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Generalized Adrien numbers

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Abstract. In this paper, we introduce and investigate the generalized Adrien sequences and we deal with, in detail, two special cases, namely, Adrien and Adrien-Lucas sequences. We present Binet's formulas, generating functions, Simson formula and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Adrien, Adrien-Lucas and third order Pell, third order Pell-Lucas numbers.

1. Introduction

Third-order Pell sequence $\{P_n\}_{n\geq 0}$ (OEIS: A077939, [6]) and third-order Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ (OEIS: A276225, [6]) are defined, respectively, by the third-order recurrence relations

$$P_n = 2P_{n-1} + P_{n-2} + P_{n-3}, \quad P_0 = 0, P_1 = 1, P_2 = 2, \tag{1}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2} + Q_{n-3}, \quad Q_0 = 3, Q_1 = 2, Q_2 = 6. \tag{2}$$

Here, OEIS stands for On-line Encyclopedia of Integer Sequences and A077939 is the index entry of third-order Pell sequence in the site OEIS, A276225 is the index entry of third-order Pell-Lucas sequence in the site OEIS.

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} P_{-n} &= -P_{-(n-1)} - 2P_{-(n-2)} + P_{-(n-3)}, \\ Q_{-n} &= -Q_{-(n-1)} - 2Q_{-(n-2)} + Q_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1) and (2) hold for all integer n. For more information on generalized third-order Pell numbers, see Soykan [13].

Now, we define two sequences related to third order Pell and third order Pell-Lucas numbers. Adrien and Adrien-Lucas numbers are defined as

$$A_n = 2A_{n-1} + A_{n-2} + A_{n-3} + 1, \quad \text{with } A_0 = 0, A_1 = 1, A_2 = 3, \quad n \geq 3,$$

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and

$$B_n = 2B_{n-1} + B_{n-2} + B_{n-3} - 3, \quad \text{with } B_0 = 4, B_1 = 3, B_2 = 7, \qquad n \geq 3,$$

respectively.

The first few values of Adrien and Adrien-Lucas numbers are

$$0, 1, 3, 8, 21, 54, 138, 352, 897, 2285, 5820, 14823, 37752, 96148, ...$$

and

$$4, 3, 7, 18, 43, 108, 274, 696, 1771, 4509, 11482, 29241, 74470, 189660, ...$$

respectively.

The sequences $\{A_n\}$ and $\{B_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{array}{rcl} A_n &=& 3A_{n-1} - A_{n-2} - A_{n-4}, \quad A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8, \quad n \geq 4, \\ B_n &=& 3B_{n-1} - B_{n-2} - B_{n-4}, \quad B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18, \quad n \geq 4. \end{array}$$

There are close relations between Adrien, Adrien-Lucas and third order Pell, third order Pell-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{array}{rcl} P_{n+1} &=& A_{n+1} - A_n, \\ 87P_n &=& 11B_{n+2} - 12B_{n+1} - 14B_n + 15, \\ Q_n &=& -3A_{n+3} + 8A_{n+2} + 3A_{n+1} - 8A_n, \\ 3Q_n &=& B_{n+3} - 2B_{n+2} - B_{n+1} + 2B_n, \end{array}$$

and

$$\begin{array}{rcl} 3A_n &=& P_{n+2} - P_{n+1} + P_n - 1, \\ B_n &=& -P_{n+2} + 5P_{n+1} - 3P_n + 1, \\ 87A_n &=& 6Q_{n+2} + 4Q_{n+1} - 5Q_n - 29, \\ B_n &=& Q_n + 1. \end{array}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Adrien, Adrien-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \; n \geq 4 \tag{3}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1],[3],[4],[5],[8],[10],[11],[14],[15]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (3) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{4}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers n, Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{5}$$

where

$$\begin{aligned} p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (4) has at least one real (say α) solution. Note that the Binet form of a sequence satisfying (4) for non-negative integers is valid for all integers n (see [2]).

Next, we consider two special cases of the generalized (r, s, t, u) sequence $\{W_n\}$ which we call them (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas sequences. (r, s, t, u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and (r, s, t, u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, & (6) \\ G_0 &= 0, G_1 = 1, G_2 = r, G_3 = r^2 + s, \\ H_{n+4} &= rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, & (7) \\ H_0 &= 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (6) and (7) hold for all integers n.

For all integers n, (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas numbers (using initial conditions in (6) or (7)) can be expressed using Binet's formulas as in the following corollary.

Corollary 1.2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formulas of (r, s, t, u)-Fibonacci and (r, s, t, u)-Lucas numbers are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

respectively.

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1.1, respectively. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 1.3. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized* (r, s, t, u) *sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \tag{8}$$

Proof. For a proof, see Soykan [8, Lemma 1]. □

The following theorem presents Simson's formula of generalized (r, s, t, u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

Theorem 1.4 (Simson's Formula of Generalized (r, s, t, u) Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{9}$$

Proof. (9) is given in Soykan [7]. □

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.5. *For* $n \in \mathbb{Z}$, *for the generalized Tetranacci sequence (or generalized* (r, s, t, u)*-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_n W_{2n} - 3H_n^2 W_n + 3H_{2n} W_n + W_0 H_n^3 + 2W_0 H_{3n} - 3W_0 H_n H_{2n}) \\ &= (-1)^{-n-1} u^{-n}(W_{3n} - H_n W_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [9, Theorem 1.]. □

Using Theorem 1.5, we have the following corollary, see Soykan [9, Corollary 4].

Corollary 1.6. *For* $n \in \mathbb{Z}$, *we have*

(a) $2(-u)^{n+4} G_{-n} = -(3ru^2 + t^3 - 3stu)^2 G_n^3 - (2su - t^2)^2 G_{n+3}^2 G_n - (-rt^2 - tu + 2rsu)^2 G_{n+2}^2 G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2 G_{n+1}^2 G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1}) G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n.$

(b) $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 1.5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_n G_{2n} - 3H_n^2 G_n + 3H_{2n} G_n), \tag{10}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right), \tag{11}$$

respectively.

If we define the square matrix M of order 4 as

$$M = M_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$S_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 1.7. *For all integers m, n, we have*

(a) $S_n = M^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b) $U_1 M^n = M^n U_1$.

(c) $U_{n+m} = U_n S_m = S_m U_n$.

Proof. For the proof, see Soykan [8, Theorem 19]. □

Theorem 1.8. *For all integers m, n, we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \quad (12)$$

Proof. For the proof, see Soykan [8, Theorem 20.]. □

In the next sections, we present new results.

2. Generalized Adrien Sequence

In this paper, we consider the case $r = 3, s = -1, t = 0, u = -1$. A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4} \quad (13)$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (13) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\delta &= 1
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The first few generalized Adrien numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Adrien numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_2 - W_1 - W_3$
2	W_2	$3W_1 - W_0 - W_2$
3	W_3	$3W_0 - 3W_2 + W_3$
4	$3W_3 - W_2 - W_0$	$10W_2 - 6W_1 - 3W_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0$	$10W_1 - 6W_0 - 3W_2$
6	$21W_3 - 3W_1 - 9W_2 - 8W_0$	$10W_0 + 3W_1 - 18W_2 + 6W_3$
7	$54W_3 - 8W_1 - 24W_2 - 21W_0$	$3W_0 - 28W_1 + 36W_2 - 10W_3$
8	$138W_3 - 21W_1 - 62W_2 - 54W_0$	$33W_1 - 28W_0 - W_2 - 3W_3$
9	$352W_3 - 54W_1 - 159W_2 - 138W_0$	$33W_0 + 27W_1 - 87W_2 + 28W_3$
10	$897W_3 - 138W_1 - 406W_2 - 352W_0$	$27W_0 - 120W_1 + 127W_2 - 33W_3$
11	$2285W_3 - 352W_1 - 1035W_2 - 897W_0$	$100W_1 - 120W_0 + 48W_2 - 27W_3$
12	$5820W_3 - 897W_1 - 2637W_2 - 2285W_0$	$100W_0 + 168W_1 - 387W_2 + 120W_3$
13	$14\,823W_3 - 2285W_1 - 6717W_2 - 5820W_0$	$168W_0 - 487W_1 + 420W_2 - 100W_3$

Note that the sequences $\{A_n\}$ and $\{B_n\}$ which are defined in the section Introduction, are the special cases of the generalized Adrien sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$, in this section as well. Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} A_n &= 3A_{n-1} - A_{n-2} - A_{n-4}, \quad A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8, \quad n \geq 4, \\ B_n &= 3B_{n-1} - B_{n-2} - B_{n-4}, \quad B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18, \quad n \geq 4. \end{aligned}$$

The sequences $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\ B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Next, we present the first few values of the Adrien and Adrien-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
A_n	0	1	3	8	21	54	138	352	897	2285	5820	14823	37752	96148
A_{-n}	0	0	0	−1	0	1	−3	0	6	−10	−3	28	−33	−27
B_n	4	3	7	18	43	108	274	696	1771	4509	11482	29241	74470	189660
B_{-n}	4	0	−2	9	−2	−15	31	0	−74	108	43	−330	355	351

(5) can be used to obtain the Binet formula of generalized Adrien numbers. Binet's formula of generalized Adrien numbers can be given as follows:

Theorem 2.1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of generalized Adrien numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - 1W_0)\alpha^n}{4\alpha^2 + 3\alpha - 1} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - 1W_0)\beta^n}{4\beta^2 + 3\beta - 1} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - 1W_0)\gamma^n}{4\gamma^2 + 3\gamma - 1} \\ &+ \frac{W_3 - 2W_2 - W_1 - W_0}{-3}. \end{aligned}$$

Adrien and Adrien-Lucas numbers can be expressed using Binet's formulas as follows.

Corollary 2.2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formula of Adrien and Adrien-Lucas numbers are*

$$A_n = \frac{(2\alpha^2 + \alpha + 1)\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3},$$

and

$$B_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

Note that Binet's formulas of third order Pell and third order Pell-Lucas numbers, respectively, are

$$\begin{aligned} P_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ Q_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

see, Soykan [13] for more details.

So, by using Binet's formulas of Adrien, Adrien-Lucas and third order Pell, third order Pell-Lucas numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

Lemma 2.3. *For all integers n, the following equalities (identities) are true:*

(a)

- $P_{n+3} = A_{n+3} - A_{n+2}$.
- $P_n = A_{n+3} - 3A_{n+2} + A_{n+1} + A_n$.
- $3A_{n+4} = 25P_{n+2} + 14P_{n+1} + 10P_n - 1$.
- $3A_n = P_{n+2} - P_{n+1} + P_n - 1$.
- $P_n = -A_{n+2} + 2A_{n+1} + 2A_n + 1$.
- $P_{n+1} = A_{n+1} - A_n$.

(b)

- $87P_{n+3} = 33B_{n+3} - 6B_{n+1} - 11B_{n+2} - 16B_n$.
- $87P_n = -5B_{n+3} + 21B_{n+2} - 7B_{n+1} - 9B_n$.
- $B_{n+4} = 17P_{n+2} + 8P_{n+1} + 6P_n + 1$.
- $B_n = -P_{n+2} + 5P_{n+1} - 3P_n + 1$.
- $87P_n = 11B_{n+2} - 12B_{n+1} - 14B_n + 15$.
- $B_{n+1} + 3B_n = 11P_{n+1} - 10P_n + 4$.

(c)

- $Q_{n+3} = 2A_{n+3} + A_{n+1} - 3A_n$.
- $Q_n = -3A_{n+3} + 8A_{n+2} + 3A_{n+1} - 8A_n$.
- $87A_{n+4} = 225Q_{n+2} + 121Q_{n+1} + 88Q_n - 29$.
- $87A_n = 6Q_{n+2} + 4Q_{n+1} - 5Q_n - 29$.
- $Q_n = 2A_{n+2} - 11A_n - 3$.
- $3(3A_{n+1} - 8A_n) = 2Q_n - Q_{n+1} + 5$.

(d)

- $3Q_{n+3} = 4B_{n+3} - 2B_{n+2} - B_{n+1} - B_n$.
- $3Q_n = B_{n+3} - 2B_{n+2} - B_{n+1} + 2B_n$.
- $B_{n+4} = 5Q_{n+2} + 3Q_{n+1} + 2Q_n + 1$.
- $B_n = Q_n + 1$.
- $Q_n = B_n - 1$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 2.4. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Adrien sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)z + (W_2 - 3W_1 + W_0)z^2 + (W_3 - 3W_2 + W_1)z^3}{1 - 3z + z^2 + z^4}.$$

Proof. Take $r = 3, s = -1, t = 0, u = -1$ in Lemma 1.3.
The previous lemma gives the following results as particular examples.

Corollary 2.5. *Generating functions of Adrien and Adrien-Lucas numbers are*

$$\sum_{n=0}^{\infty} A_n z^n = \frac{z}{1-3z+z^2+z^4} = \frac{z}{(-1+2z+z^2+z^3)(z-1)},$$
$$\sum_{n=0}^{\infty} B_n z^n = \frac{4-9z+2z^2}{1-3z+z^2+z^4} = \frac{4-9z+2z^2}{(-1+2z+z^2+z^3)(z-1)},$$

respectively.

3. Simson Formulas

Now, we present Simson's formula of generalized Adrien numbers.

Theorem 3.1 (Simson's Formula of Generalized Adrien Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (W_0+W_1+2W_2-W_3)(-W_3^3+5W_2^3+W_1^3+W_0^3-(W_0+3W_1-7W_2)W_3^2+(3W_0-4W_1-14W_3)W_2^2+(2W_0+W_2-6W_3)W_1^2-(W_1+2W_3)W_0^2+13W_1W_2W_3+W_0W_2W_3+5W_0W_1W_3-7W_0W_1W_2).$$

Proof. Take $r = 3, s = -1, t = 0, u = -1$ in Theorem 1.4. □
The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers n, the Simson's formulas of Adrien and Adrien-Lucas numbers are given as*

$$\begin{vmatrix} A_{n+3} & A_{n+2} & A_{n+1} & A_n \\ A_{n+2} & A_{n+1} & A_n & A_{n-1} \\ A_{n+1} & A_n & A_{n-1} & A_{n-2} \\ A_n & A_{n-1} & A_{n-2} & A_{n-3} \end{vmatrix} = 1,$$

$$\begin{vmatrix} B_{n+3} & B_{n+2} & B_{n+1} & B_n \\ B_{n+2} & B_{n+1} & B_n & B_{n-1} \\ B_{n+1} & B_n & B_{n-1} & B_{n-2} \\ B_n & B_{n-1} & B_{n-2} & B_{n-3} \end{vmatrix} = -783,$$

respectively.

4. Some Identities

In this section, we obtain some identities of Adrien and Adrien-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{A_n\}$.

Lemma 4.1. *The following equalities are true:*

(a) $W_n = (10W_2-6W_1-3W_3)A_{n+5}+(3W_0+18W_1-33W_2+10W_3)A_{n+4}+(18W_2-3W_1-10W_0-6W_3)A_{n+3}+(6W_0-10W_1+3W_2)A_{n+2}$.

(b) $W_n = (3W_0-3W_2+W_3)A_{n+4}+(3W_1-10W_0+8W_2-3W_3)A_{n+3}+(6W_0-10W_1+3W_2)A_{n+2}+(6W_1-10W_2+3W_3)A_{n+1}$.

(c) $W_n = (3W_1-W_0-W_2)A_{n+3}+(3W_0-10W_1+6W_2-W_3)A_{n+2}+(6W_1-10W_2+3W_3)A_{n+1}+(3W_2-3W_0-W_3)A_n$.

(d) $W_n = (3W_2 - W_1 - W_3)A_{n+2} + (W_0 + 3W_1 - 9W_2 + 3W_3)A_{n+1} + (3W_2 - 3W_0 - W_3)A_n + (W_0 - 3W_1 + W_2)A_{n-1}$.

(e) $W_n = W_0 A_{n+1} + (W_1 - 3W_0)A_n + (W_0 - 3W_1 + W_2)A_{n-1} + (W_1 - 3W_2 + W_3)A_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times A_{n+5} + b \times A_{n+4} + c \times A_{n+3} + d \times A_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times A_5 + b \times A_4 + c \times A_3 + d \times A_2 \\ W_1 &= a \times A_6 + b \times A_5 + c \times A_4 + d \times A_3 \\ W_2 &= a \times A_7 + b \times A_6 + c \times A_5 + d \times A_4 \\ W_3 &= a \times A_8 + b \times A_7 + c \times A_6 + d \times A_5 \end{aligned}$$

we find that $a = 10W_2 - 6W_1 - 3W_3, b = 3W_0 + 18W_1 - 33W_2 + 10W_3, c = 18W_2 - 3W_1 - 10W_0 - 6W_3, d = 6W_0 - 10W_1 + 3W_2$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{B_n\}$.

Lemma 4.2. *The following equalities are true:*

(a) $261W_n = -(16W_0 - 110W_1 + 272W_2 - 91W_3)B_{n+5} - (43W_0 + 346W_1 - 835W_2 + 272W_3)B_{n+4} + (256W_0 + 67W_1 - 346W_2 + 110W_3)B_{n+3} - (110W_0 - 256W_1 + 43W_2 + 16W_3)B_{n+2}$.

(b) $261W_n = -(91W_0 + 16W_1 - 19W_2 - W_3)B_{n+4} + (272W_0 - 43W_1 - 74W_2 + 19W_3)B_{n+3} - (110W_0 - 256W_1 + 43W_2 + 16W_3)B_{n+2} + (16W_0 - 110W_1 + 272W_2 - 91W_3)B_{n+1}$.

(c) $261W_n = -(W_0 + 91W_1 + 17W_2 - 22W_3)B_{n+3} - (19W_0 - 272W_1 + 62W_2 + 17W_3)B_{n+2} + (16W_0 - 110W_1 + 272W_2 - 91W_3)B_{n+1} + (91W_0 + 16W_1 - 19W_2 - W_3)B_n$.

(d) $261W_n = -(22W_0 + W_1 + 113W_2 - 49W_3)B_{n+2} + (17W_0 - 19W_1 + 289W_2 - 113W_3)B_{n+1} + (91W_0 + 16W_1 - 19W_2 - W_3)B_n + (W_0 + 91W_1 + 17W_2 - 22W_3)B_{n-1}$.

(e) $261W_n = -(49W_0 + 22W_1 + 50W_2 - 34W_3)B_{n+1} + (113W_0 + 17W_1 + 94W_2 - 50W_3)B_n + (W_0 + 91W_1 + 17W_2 - 22W_3)B_{n-1} + (22W_0 + W_1 + 113W_2 - 49W_3)B_{n-2}$.

Now, we give a few basic relations between $\{A_n\}$ and $\{B_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{aligned} 261A_n &= 22B_{n+5} - 17B_{n+4} - 91B_{n+3} - B_{n+2}, \\ 261A_n &= 49B_{n+4} - 113B_{n+3} - B_{n+2} - 22B_{n+1}, \\ 261A_n &= 34B_{n+3} - 50B_{n+2} - 22B_{n+1} - 49B_n, \\ 261A_n &= 52B_{n+2} - 56B_{n+1} - 49B_n - 34B_{n-1}, \\ 261A_n &= 100B_{n+1} - 101B_n - 34B_{n-1} - 52B_{n-2}, \end{aligned}$$

and

$$\begin{aligned} B_n &= -2A_{n+5} + 15A_{n+4} - 31A_{n+3} + 15A_{n+2}, \\ B_n &= 9A_{n+4} - 29A_{n+3} + 15A_{n+2} + 2A_{n+1}, \\ B_n &= -2A_{n+3} + 6A_{n+2} + 2A_{n+1} - 9A_n, \\ B_n &= 4A_{n+1} - 9A_n + 2A_{n-1}. \end{aligned}$$

5. Relations Between Special Numbers

In this section, we present identities on Adrien, Adrien-Lucas numbers and third order Pell, third order Pell-Lucas numbers. We know that

$$\begin{aligned} 3A_n &= P_{n+2} - P_{n+1} + P_n - 1, \\ B_n &= Q_n + 1. \end{aligned}$$

Note also that from Lemma 4.1 and Lemma 4.2 , we have the formulas of W_n as

$$\begin{aligned} W_n &= (3W_1 - W_0 - W_2)A_{n+3} + (3W_0 - 10W_1 + 6W_2 - W_3)A_{n+2} \\ &\quad +(6W_1 - 10W_2 + 3W_3)A_{n+1} + (3W_2 - 3W_0 - W_3)A_n, \\ 261W_n &= -(W_0 + 91W_1 + 17W_2 - 22W_3)B_{n+3} - (19W_0 - 272W_1 + 62W_2 + 17W_3)B_{n+2} \\ &\quad +(16W_0 - 110W_1 + 272W_2 - 91W_3)B_{n+1} + (91W_0 + 16W_1 - 19W_2 - W_3)B_n. \end{aligned}$$

Using the above identities, we obtain the relation of generalized Adrien numbers and third order Pell, third order Pell-Lucas numbers in the following forms:

Lemma 5.1. *For all integers n, we have the following identities:*

(a) $3W_n = (-2W_3 + 7W_2 - 4W_1 - W_0)P_{n+2} + (5W_3 - 16W_2 + 7W_1 + 4W_0)P_{n+1} + (W_3 + 8W_1 - 5W_2 - 4W_0)P_n - W_3 + 2W_2 + W_1 + W_0.$

(b) $87W_n = (9W_3 - 32W_2 + 30W_1 - 7W_0)Q_{n+2} + (-23W_3 + 85W_2 - 67W_1 + 5W_0)Q_{n+1} + (7W_3 - 12W_2 - 25W_1 + 30W_0)Q_n - 29W_3 + 58W_2 + 29W_1 + 29W_0.$

6. On the Recurrence Properties of Generalized Adrien Sequence

Taking $r = 3, s = -1, t = 0, u = -1$ in Theorem 1.5, we obtain the following Proposition.

Proposition 6.1. *For $n \in \mathbb{Z}$, generalized Adrien numbers (the case $r = 3, s = -1, t = 0, u = -1$) have the following identity:*

$$W_{-n} = \frac{1}{6}(-6W_{3n} + 6B_nW_{2n} - 3B_n^2W_n + 3B_{2n}W_n + W_0B_n^3 + 2W_0B_{3n} - 3W_0B_nB_{2n})$$

From the above Proposition 6.1 (or by taking $G_n = A_n$ and $H_n = B_n$ in (10) and (11) respectively), we have the following corollary which gives the connection between the special cases of generalized Adrien sequence at the positive index and the negative index: for Adrien and Adrien-Lucas and Adrien numbers: take $W_n = A_n$ with $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8$ and take $W_n = B_n$ with $B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18$, respectively. Note that in this case $H_n = B_n$.

Corollary 6.2. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Adrien sequence:*

$$A_{-n} = \frac{1}{6}(-6A_{3n} + 6B_nA_{2n} - 3B_n^2A_n + 3B_{2n}A_n).$$

(b) *Adrien-Lucas sequence:*

$$B_{-n} = \frac{1}{6}\left(B_n^3 + 2B_{3n} - 3B_{2n}B_n\right).$$

We can also present the formulas of A_{-n} and B_{-n} in the following forms.

Corollary 6.3. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) $A_{-n} = \frac{1}{6}(-6A_{3n} + 6(-2A_{n+3} + 6A_{n+2} + 2A_{n+1} - 9A_n)A_{2n} - 3(-2A_{n+3} + 6A_{n+2} + 2A_{n+1} - 9A_n)^2 A_n + 3(-2A_{2n+3} + 6A_{2n+2} + 2A_{2n+1} - 9A_{2n})A_n)$.

(b) $A_{-n} = \frac{1}{3}(3P_n^2 - 3P_{n-1}^2 + 3P_{n-2}^2 + P_{2n} - P_{2n-2} + P_{2n-4} + (P_{n+2} - 5P_{n+1} + 5P_{n-1} + P_{n-2})P_n - (P_{n+1} + 5P_{n-2})P_{n-1} - 1)$.

(c) $B_{-n} = \frac{1}{2}(Q_n^2 - Q_{2n} + 2)$.

Proof.

(a) By using the identity $B_n = -2A_{n+3} + 6A_{n+2} + 2A_{n+1} - 9A_n$ and Corollary 6.2, (or by using Corollary 1.6 (a)), we get (a).

(b) Since $A_n = \frac{1}{3}(P_{n+2} - P_{n+1} + P_n - 1)$ and $P_{-n} = 3P_n^2 + P_{2n} + P_{n+2}P_n - 5P_{n+1}P_n$ (see, for example Soykan [12]), we get (b).

(c) Since $B_n = Q_n + 1$ and $Q_{-n} = \frac{1}{2}(Q_n^2 - Q_{2n})$ (see, for example Soykan [12]), we obtain

$$B_{-n} = \frac{1}{2}(Q_n^2 - Q_{2n} + 2). \quad \square$$

7. Sum Formulas

The following Corollary gives sum formulas of third order Pell and third order Pell-Lucas numbers.

Corollary 7.1. *For $n \geq 0$, third order Pell and third order Pell-Lucas numbers have the following properties:*

(a)

- **(i)** $\sum_{k=0}^{n} P_k = \frac{1}{3}\left(P_{n+3} - P_{n+2} - 2P_{n+1} - 1\right)$.
- **(ii)** $\sum_{k=0}^{n} P_{2k} = \frac{1}{3}\left(P_{2n+1} + P_{2n} - 1\right)$.
- **(iii)** $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{3}\left(P_{2n+2} + P_{2n+1}\right)$.

(b)

- **(i)** $\sum_{k=0}^{n} Q_k = \frac{1}{3}\left(Q_{n+3} - Q_{n+2} - 2Q_{n+1} + 2\right)$.
- **(ii)** $\sum_{k=0}^{n} Q_{2k} = \frac{1}{3}\left(Q_{2n+1} + Q_{2n} + 4\right)$.
- **(iii)** $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{3}\left(Q_{2n+2} + Q_{2n+1} - 2\right)$.

Proof. It is given in Soykan [13, Corollary 14 and Corollary 15]. □
The following Corollary presents sum formulas of Adrien and Adrien-Lucas numbers.

Corollary 7.2. *For $n \geq 0$, Adrien and Adrien-Lucas numbers have the following properties:*

(a)

- **(i)** $\sum_{k=0}^{n} A_k = \frac{1}{9}(4P_{n+2} - P_{n+1} + P_n - 3n - 7)$.
- **(ii)** $\sum_{k=0}^{n} A_{2k} = \frac{1}{9}(2P_{2n+2} + P_{2n+1} + 2P_{2n} - 3n - 5)$.
- **(iii)** $\sum_{k=0}^{n} A_{2k+1} = \frac{1}{9}(5P_{2n+2} + 4P_{2n+1} + 2P_{2n} - 3n - 5)$.

(b)

- **(i)** $\sum_{k=0}^{n} B_k = \frac{1}{3}(Q_{n+2} - Q_{n+1} + Q_n + 3n + 5)$.

(ii) $\sum_{k=0}^{n} B_{2k} = \frac{1}{3}(Q_{2n+1} + Q_{2n} + 3n + 7)$.

(iii) $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{3}(Q_{2n+2} + Q_{2n+1} + 3n + 1)$.

Proof. The proof follows from Corollary 7.1 and the identities

$$\begin{aligned} A_n &= \frac{1}{3}(P_{n+2} - P_{n+1} + P_n - 1), \\ B_n &= Q_n + 1. \square \end{aligned}$$

8. Matrices and Identities Related With Generalized Adrien Numbers

If we define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$S_n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & -W_n - W_{n-2} & -W_{n-1} & -W_n \\ W_n & -W_{n-1} - W_{n-3} & -W_{n-2} & -W_{n-1} \\ W_{n-1} & -W_{n-2} - W_{n-4} & -W_{n-3} & -W_{n-2} \\ W_{n-2} & -W_{n-3} - W_{n-5} & -W_{n-4} & -W_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 8.1. *For all integers m, n, we have*

(a) $S_n = M^n$.

(b) $U_1 M^n = M^n U_1$.

(c) $U_{n+m} = U_n S_m = S_m U_n$.

Proof. Take $r = 3, s = -1, t = 0, u = -1$ in Theorem 1.7. $\square$

Corollary 8.2. *For all integers n, we have the following formulas for the Adrien and Adrien-Lucas numbers.*

(a) *Adrien Numbers.*

$$M^n = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}.$$

(b) *Adrien-Lucas Numbers.*

$$M^n = \frac{1}{261}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= 34B_{n+4} - 50B_{n+3} - 22B_{n+2} - 49B_{n+1} \\ a_{21} &= 34B_{n+3} - 50B_{n+2} - 22B_{n+1} - 49B_{n} \\ a_{31} &= 34B_{n+2} - 50B_{n+1} - 22B_{n} - 49B_{n-1} \\ a_{41} &= 34B_{n+1} - 50B_{n} - 22B_{n-1} - 49B_{n-2} \end{aligned}$$

$$\begin{aligned} a_{12} &= 99B_n + 22B_{n-1} - 12B_{n+1} + 49B_{n-2} + 50B_{n+2} - 34B_{n+3} \\ a_{22} &= 99B_{n-1} - 12B_n + 50B_{n+1} + 22B_{n-2} - 34B_{n+2} + 49B_{n-3} \\ a_{32} &= 50B_n - 12B_{n-1} - 34B_{n+1} + 99B_{n-2} + 22B_{n-3} + 49B_{n-4} \\ a_{42} &= 50B_{n-1} - 34B_n - 12B_{n-2} + 99B_{n-3} + 22B_{n-4} + 49B_{n-5} \end{aligned}$$

$$\begin{aligned} a_{13} &= -(34B_{n+2} - 50B_{n+1} - 22B_n - 49B_{n-1}) \\ a_{23} &= -(34B_{n+1} - 50B_n - 22B_{n-1} - 49B_{n-2}) \\ a_{33} &= -(34B_n - 50B_{n-1} - 22B_{n-2} - 49B_{n-3}) \\ a_{43} &= -(34B_{n-1} - 50B_{n-2} - 22B_{n-3} - 49B_{n-4}) \end{aligned}$$

$$\begin{aligned} a_{14} &= -(34B_{n+3} - 50B_{n+2} - 22B_{n+1} - 49B_n) \\ a_{24} &= -(34B_{n+2} - 50B_{n+1} - 22B_n - 49B_{n-1}) \\ a_{34} &= -(34B_{n+1} - 50B_n - 22B_{n-1} - 49B_{n-2}) \\ a_{44} &= -(34B_n - 50B_{n-1} - 22B_{n-2} - 49B_{n-3}) \end{aligned}$$

Proof.

(a) It is given in Theorem 8.1 (a).

(b) Note that, from Lemma 4.3, we have

$$261A_n = 34B_{n+3} - 50B_{n+2} - 22B_{n+1} - 49B_n.$$

Using the last equation and (a), we get the required result. □

Using the above last Corollary and the identity

$$3A_n = P_{n+2} - P_{n+1} + P_n - 1,$$

we obtain the following formula for third order Pell numbers.

Corollary 8.3. *For all integers n, we have the following formula for third order Pell numbers.*

$$M^n = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{3}\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

where

$$\begin{aligned} b_{11} &= P_{n+2} + 2P_{n+1} + P_n - 1 \\ b_{21} &= P_{n+2} - P_{n+1} + P_n - 1 \\ b_{31} &= P_{n+2} - P_{n+1} - 2P_n - 1 \\ b_{41} &= -2P_{n+2} + 5P_{n+1} + P_n - 1 \end{aligned}$$

$$b_{12} = P_{n+2} - 4P_{n+1} - 2P_n + 2$$
$$b_{22} = -2P_{n+2} + 5P_{n+1} - 2P_n + 2$$
$$b_{32} = -2P_{n+2} + 2P_{n+1} + 7P_n + 2$$
$$b_{42} = 7P_{n+2} - 16P_{n+1} - 5P_n + 2$$

$$b_{13} = -(P_{n+2} - P_{n+1} - 2P_n - 1)$$
$$b_{23} = -(-2P_{n+2} + 5P_{n+1} + P_n - 1)$$
$$b_{33} = -(P_{n+2} - 4P_{n+1} + 4P_n - 1)$$
$$b_{43} = -(4P_{n+2} - 7P_{n+1} - 8P_n - 1)$$

$$b_{14} = -(P_{n+2} - P_{n+1} + P_n - 1)$$
$$b_{24} = -(P_{n+2} - P_{n+1} - 2P_n - 1)$$
$$b_{34} = -(-2P_{n+2} + 5P_{n+1} + P_n - 1)$$
$$b_{44} = -(P_{n+2} - 4P_{n+1} + 4P_n - 1)$$

Next, we present an identity for W_{n+m}.

Theorem 8.4. *For all integers m, n, we have*

$$W_{n+m} = W_n A_{m+1} + W_{n-1}(-A_m - A_{m-2}) - W_{n-2}A_{m-1} - W_{n-3}A_m.$$

Proof. Take $r = 3, s = -1, t = 0, u = -1$ in Theorem 1.8. □
As particular cases of the above theorem, we give identities for A_{n+m} and B_{n+m}.

Corollary 8.5. *For all integers m, n, we have*

$$\begin{aligned} A_{n+m} &= A_n A_{m+1} + A_{n-1}(-A_m - A_{m-2}) - A_{n-2}A_{m-1} - A_{n-3}A_m, \\ B_{n+m} &= B_n A_{m+1} + B_{n-1}(-A_m - A_{m-2}) - B_{n-2}A_{m-1} - B_{n-3}A_m. \end{aligned}$$

Taking $m = n$ in the last corollary, we obtain the following identities:

$$\begin{aligned} A_{2n} &= A_n A_{n+1} - A_n A_{n-1} - A_n A_{n-3} - 2A_{n-1}A_{n-2}, \\ B_{2n} &= B_n A_{n+1} - A_n B_{n-1} - A_n B_{n-3} - A_{n-1}B_{n-2} - A_{n-2}B_{n-1}. \end{aligned}$$

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