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Generalized Gaussian Jacobsthal Polynomials and Generalized Gaussian Jacobsthal Lucas Polynomials

Snežana S. Djordjević[a], **Gospava B. Djordjević**[a]

[a] *University of Niš, Faculty of Technology, Leskovac, Serbia*

Abstract. In this paper we introduce and study generalized Gaussian Jacobsthal polynomials and generalized Gaussian Jacobsthal Lucas polynomials, where $m \geq 2$. For $m = 2$ these polynomials are Gaussian Jacobsthal polynomials and Gaussian Jacobsthal Lucas polynomials, respectively. We find the generating functions, explicit representations and some interesting properties for these generalized polynomials.

1. Introduction

Generalized Jacobsthal polynomials $J_{n,m}(x)$ and generalized Jacobsthal Lucas polynomials $j_{n,m}(x)$ $(m \geq 2)$ are given by the following recurrence relations, respectively (G. B. Djordjevic [3], G. B. Djordjevic [4], G. B. Djordjevic, H. M. Srivastava [5]):

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \quad n \geq m, \tag{1}$$

with $J_{0,m}(x) = 0, \quad J_{n,m}(x) = 1, \ n = 1, \dots, m-1.$

Similarly,

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \quad n \geq m, \tag{2}$$

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Email addresses: snezanadjordjevicle@gmail.com (Snežana S. Djordjević), gospava48@gmail.com (Gospava B. Djordjević)

with $j_{0,m}(x) = 2, \quad j_{n,m}(x) = 1, \ n = 1, \ldots, m-1.$

For $x = 1$ we get the generalized Jacobsthal numbers $J_{n,m}(1)$ and the generalized Jacobsthal Lucas numbers $j_{n,m}(1)$, respectively.

In paper Asci M., Gurel E. [1] the authors introduce and study Gaussian Jacobsthal polynomials $GJ_n(x)$ and Gaussian Jacobsthal Lucas polynomials $Gj_n(x)$. Also, in Asci M., Gurel E. [2], the authors consider Gaussian Jacobsthal numbers $GJ_n(1)$ and Gaussian Jacobsthal Lucas numbers $Gj_n(1)$. ore information about Jacobsthal polynomials and Jacobsthal Lucas polynomials can be found in Horadam A., P. Filipponi [7], Horadam A. [8, 9]. Further, Jordan J. H. [10] studied Gaussian Fibonacci and Lucas numbers.

In this paper we introduce and study generalized Gaussian Jacobsthal polynomials $GJ_{n,m}(x)$ and generalized Gaussian Jacobsthal Lucas polynomials $Gj_{n,m}(x)$, where $m > 2$. For $m = 2$ these polynomials are known polynomials $GJ_n(x)$ and $Gj_n(x)$.

Now we define generalized polynomials $GJ_{n,m}(x)$ and $Gj_{n,m}(x)$.

Definition 1.1. *Generalized Gaussian Jacobsthal polynomials $GJ_{n,m}(x)$ are defined by*

$$GJ_{n,m}(x) = GJ_{n-1,m}(x) + 2xGJ_{n-m,m}(x), \ n \geq m, \tag{3}$$

with

$$GJ_{0,m}(x) = \frac{i}{2}, \ GJ_{n,m}(x) = 1, \ n = 1, \ldots, m-1, \tag{4}$$

Definition 1.2. *Generalized Gaussian Jacobsthal Lucas polynomials $Gj_{n,m}(x)$ are defined by*

$$Gj_{n,m}(x) = Gj_{n-1,m}(x) + 2xGj_{n-m,m}(x), \ n \geq m. \tag{5}$$

with

$$Gj_{0,m}(x) = 2 - \frac{i}{2}, \ Gj_{n,m}(x) = 1, \ n = 1, \ldots, m-2, \ Gj_{m-1,m}(x) = 1 + 2xi, \tag{6}$$

where $i^2 = -1$.

2. Properties of the generalized Gaussian polynomials

Using the relations (1), (3) and (4), in Table 1 we give some initial members of the generalized Jacobsthal polynomials $J_{n,m}(x)$ and the generalized Gaussian Jacobsthal polynomials $GJ_{n,m}(x)$, respectively.

Next, using the recurrence relations (2), (5) and (6), we find some initial members of the generalized Jacobsthal-Lucas polynomials $j_{n,m}(x)$ and the generalized Gaussian Jacobsthal Lucas polynomials $Gj_{n,m}(x)$, which are given in Table 2.

Remark 1. It is not difficult to see from Table 1 and Table 2 that the following relations hold

$$GJ_{n,m}(x) = J_{n,m}(x) + xiJ_{n+1-m,m}(x), \ n \geq m-1, \tag{7}$$

$$Gj_{n,m}(x) = j_{n,m}(x) + xij_{n+1-m,m}(x), \ n \geq m-1. \tag{8}$$

$$j_{n,m}(x) = J_{n+1,m}(x) + 2xJ_{n+1-m,m}(x), \ n \geq m-1. \tag{9}$$

Table 1:

n	$J_{n,m}(x)$	$GJ_{n,m}(x)$
0	0	$i/2$
1	1	1
2	1	1
....	....	....
$m-1$	1	1
m	1	$1+xi$
$m+1$	$1+2x$	$1+2x+xi$
$m+2$	$1+4x$	$1+4x+xi$
$m+3$	$1+6x$	$1+6x+xi$
$m+4$	$1+8x$	$1+8x+xi$
....	....	....
$2m$	$1+2mx$	$1+2mx+xi(1+2x)$
$2m+1$	$1+(m+1)2x$	$1+(m+1)2x+4x^2+xi(1+4x)$
$2m+2$	$1+(m+2)2x+4x^2$	$1+(m+2)2x+12x^2+xi(1+6x)$
$2m+3$	$1+(m+3)2x+12x^2$	$1+(m+3)2x+24x^2+xi(1+8x)$
$2m+4$	$1+(m+4)2x+24x^2$	$1+(m+4)2x+40x^2+xi(1+10x)$

The next theorem is directly verifiable.

Theorem 2.1. *The explicit formulae for the generalized Gaussian Jacobsthal polynomials and for the generalized Gaussian Jacobsthal Lucas polynomials, respectively, are*

$$GJ_{n,m}(x) = \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} (2x)^k + i \sum_{k=0}^{[(n-m)/m]} \binom{n-m-(m-1)k}{k} 2^k x^{k+1}, \ n \geq m,$$

$$Gj_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k + i \sum_{k=0}^{[(n+1-m)/m]} \frac{n+1-m-(m-2)k}{n+1-m-(m-1)k} \binom{n+1-m-(m-1)k}{k} 2^k x^{k+1}, \ n \geq m.$$

Remark 2. When $m = 2$ these relations become, respectively, the explicit formulas for Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials (see Asci M., Gurel E. [1]):

Table 2:

n	$j_{n,m}(x)$	$Gj_{n,m}(x)$
0	2	$2-i/2$
1	1	1
2	1	1
....	....	....
$m-1$	1	$1+2xi$
m	$1+4x$	$1+4x+xi$
$m+1$	$1+6x$	$1+6x+xi$
$m+2$	$1+8x$	$1+8x+xi$
$m+3$	$1+10x$	$1+10x+xi$
$m+4$	$1+12x$	$1+12x+xi$
....	....	....
$2m$	$1+(2+m)2x+4x^2$	$1+(2+m)2x+4x^2+xi(1+6x)$
$2m+1$	$1+(m+3)2x+12x^2$	$1+(m+3)2x+16x^2+xi(1+8x)$
$2m+2$	$1+(m+4)2x+24x^2$	$1+(m+4)2x+32x^2+xi(1+10x)$
....	....	....

$$\begin{aligned} GJ_n(x) = & \sum_{k=0}^{[(n-1)/2]} \binom{n-1-k}{k} (2x)^k \\ & + i \sum_{k=0}^{[(n-2)/2]} \binom{n-2-k}{k} 2^k x^{k+1} \end{aligned}$$

and

$$\begin{aligned} Gj_n(x) = & \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (2x)^k \\ & + i \sum_{k=0}^{[(n-1)/2]} \frac{n-1}{n-1-k} \binom{n-1-k}{k} 2^k x^{k+1}. \end{aligned}$$

Theorem 2.2. *The polynomials $GJ_{n,m}(x)$ and $Gj_{n,m}(x)$ satisfy the following relation*

$$Gj_{n,m}(x) = GJ_{n+1,m}(x) + 2xGJ_{n+1-m,m}(x), \quad n \geq m-1. \tag{10}$$

PROOF. Starting from the relation (3), we find

$$\begin{aligned}
&GJ_{n+1,m}(x) + 2xGJ_{n+1-m,m}(x) \\
&\qquad = J_{n+1,m}(x) + xiJ_{n+2-m,m}(x) \\
&\qquad\quad + 2x(J_{n+1-m,m}(x) + xiJ_{n+2-2m,m}(x)) \;\; (\text{by } 7) \\
&\qquad = J_{n+1,m}(x) + 2xJ_{n+1-m,m}(x) \\
&\qquad\quad + xi(J_{n+2-m,m}(x) + 2xJ_{n+2-2m,m}(x)) \\
&\qquad = j_{n,m}(x) + xij_{n+1-m,m}(x) \;\; (\text{by } (9)) \\
&\qquad = Gj_{n,m}(x) \;\; (\text{by } (8)).
\end{aligned}$$

So, we have proved the relation (10). □

Theorem (2.1) can be further generalized in the following sense. If we choose the initial values to be

$$GJ_{0,m}(x) = A, \;\; GJ_{n,m}(x) = 1, \;\; n = 1, \ldots, m-1,$$

then for $n \geq m$ we get

$$\begin{aligned}
GJ_{n,m}(x) = GJ_{n,m}(x, A) = &\sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} (2x)^k \\
+ 2Ax &\sum_{k=0}^{[(n-m)/m]} \binom{n-m-(m-1)k}{k} (2x)^k.
\end{aligned}$$

Additionally, if we choose the initial values to be

$$Gj_{0,m}(x) = 2 - A, \;\; Gj_{n,m} = 1, \;\; n = 1, \ldots, m-2, \;\; Gj_{m-1,m} = 1 + 4Ax,$$

then combined with (10), for $n \geq m$ we have

$$\begin{aligned}
Gj_{n,m}(x) = Gj_{n,m}(x, A) = &\sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k \\
+ 2Ax &\sum_{k=0}^{[(n+1-m)/m]} \frac{n+1-m-(m-1)k}{n+1-m-(m-1)k} \binom{n+1-m-(m-1)k}{k} (2x)^k.
\end{aligned}$$

Theorem 2.3. *The generating function $g_m(x,t)$ for the generalized Gaussian Jacobsthal $GJ_{n,m}(x)$ polynomials is*

$$g_m(x,t) = \sum_{n=0}^{\infty} GJ_{n,m}(x) t^n = \frac{2t + i(1-t)}{2(1 - t - 2xt^m)}, \;\; i^2 = -1, \tag{11}$$

and the generating function $h_m(x,t)$ for the generalized Gaussian Jacobsthal Lucas polynomials $Gj_{n,m}(x)$ is

$$h_m(x,t) = \sum_{n=0}^{\infty} Gj_{n,m}(x)t^n = \frac{4-2t+i(4xt^{m-1}+t-1)}{2(1-t-2xt^m)}, \quad i^2=-1. \tag{12}$$

Proof. Since

$$\begin{aligned} g_m(x,t) &= GJ_{0,m}(x) + GJ_{1,m}(x)t + GJ_{2,m}(x)t^2 + \cdots + GJ_{n,m}(x)t^n + \cdots - \\ -tg_m(x,t) &= -GJ_{0,m}(x)t - GJ_{1,m}(x)t^2 - GJ_{2,m}(x)t^3 - \cdots - GJ_{n,m}(x)t^{n+1} - \cdots - \\ -2xt^m g_m(x,t) &= -2xGJ_{0,m}(x)t^m - 2xGJ_{1,m}(x)t^{m+1} - 2xGJ_{2,m}(x)t^{m+2} \\ &\quad - \cdots - 2xGJ_{n,m}(x)t^{n+m} - \ldots, \end{aligned}$$

it follows that

$$\begin{aligned} g_m(x,t)\cdot(1-t-2xt^m) &= GJ_{0,m}(x) + t(GJ_{1,m}(x) - GJ_{0,m}(x)) + t^2(GJ_{2,m}(x) \\ &\quad - GJ_{1,m}(x)) + \ldots t^{m-1}(GJ_{m-1,m}(x) - GJ_{m-2,m}(x)) \\ &\quad + \sum_{n=m}^{\infty}(GJ_{n,m}(x) - GJ_{n-1,m}(x) - 2xGJ_{n-m,m}(x))t^n. \end{aligned}$$

So, from the recurrence relations (3) and (4), we conclude that (11) is correct.

In the same way, using the relations (5) and (6), we conclude that (12) holds. □

Remark 3. For $m=2$ the generating function $g_2(x,t)$ yields $g(x,t)$ - the generating function for the Gaussian Jacobsthal polynomials $GJ_n(x)$, and the generating function $h_2(x,t)$ yields $h(x,t)$ - the generating function for the Gaussian Jacobsthal Lucas polynomials $Gj_n(x)$ (see Asci M., Gurel E. [1]).

Theorem 2.4. *For $n \geq m-1$ $(m \geq 2)$, the polynomials $GJ_{n,m}(x)$ and $Gj_{n,m}(x)$ satisfy the following relation*

$$Gj_{n,m}(x) = GJ_{n+1,m}(x) + 2xGJ_{n+1-m,m}(x), \quad n \geq m-1. \tag{13}$$

Proof. We use induction on n. Namely, it is easily to check the statement (13) for $n=m-1$, using the Table 1 and Table 2. Suppose that (13) is correct for n $(n \geq m-1)$, then, for $n+1$ we get

$$\begin{aligned} Gj_{n+1,m}(x) &= Gj_{n,m}(x) + 2xGj_{n+1-m,m}(x) \ (by\ (5)) \\ &= GJ_{n+1,m}(x) + 2xGJ_{n+1-m,m}(x) \\ &\quad + 2x\left(GJ_{n+2-m,m}(x) + 2xGJ_{n+2-2m,m}(x)\right) \\ &= GJ_{n+1,m}(x) + 2xGJ_{n+2-m,m}(x) \\ &\quad + 2x\left(GJ_{n+1-m,m}(x) + 2xGJ_{n+2-2m,m}(x)\right) \\ &= GJ_{n+2,m}(x) + 2xGJ_{n+2-m,m}(x). \end{aligned}$$

□

Theorem 2.5. *The polynomials $GJ_{n,m}(x)$ and $Gj_{n,m}(x)$ satisfy the following relations, respectively*

$$\sum_{k=0}^{n} GJ_{k,m}(x) = \frac{1}{2x}\left[GJ_{n+m,m}(x) - 1\right] \tag{14}$$

and

$$\sum_{k=0}^{n} Gj_{k,m}(x) = \frac{1}{2x}\left[Gj_{n+m,m}(x) - (1+2xi)\right]. \tag{15}$$

Proof. We are going to prove the relation (14). Starting from the recurrence relation

$$GJ_{n,m}(x) = GJ_{n-1,m}(x) + 2xGJ_{n-m,m}(x)$$

or

$$GJ_{n+m,m}(x) = GJ_{n+m-1,m}(x) + 2xGJ_{n,m}(x)$$

we get

$$GJ_{n,m}(x) = \frac{1}{2x}\left[GJ_{n+m,m}(x) - GJ_{n+m-1,m}(x)\right].$$

Next we obtain

$$\begin{aligned} GJ_{0,m}(x) &= \frac{1}{2x}\left[GJ_{m,m}(x) - GJ_{m-1,m}(x)\right] \\ GJ_{1,m}(x) &= \frac{1}{2x}\left[GJ_{m+1,m}(x) - GJ_{m,m}(x)\right] \\ GJ_{2,m}(x) &= \frac{1}{2x}\left[GJ_{m+2,m}(x) - GJ_{m+1,m}(x)\right] \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ GJ_{n,m}(x) &= \frac{1}{2x}\left[GJ_{n+m,m}(x) - GJ_{n+m-1,m}(x)\right]. \end{aligned}$$

Summing the last equalities we get

$$\begin{aligned} \sum_{k=0}^{n} GJ_{k,m}(x) &= \frac{1}{2x}\left[GJ_{n+m,m}(x) - GJ_{m-1,m}(x)\right] \\ &= \frac{1}{2x}\left[GJ_{n+m,m}(x) - 1\right] \quad (by\ 4). \end{aligned}$$

The relation (15) can be proved in a similar manner, starting from recurrence relations (5) and (6). □

Recall that the convolutions of the generalized Jacobsthal polynomials are considered in G. B. Djordjevic [6].

Theorem 2.6. *Let $D^s \equiv \dfrac{d^s}{dx^s}$ $(s \geq 1)$. Then the following relation*

$$D^s\{GJ_{n,m}(x)\} = (2s)!! \sum_{k=0}^{n+1-ms} J^{s-1}_{n+1-ms-k,m}(x) GJ_{k,m}(x), \tag{16}$$

holds, where $J^{s-1}_{n,m}(x)$ is the $s-1$ - convolution of the generalized Jacobsthal polynomials.

Proof. Differentiating the generating function $g_m(x,t)$ (11) with respect to x, s-times, one by one, we find

$$\frac{\partial^s\{g_m(x,t)\}}{\partial x^s} = \frac{(2t + i(1-t)) \cdot s! \cdot (4t^m)^s}{(2(1-t-2xt^m))^{s+1}} \tag{17}$$

Next, the relation (17) can be written in the following form

$$\begin{aligned}\frac{\partial^s\{g_m(x,t)\}}{\partial x^s} &= \frac{2t + i(1-t)}{2(1-t-2xt^m)} \cdot \frac{s!4^s t^{ms}}{2^s(1-t-2xt^m)^s} \\ &= 2^s\, s!\, t^{ms} \sum_{n=0}^{\infty} J^{s-1}_{n+1,m}(x)t^n \cdot \sum_{n=0}^{\infty} GJ_{n,m}(x)t^n \\ &= (2s)!! \sum_{n=0}^{\infty} J^{s-1}_{n+1-ms-k,m}(x) GJ_{k,m}(x) t^n.\end{aligned}$$

So, we get the relation (16). □

3. Generalized Gaussian numbers

For $x = 1$ we get $GJ_{n,m}(1)$ - generalized Gaussian Jacobsthal numbers and $Gj_{n,m}(1)$ - generalized Gaussian Jacobsthal Lucas numbers. From the relations (3) and (4), we obtain

$$GJ_{n,m} = GJ_{n-1,m} + 2GJ_{n-m,m}, \;\; n \geq m,$$

with $GJ_{0,m} = i/2$, $GJ_{n,m} = 1$, $n = 1, \ldots, m-1$, where we have denoted $GJ_{n,m}(1) \equiv GJ_{n,m}$.

From the relation (7), for $x = 1$, we find that

$$GJ_{n,m} = J_{n,m} + iJ_{n+1-m,m}$$

where $J_{n,m}$ are the generalized Jacobsthal numbers (see G. B. Djordjevic, H. M. Srivastava [5]).

Remark 4. For $m = 2$ we get (Asci M., Gurel E. [1, 2])

$$GJ_n = GJ_{n-1} + 2GJ_{n-2}, \; GJ_0 = i/2, \; GJ_1 = 1,$$

and

$$GJ_n = J_n + iJ_{n-1}.$$

The generalized Gaussian Jacobsthal Lucas numbers $Gj_{n,m}(1) \equiv Gj_{n,m}$ are defined by

$$Gj_{n,m} = Gj_{n-1,m} + 2Gj_{n-m,m}, \;\; n \geq m,$$

with $Gj_{0,m} = 2 - i/2$, $Gj_{n,m} = 1, \;\; n = 1, \ldots, m-2$, $Gj_{m-1,m} = 1 + 2i$. These numbers satisfy the following relation (by (8))

$$Gj_{n,m} = j_{n,m} + ij_{n+1-m,m}, \;\; n \geq m-1.$$

Remark 5. For $m = 2$ we have (Asci M., Gurel E. [1, 2])

$$Gj_n = Gj_{n-1} + 2Gj_{n-2}, \;\; n \geq 2$$

and

$$Gj_n = j_n + iJ_{n-1}, \;\; n \geq 1.$$

4. Binet formulas

Let $\alpha_1, \;\alpha_2, \ldots, \alpha_m$ be the roots to the characteristic equation

$$t^m - t - 2 = 0$$

of the recurrence relation (3), for $x = 1$. Then it holds

$$GJ_{n,m} = C_1\alpha_1^n + C_2\alpha_2^n + \cdots + C_m\alpha_m^n,$$

where $C_1, C_2, \ldots, C_m$ are the solution to the following system of linear equations

$$\begin{aligned} C_1 + C_2 + \cdots + C_m &= \frac{i}{2} \\ C_1\alpha_1 + C_2\alpha_2 + \cdots + C_m\alpha_m &= 1 \\ C_1\alpha_1^2 + C_2\alpha_2^2 + \cdots + C_m\alpha_m^2 &= 1 \\ &\cdots\cdots\cdots\cdots\cdots \\ C_1\alpha_1^{m-1} + C_2\alpha_2^{m-1} + \cdots + C_m\alpha_m^{m-1} &= 1. \end{aligned}$$

Example 1. If $m = 2$ we get the Binet formula for the Gaussian Jacobsthal numbers GJ_n in Asci M., Gurel E. [2]

$$GJ_n = \frac{2+i}{3} \cdot 2^{n-1} + \frac{i-1}{3} \cdot (-1)^n.$$

Also, the Binet formula for the generalized Gaussian Jacobsthal Lucas sequence of numbers $Gj_{n,m}$ is

$$Gj_{n,m} = D_1\alpha_1^n + D_2\alpha_2^n + \cdots + D_m\alpha_m^n, \; m \geq 2, \tag{18}$$

where $D_1,\ D_2,\ldots,D_m$ are the solution to the following system of linear equations

$$\begin{aligned} D_1+D_2+\cdots+D_m &= 2-\frac{i}{2}\\ D_1\alpha_1+D_2\alpha_2+\cdots+D_m\alpha_m &= 1\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ D_1\alpha_1^{m-2}+D_2\alpha_2^{m-2}+\cdots+D_m\alpha_m^{m-2} &= 1\\ D_1\alpha_1^{m-1}+D_2\alpha_2^{m-1}+\cdots+D_m\alpha_m^{m-1} &= 1+2i. \end{aligned}$$

Example 2. For $m=2$ we get the Binet formula of the Gaussian Jacobsthal Lucas numbers Gj_n

$$Gj_n=(2+i)\cdot 2^{n-1}+(1-i)\cdot(-1)^n.$$

Example 3. For $m=3$ we get the following characteristic equation

$$t^3-t-2=0 \tag{19}$$

of the recurrence relation

$$GJ_{n,3}=GJ_{n-1,3}+2GJ_{n-3,3}.$$

The solution to the equation (19) is

$$\begin{aligned} \alpha_1&=u_0+v_0,\\ \alpha_2&=\varepsilon_1\cdot u_0+\varepsilon_2\cdot v_0,\\ \alpha_3&=\varepsilon_2\cdot u_0+\varepsilon_1\cdot v_0, \end{aligned}$$

where

$$u_0=\left(\frac{9+\sqrt{78}}{9}\right)^{1/3},\qquad v_0=\left(\frac{9-\sqrt{78}}{9}\right)^{1/3},$$

and ε_1 and ε_2 are the third roots of 1, or $\varepsilon_{1,2}=\frac{-1}{2}\pm\frac{i\sqrt{3}}{2}$.

So we have

$$GJ_{n,3}=C_1\alpha_1^n+C_2\alpha_2^n+C_3\alpha_3^n,$$

where

$$\begin{aligned} C_1+C_2+C_3&=\frac{i}{2}\\ C_1\alpha_1+C_2\alpha_2+C_3\alpha_3&=1\\ C_1\alpha_1^2+C_2\alpha_2^2+C_3\alpha_3^2&=1. \end{aligned}$$

Since

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = -1, \quad \alpha_1\alpha_2\alpha_3 = 2,$$

we get

$$C_1 = \frac{1 + 2\alpha_1 - i(1 - 5\alpha_1^2)}{2(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)},$$
$$C_2 = \frac{2 + 2\alpha_2 + i(1 - \alpha^2)}{2(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1)},$$
$$C_3 = \frac{2 + 2\alpha_3 + i(-1 + \alpha_3^2)}{2(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}.$$

Example 4. For $m = 3$, from (18), we get

$$Gj_{n,3} = D_1\alpha_1^n + D_2\alpha_2^n + D_3\alpha_3^n,$$

where

$$D_1 = \frac{-2 + \alpha_1 + 4\alpha_1^2 + i(3 + 5\alpha_1^2)}{2(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)},$$
$$D_2 = \frac{-2 + 2\alpha_2 + 4\alpha_2^2 + i(5 - \alpha_2^2)}{2(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_1)},$$
$$D_3 = \frac{-2 + 2\alpha_3 + 4\alpha_3^2 + i(5 - \alpha_3^2)}{2(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}.$$

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