
\int_0^{d(x_2,x_3)} \varphi(t)dt &\leq \left(\frac{p+r}{1-q-f}\right)\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi^2 \\
&\leq \xi\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi^2.
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\begin{aligned}\int_0^{d(x_2,x_3)}\varphi(t)dt &\le \xi\int_0^{d(x_1,x_2)}\varphi(t)dt+\xi^2\\ &\le \xi\left[\xi\int_0^{d(x_0,x_1)}\varphi(t)dt+\xi\right]+\xi^2\\ &\le \xi^2\int_0^{d(x_0,x_1)}\varphi(t)dt+2\xi^2.\end{aligned}$$

Continuing this iteration repeatedly, we have

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt\le\xi^n\int_0^{d(x_0,x_1)}\varphi(t)dt+n\xi^n,\quad n\in\mathbb{N}.$$

Hence, for $n>m\ge 1$, we get

$$\begin{aligned}\int_0^{d(x_m,x_n)}\varphi(t)dt &\le \int_0^{d(x_m,x_{m+1})}\varphi(t)dt+\int_0^{d(x_{m+1},x_{m+2})}\varphi(t)dt+\cdots+\int_0^{d(x_{n-1},x_n)}\varphi(t)dt\\ &\le \xi^m\int_0^{d(x_0)}\varphi(t)dt+m\xi^m+\xi^{m+1}\int_0^{d(x_0,x_1)}\varphi(t)dt\\ &\quad+(m+1)\xi^{m+1}+\cdots+\xi^{n-1}\int_0^{d(x_0,x_1)}\varphi(t)dt+(n-1)\xi^{n-1}\\ &\le \left(\xi^m+\xi^{m+1}+\cdots+\xi^{n-1}\right)\int_0^{d(x_0,x_1)}\varphi(t)dt\\ &\quad+\left(m\xi^m+(m+1)\xi^{m+1}+\cdots+(n-1)\xi^{n-1}\right)\\ &\le \sum_{i=m}^{n-1}\xi^i\int_0^{d(x_0,x_1)}\varphi(t)dt+\sum_{m}^{n-1}i\xi^i\\ &\le \frac{\xi^m}{1-\xi}\int_0^{d(x_0,x_1)}\varphi(t)dt+\sum_{i=m}^{n-1}i\xi^i.\end{aligned}$$

Observe that $(u_n)^{\frac{1}{n}}=n^{\frac{1}{n}}\xi<1$ as $n\longrightarrow\infty$. Hence, by Cauchy's root test, the series $\sum_{i=m}^{n-1}i\xi^i$ is convergent. It follows that $d(x_m,x_n)\longrightarrow 0$ as $n,m\longrightarrow\infty$. This shows that $\{x_n\}$ is a Cauchy sequence of points of (X,d). By completeness of (X,d), there exists $\rho\in(X,d)$ such that $x_n\longrightarrow\rho$ as $n\longrightarrow\infty$. By Lemma 2.1, we get

$$\begin{aligned}d(\rho,[S\rho]_{\alpha_L}) &\le d(\rho,x_{2n})+d(x_{2n},[S\rho]_{\alpha_L})\\ &\le d(\rho,x_{2n})+H\left([S\rho]_{\alpha_L},[Tx_{2n-1}]_{\alpha_L}\right)\end{aligned}\tag{3.13}$$

From (3.17) and (3.13), we have

$$\begin{aligned}\int_0^{d(\rho,[S\rho]_{\alpha_L})}\varphi(t)dt &\le \int_0^{d(\rho,x_{2n})}\varphi(t)dt+p\int_0^{d(\rho,x_{2n-1})}\varphi(t)dt\\ &+q\int_0^{d(\rho,[S\rho]_{\alpha_L})}\varphi(t)dt+r\int_0^{d(x_{2n-1},[Tx_{2n-1}]_{\alpha_L})}\varphi(t)dt\\ &+f\int_0^{\frac{d(\rho,[S\rho]_{\alpha_L})d(d(x_{2n-1},[Tx_{2n-1}]_{T(x_{2n-1})}))}{1+d(\rho,x_{2n-1})}}\varphi(t)dt\\ &\le \int_0^{d(\rho,x_{2n-1})}\varphi(t)dt+p\int_0^{d(\rho,x_{2n-1})}\varphi(t)dt\\ &+q\int_0^{d(\rho,[S\rho]_{\alpha_L})}\varphi(t)dt+r\int_0^{d(x_{2n-1},x_{2n})}\varphi(t)dt\\ &+f\int_0^{\frac{d(\rho,[S\rho]_{\alpha_L})d(x_{2n-1},x_{2n})}{1+d(\rho,x_{2n-1})}}\varphi(t)dt\end{aligned}$$

As $n \longrightarrow \infty$, the above expression reduces to

$$\int_0^{d(\rho,[S\rho]_{\alpha_L})}\varphi(t)dt \le b\int_0^{[S\rho]_{\alpha_L}}\varphi(t)dt.$$

Hence, $\rho \in [S\rho]_{\alpha_L}$.

On similar steps, one can show that $\rho \in [T\rho]_{\alpha_L}$. Consequently,

$$\rho \in [S\rho]_{\alpha_L} \cap [T\rho]_{\alpha_L}.$$

□

In what follows, we provide an example to support the hypotheses of Theorem 3.1.

Example 3.1. Let $L=\{a,b,c,g,s,m,n,v\}$ be such that $a \preceq_L s \preceq_L c \preceq_L v$, $a \preceq_L g \preceq_L b \preceq_L v$, $s \preceq_L m \preceq_L v$, $g \preceq_L m \preceq_L v$, $n \preceq_L b \preceq_L v$; and each elements of the doubletons $\{c,m\},\{m,b\},\{s,n\},\{n,g\}$ are not comparable. It follows that $(L,\preceq_L)$ is a complete distributive lattice. Let $X=[0,1]$ and define $d: X\times X \longrightarrow \mathbb{R}$ by $d(x,y)=|x-y|$, for all $x,y \in X$. Clearly, (X,d) is a complete metric space. Let $S,T: X \longrightarrow L^X$ be two L-fuzzy mappings defined as follows:

$$S(x)(t)=\begin{cases} v, & \text{if } 0\le t\le \frac{x}{60}\\ s, & \text{if } \frac{x}{60}<t\le\frac{x}{20}\\ m & \text{if } \frac{x}{20}\le t<\frac{x}{9}\\ g, & \text{if } \frac{x}{9}<t\le 1,\end{cases} \qquad T(x)(t)=\begin{cases} v, & \text{if } 0\le t\le \frac{x}{60}\\ u, & \text{if } \frac{x}{60}<t\le\frac{x}{10}\\ a & \text{if } \frac{x}{10}< t\le\frac{x}{5}\\ n, & \text{if } \frac{x}{5}<t\le 1.\end{cases}$$

Take $\alpha_L=v$, then for all $x\in X$, there exists $\alpha_L \in L\setminus\{0_L\}$ such that

$$[Sx]_{\alpha_L}=\left[0,\frac{x}{60}\right]=[Tx]_{\alpha_L}\in CB(X).$$

Hence, for $x\neq y$, we have

$$H\left([Sx]_{\alpha_L},[Ty]_{\alpha_L}\right)\le\frac{1}{60}|x-y|=\frac{1}{60}d(x,y).$$

Define $\varphi:[0,\infty)\longrightarrow[0,\infty)$ by $\varphi(t)=\frac{\ln(1+t)}{1+t}$, for all $t\in[0,\infty)$. Clearly, $\varphi\in\psi$, and

$$\int_0^{d(x,y)}\varphi(t)dt=\frac{1}{2}\left[\ln^2(1+t)\right]_0^{d(x,y)}.$$

By taking $p = \frac{1}{2}$ and $q = r = f = 0$ in Theorem 3.1, we see that

$$\begin{aligned} \int_0^{H\left([Sx]_{\alpha_L},[Ty]_{\alpha_L}\right)} \varphi(t)dt &\leq \int_0^{\frac{1}{60}d(x,y)} \varphi(t)dt \\ &= \frac{1}{2}\left[\ln^2(1+t)\right]_0^{\frac{d(x,y)}{60}} \\ &\leq \frac{1}{2}\int_0^{d(x,y)} \varphi(t)dt = \frac{1}{4}\left[\ln^2(1+t)\right]_0^{d(x,y)}. \end{aligned}$$

Hence, all the conditions of Theorem 3.1 are satisfied. In this case, it is easy to see that there exists $\rho = 0 \in X$ such that $0 \in [S0]_v \cap [T0]_v$.

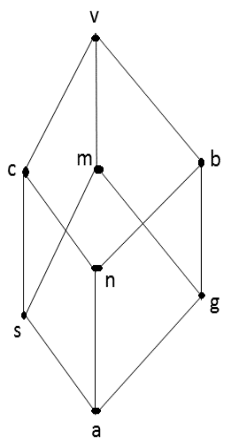


Figure 5: The Lattice in Example 3.1

3.1. Consequences

Here, we deduce some consequences of our main result.

Corollary 3.1. *Let (X,d) be a complete metric space, $T : X \longrightarrow L^X$ be an L-fuzzy mapping, $\varphi \in \psi$ and for $x \in X$, there exists $\alpha_L \in L\setminus\{0_L\}$ such that $[Tx]_{\alpha_L} \in CB(X)$ for all $x \in X$. If*

$$\begin{aligned} \int_0^{H\left([Tx]_{\alpha_L},[Ty]_{\alpha_L}\right)} \varphi(t)dt \leq\; & p\int_0^{d(x,y)} \varphi(t)dt \\ & +q\int_0^{d(x,[Tx]_{\alpha_L})} \varphi(t)dt \\ & +r\int_0^{d(y,[Ty]_{\alpha_L})} \varphi(t)dt \\ & +f\int_0^{\left(\frac{d(x,[Tx]_{\alpha_L})d(y,[Ty]_{\alpha_L})}{1+d(x,y)}\right)} \varphi(t)dt, \end{aligned} \tag{3.14}$$

where p,q,r,f, are nonnegative real numbers with $p+q+r+f < 1$, then there exists $\rho \in (X,d)$ such that $\rho \in [T\rho]_{\alpha_L}$,for some $\alpha_L \in L\setminus\{0_L\}$.

Proof. By setting $S = T$ in Theorem 3.1, the proof follows. □

Corollary 3.2. *Let (X,d) be a complete metric space, $T : X \longrightarrow L^X$ be an L-fuzzy mapping, $\varphi \in \psi$ and for $x \in X$, there exists $\alpha_L \in L \setminus \{0_L\}$ such that $[Tx]_{\alpha_L} \in CB(X)$ for all $x \in X$. If*

$$\int_0^{H\left([Tx]_{\alpha_L},[Ty]_{\alpha_L}\right)} \varphi(t)dt \quad \leq \quad p\int_0^{d(x,y)} \varphi(t)dt \tag{3.15}$$

where $p \in (0,1)$, then there exists $\rho \in (X,d)$ such that $\rho \in [T\rho]_{\alpha_L}$,for some $\alpha_L \in L \setminus \{0_L\}$.

Proof. Put $q = r = f = 0$ and $S = T$ in Theorem 3.1, the proof is complete. □

Remark 3.1. (i) Notice that even when $\varphi = 1$, Theorem 3.1 is not a consequence of Theorem 1.1 due to Azam et al. [5].

(ii) By putting $\varphi = 1$ in Corollary 3.2, one can establish a Nader type L-fuzzy fixed point theorem.

Recall that every L-fuzzy set reduces to a fuzzy set when $L = [0,1]$. This fact leads to the following corollaries.

Corollary 3.3. *Let (X,d) be a complete metric space, $T : X \longrightarrow I^X$ be a fuzzy mapping, $\varphi \in \psi$ and for $x \in X$, there exists $\alpha \in (0,1]$ such that $[Tx]_\alpha \in CB(X)$ for all $x \in X$. If*

$$\begin{aligned} \int_0^{H([Tx]_\alpha,[Ty]_\alpha)} \varphi(t)dt \quad \leq \quad & p\int_0^{d(x,y)} \varphi(t)dt \\ & +q\int_0^{d(x,[Tx]_\alpha)} \varphi(t)dt \\ & +r\int_0^{d(y,[Ty]_\alpha)} \varphi(t)dt \\ & +f\int_0^{\left(\frac{d(x,[Tx]_\alpha)d(y,[Ty]_\alpha)}{1+d(x,y)}\right)} \varphi(t)dt, \end{aligned} \tag{3.16}$$

where p,q,r,f, are nonnegative real numbers with $p+q+r+f < 1$, then there exists $\rho \in (X,d)$ such that $\rho \in [T\rho]_\alpha$,for some $\alpha \in (0,1]$.

Corollary 3.4. *Let (X,d) be a complete metric space, $T : X \longrightarrow I^X$ be a fuzzy mapping, $\varphi \in \psi$ and for $x \in X$, there exists $\alpha \in (0,1]$ such that $[Tx]_\alpha \in CB(X)$ for all $x \in X$. If*

$$\int_0^{H([Tx]_\alpha,[Ty]_\alpha)} \varphi(t)dt \quad \leq \quad p\int_0^{d(x,y)} \varphi(t)dt \tag{3.17}$$

where $p \in (0,1)$, then there exists $\rho \in (X,d)$ such that $\rho \in [T\rho]_\alpha$,for some $\alpha \in (0,1]$.

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