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On the average value of a function of generalized mean

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Abstract. In this short note we find the exact formula for

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n} f(M_{n,p}(x_1,\ldots,x_n))\,dx_1\ldots dx_n,$$

where $M_{n,p}(x_1,\ldots,x_n)$, $p\in[-\infty,\infty]$ is the generalized mean and f is an arbitrary continuous function.

1. Introduction and preliminaries

The Miklós Schweitzer competition is an annual Hungarian mathematics competition for university students, established in 1949. It is named after Miklós Schweitzer, a young mathematician who died in the Second World War. The competition consists of ten to twelve problems written by prominent Hungarian mathematicians. Competitors are allowed ten days to come up with solutions and they can use any tools and literature they want. The Schweitzer competition is one of the most unique in the world. Winners of the contests have gone on to become world-class scientists. The contests serve as reflections of Hungarian mathematical trends and as starting points for many interesting research problems in mathematics, [1]. In 1967 competition the following problem was given.

Problem 1.1. *(See problem P.6 in [1].) Let f be a continuous function on the unit interval $[0,1]$. Show that*

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n} f\left(\frac{x_1+\cdots+x_n}{n}\right)dx_1\ldots dx_n = f\left(\frac{1}{2}\right)$$

and

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n} f\left(\sqrt[n]{x_1\cdots x_n}\right)dx_1\ldots dx_n = f\left(\frac{1}{e}\right).$$

The problem is solved in [1] in two different ways. The first one uses some combinatorial arguments among others, while the other one is based on the strong law of large numbers and Lebesgue's theorem of dominant

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convergence. Our aim is to solve the problem in the most general case when the arithmetic (geometric) mean is replaced by the generalized mean with exponent $p \in [-\infty, \infty]$. Our proof is quite elementary and it is intended to be accessible to undergraduate students of mathematics.

Recall some definitions and notations. The generalized mean with exponent $p \in [-\infty, \infty]$ of positive real numbers $x_1, \dots, x_n$ is defined by

$$M_{n,p}(x_1, \dots, x_n) = \begin{cases} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}, & p \in \mathbb{R} \setminus \{0\} \\ \sqrt[n]{x_1 \cdots x_n}, & p = 0 \\ \min\{x_1, \dots, x_n\}, & p = -\infty \\ \max\{x_1, \dots, x_n\}, & p = \infty \end{cases}. \tag{1}$$

As we know, $\lim_{p \to 0} M_{n,p} = M_{n,0}$, $\lim_{p \to \infty} M_{n,p} = M_{n,\infty}$ and $\lim_{p \to -\infty} M_{n,p} = M_{n,-\infty}$

For $k \in \mathbb{N} \cup \{0\}$, denote by p_k the power function $p_k(x) = x^k$, $p_0(x) = 1$, $x \in \mathbb{R}$.

As usual, for a compact set $K \subset \mathbb{R}$ we denote by $C(K)$ the Banach space of all real-valued continuous functions $f : K \to \mathbb{R}$ with the supremum norm $\|f\| = \sup_{x \in K} |f(x)|$.

2. Main result

We need the following result. Its proof is quite elementary and can be found in [3] Theorem 7.9 or in [2] Lemma 8.14. Note that its converse is a consequence of the uniform boundedness principle.

Lemma 2.1. *Let X be a normed space and Y be a Banach space and let M be a subset of X whose linear span is dense in X. Suppose for a sequence $(A_n)_{n \in \mathbb{N}} \subset B(X, Y)$ the following hold:*

1. *The sequence $(\|A_n\|)_{n \in \mathbb{N}}$ is bounded.*
2. *For each $x \in M$, $\lim_n A_n x$ exists.*

Then $(A_n x)_{n \in \mathbb{N}}$ converges for each $x \in X$ and the map $Ax := \lim_n A_n x$ belongs to $B(X, Y)$.

Corollary 2.2. *Let $K \subset \mathbb{R}$ be a compact set and let I be an interval of length one. For a sequence of continuous functions $\varphi_n : I^n \to K$, let $A_n : C(K) \to \mathbb{R}$ be the sequence of maps defined by*

$$A_n f = \int \cdots \int_{I^n} f(\varphi_n(x_1, \dots, x_n)) dx_1 \dots dx_n, \quad f \in C(K).$$

Suppose there is a constant $c \in \mathbb{R}$ such that $\lim_n A_n p_k = p_k(c)$, for every $k = 0, 1, 2, \dots$. Then the map $Af := \lim_n A_n f$ is well defined, $A \in B(C(K), \mathbb{R})$ and $Af = f(c)$ for every $f \in C(K)$.

Proof. By Weierstrass approximation theorem we know that the set of polynomials $\mathrm{span}\{p_k : k \in \mathbb{N} \cup \{0\}\}$ is dense in $C(K)$. Since the length of interval I is equal to one we have

$$\begin{aligned} \|A_n f\| = |A_n f| &\le \int \cdots \int_{I^n} |f(\varphi_n(x_1, \dots x_n))| dx_1 \dots dx_n \\ &\le \int \cdots \int_{I^n} \sup_{x \in K} |f(x)| dx_1 \dots dx_n = \|f\|, \end{aligned}$$

so $\|A_n\| \le 1$. Also, $\mathbb{R}$, as a vector space over $\mathbb{R}$, is a Banach space. We can now apply Lemma 2.1 to conclude that the functional $Af = \lim_n A_n f$ is well defined and $A \in B(C(K), \mathbb{R})$. Since A_n is linear and $\lim_n A_n p_k = p_k(c)$, we have $Aq = \lim_n A_n q = q(c)$ for every polynomial q. For arbitrary $f \in C(K)$ there exists a sequence of polynomials $(f_n)_n$ such that $f_n \to f$ in the supremum norm. By the continuity of A we obtain

$$Af = A(\lim_n f_n) = \lim_n Af_n = \lim_n f_n(c) = f(c).$$

□

Theorem 2.3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function and let $M_{n,p}(x_1,\dots,x_n)$, $-\infty \le p \le \infty$ be the generalized mean defined by (1). Then*

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n} f\left(M_{n,p}(x_1,\dots,x_n)\right)dx_1\dots dx_n$$
$$= \begin{cases} f(1), & p=\infty \\ f\left(\frac{1}{(p+1)^{1/p}}\right), & -1<p<\infty,\ p\neq 0 \\ f\left(\frac{1}{e}\right), & p=0 \\ f(0), & -\infty\le p\le -1 \end{cases}. \tag{2}$$

Proof. Depending on the value of the parameter p, we will divide the proof in five cases. In all cases we will first prove the adequate formula for the power function $p_k : x \to x^k$ and then we will apply Corollary 2.2.

$p = 0$: This case is quite simple. By Fubini's theorem, we have

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n}\left(\sqrt[n]{x_1\cdots x_n}\right)^k dx_1\dots dx_n$$
$$= \lim_{n\to\infty}\int\cdots\int_{[0,1]^n} x_1^{k/n}\cdots x_n^{k/n}dx_1\dots dx_n = \lim_{n\to\infty}\left(\int_0^1 x^{k/n}dx\right)^n$$
$$= \lim_{n\to\infty}\left(\frac{1}{k/n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left((1+k/n)^{n/k}\right)^k} = \frac{1}{e^k}.$$

Let $K = I = [0,1]$, $\varphi_n(x_1,\dots,x_n) = \sqrt[n]{x_1\cdots x_n} \in [0,1]$, $x_i \in [0,1]$, $c = 1/e$ and

$$A_n f := \int\cdots\int_{[0,1]^n} f\left(\sqrt[n]{x_1\cdots x_n}\right)dx_1\dots dx_n, \quad f \in C(K).$$

Then $\lim_{n\to\infty} A_n p^k = p_k(1/e)$, $\forall k \in \mathbb{N}\cup\{0\}$. The assumptions of Corollary 2.2 are satisfied, so it follows that for every $f \in C[0,1]$, $\lim_n A_n f = f\left(\frac{1}{e}\right)$.

$p = \infty$: Note that

$$[0,1]^n = \bigcup_{\sigma\in S_n}\left\{(x_1,\dots,x_n)\in[0,1]^n : 0\le x_{\sigma_1}\le x_{\sigma_2}\le\cdots\le x_{\sigma_n}\le 1\right\}, \tag{3}$$

where S_n is the set of all permutations of $\{1,2,,\dots,n\}$. Because the (n-dimensional Lebesgue) measure of the intersection of any two sets on the right hand side of (3) is equal to zero, we have

$$I_{n,k} := \int\cdots\int_{[0,1]^n}\left(\max\{x_1,\dots,x_n\}\right)^k dx_1\dots dx_n$$
$$= n!\int\cdots\int_{0\le x_1\le\cdots\le x_n\le 1} x_n^k dx_1\dots dx_n$$
$$= n!\int_0^1 x_n^k dx_n\int_0^{x_n}dx_{n-1}\int_0^{x_{n-1}}dx_{n-2}\cdots\int_0^{x_2}dx_1.$$

By induction on $n \ge 2$ it is easy to show that

$$\int_0^{x_n}dx_{n-1}\int_0^{x_{n-1}}dx_{n-2}\cdots\int_0^{x_2}dx_1 = \frac{x_n^{n-1}}{(n-1)!},$$

so we obtain

$$I_{n,k} = n!\int_0^1 x_n^k\frac{x_n^{n-1}}{(n-1)!}dx_n = \frac{n}{n+k}.$$

Thus,

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n}(\max\{x_1,\dots,x_n\})^k\,dx_1\dots dx_n=1=1^k.$$

In the same way as in the previous case, let $K=I=[0,1]$, $\varphi_n(x_1,\dots,x_n)=\max\{x_1,\dots,x_n\}\in[0,1]$, $x_i\in[0,1]$ and $c=1$. The assumptions of Corollary 2.2 are satisfied, so for every $f\in C[0,1]$,

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n}f(\max\{x_1,\dots,x_n\})\,dx_1\dots dx_n=f(1).$$

$p=-\infty$: As in the previous case, we have

$$\begin{aligned}
&\int\cdots\int_{[0,1]^n}(\min\{x_1,\dots,x_n\})^k\,dx_1\dots dx_n\\
&=n!\int\cdots\int_{0\le x_1\le\cdots\le x_n\le 1}x_1^k dx_1\dots dx_n\\
&=n!\int_0^1 dx_n\int_0^{x_n}dx_{n-1}\cdots\int_0^{x_3}dx_2\int_0^{x_2}x_1^k dx_1\\
&=n!\int_0^1\frac{k!}{(n+k-1)!}x_n^{k+n-1}dx_n=\frac{n!k!}{(n+k)!}.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n}(\min\{x_1,\dots,x_n\})^k\,dx_1\dots dx_n=0=0^k.$$

From the same reasons as in the case $p=\infty$, we conclude that

$$\lim_{n\to\infty}\int\cdots\int_{[0,1]^n}f(\min\{x_1,\dots,x_n\})\,dx_1\dots dx_n=f(0).$$

$0<p<\infty$: Let

$$I_p(n,k)=\int\cdots\int_{[0,1]^n}\left(\frac{x_1^p+\cdots+x_n^p}{n}\right)^k dx_1\dots dx_n,\quad k\in\mathbb{N}\cup\{0\}.$$

We want to derive a recurrent relation for $(I_p(n,k))_k$ but in such a way that after passage to the limit as

$n \to \infty$ we obtain a useable relation. By the symmetry and the Fubini's theorem, we have

$$\begin{aligned}
I_p(n,k) &:= \int\cdots\int_{[0,1]^n} \frac{x_1^p + \cdots + x_n^p}{n} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{k-1} dx_1 \ldots dx_n \\
&= \frac{1}{n}\sum_{i=1}^{n} \int\cdots\int_{[0,1]^n} x_i^p \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{k-1} dx_1 \ldots dx_n \\
&= \int\cdots\int_{[0,1]^n} x_n^p \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{k-1} dx_1 \ldots dx_n \\
&= \int\cdots\int_{[0,1]^n} \frac{x_n^p}{n^{k-1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \left(x_1^p + \cdots + x_{n-1}^p\right)^i (x_n^p)^{k-1-i}\, dx_1 \ldots dx_n \\
&= \sum_{i=0}^{k-1} \frac{(n-1)^i}{n^{k-1}} \binom{k-1}{i} \int\cdots\int_{[0,1]^{n-1}} \left(\frac{x_1^p + \cdots + x_{n-1}^p}{n-1}\right)^i dx_1 \ldots dx_{n-1} \cdot \int_0^1 x_n^{p(k-i)} dx_n \\
&= \sum_{i=0}^{k-1} \frac{(n-1)^i}{n^{k-1}} \binom{k-1}{i} \frac{1}{p(k-i)+1} I_p(n-1,i) \\
&= \left(\frac{n-1}{n}\right)^{k-1} \frac{1}{p+1} I_p(n-1,k-1) + \sum_{i=0}^{k-2} \frac{(n-1)^i}{n^{k-1}} \binom{k-1}{i} \frac{1}{p(k-i)+1} I_p(n-1,i)
\end{aligned}$$

where in the last row we extracted the last term from the sum. Since $0 \le x_i \le 1$ we have $0 \le ((x_1^p + \cdots + x_n^p)/n)^k \le 1$, so $0 \le I_p(n,k) \le 1$, for every $n \in \mathbb{N}$ and every $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\lim_{n\to\infty} \sum_{i=0}^{k-2} \frac{(n-1)^i}{n^{k-1}} \binom{k-1}{i} \frac{1}{p(k-i)+1} I_p(n-1,i) = 0$$

as $\lim_n (n-1)^i/n^{k-1} = 0$, for $i \in \{0, 1, \ldots k-2\}$. Note that $\lim_n ((n-1)/n)^{k-1} = 1$ and $I_p(n,0) = 1$, for every $n \in \mathbb{N}$, thus, $\lim_n I_p(n,0) = 1$. It is now easy to show by induction on $k \ge 0$ that the sequence $(I_p(n,k))_n$ is convergent for every k and

$$\lim_{n\to\infty} I_p(n,k) = \frac{1}{(p+1)^k}, \quad k \in \mathbb{N} \cup \{0\}.$$

Let $K = I = [0,1]$, $\varphi_n(x_1, \ldots, x_n) = (x_1^p + \cdots + x_n^p)/n \in [0,1]$, $x_i \in [0,1]$ and $c = 1/(p+1)$. By Corollary 2.2, it follows that for every $g \in C[0,1]$

$$\lim_{n\to\infty} \int\cdots\int_{[0,1]^n} g\left(\frac{x_1^p + \cdots + x_n^p}{n}\right) dx_1 \ldots dx_n = g\left(\frac{1}{p+1}\right).$$

For arbitrary $f \in C[0,1]$, let $g : x \to f(x^{1/p})$, $x \in [0,1]$. We obtain the desired formula (2) for $0 < p < \infty$.

$-\infty < p < 0$: The proof of the previous case is not applicable here because $\int_0^1 x^{p(k-i)}\, dx = \infty$ when $p(k-i) + 1 \le 0$. Let

$$I_p(n,k,\varepsilon) = \int\cdots\int_{[\varepsilon,1+\varepsilon]^n} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^k dx_1 \ldots dx_n,$$

where $\varepsilon > 0$ is an arbitrary positive real and $k \in \mathbb{N} \cup \{0\}$. In the same way as in the case $p > 0$ we obtain

$$I_p(n,k,\varepsilon) = \left(\frac{n-1}{n}\right)^{k-1} \int_{\varepsilon}^{1+\varepsilon} x^p\,dx \cdot I_p(n-1,k-1,\varepsilon) + \sum_{i=0}^{k-2} \frac{(n-1)^i}{n^{k-1}} \binom{k-1}{i} \int_{\varepsilon}^{1+\varepsilon} x^{p(k-i)}\,dx \cdot I_p(n-1,i,\varepsilon).$$

Since $\varepsilon \le x_i \le 1+\varepsilon$ and $p < 0$ we have $(1+\varepsilon)^{pk} \le \left((x_1^p + \cdots x_n^p)/n\right)^k \le \varepsilon^{pk}$, so $(1+\varepsilon)^{pk} \le I_p(n,k,\varepsilon) \le \varepsilon^{pk}$, for every $p < 0$, every $k \in \mathbb{N} \cup \{0\}$ and every $n \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} \sum_{i=0}^{k-2} \frac{(n-1)^i}{n^{k-1}} \binom{k-1}{i} \int_{\varepsilon}^{1+\varepsilon} x^{p(k-i)}\,dx \cdot I_p(n-1,i,\varepsilon) = 0.$$

Note that $I_p(n,0,\varepsilon) = 1$, $\forall n \in \mathbb{N}$, so $\lim_n I_p(n,0,\varepsilon) = 1$. We can now show by induction on $k \ge 0$ that the sequence $(I_p(n,k,\varepsilon))_n$ converges and that

$$\lim_{n\to\infty} I_p(n,k,\varepsilon) = (a_p(\varepsilon))^k,$$

where

$$a_p(\varepsilon) = \int_{\varepsilon}^{1+\varepsilon} x^p\,dx = \begin{cases} \frac{1}{p+1}\left((1+\varepsilon)^{p+1} - \varepsilon^{p+1}\right), & p < 0, p \ne -1 \\ \ln \frac{1+\varepsilon}{\varepsilon}, & p = -1 \end{cases}.$$

Let $I = [\varepsilon, 1+\varepsilon]$, $K = [(1+\varepsilon)^p, \varepsilon^p]$ and $c = a_p(\varepsilon)$. Then $\varphi_n(x_1,\ldots,x_n) := (x_1^p + \cdots + x_n^p)/n \in K$ when $x_i \in I$. Let

$$A_n f = \int \cdots \int_{I^n} f\left(\frac{x_1^p + \cdots + x_n^p}{n}\right) dx_1 \ldots dx_n, \quad f \in C(K).$$

We have proved that $\lim_n A_n p_k = c^k$ for every $k \in \mathbb{N} \cup \{0\}$. From Corollary 2.2, we conclude that $\lim_n A_n f = f(c)$, for every $f \in C(K)$. In the special case when $f(x) = x^{k/p}$ we obtain

$$\lim_{n\to\infty} J_p(n,k,\varepsilon) = (a_p(\varepsilon))^{\frac{k}{p}}, \tag{4}$$

where

$$J_p(n,k,\varepsilon) = \int \cdots \int_{[\varepsilon,1+\varepsilon]^n} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{k}{p}} dx_1 \ldots dx_n.$$

Let us prove that the function $J_p(n,k,\varepsilon)$ is increasing with respect to the argument $\varepsilon > 0$. Let $0 < \varepsilon_1 \le \varepsilon_2$. After the change of variables $x_i = t_i + \varepsilon_2 - \varepsilon_1$, $i = \overline{1,n}$, whose Jacobian is equal to one, we obtain

$$\begin{aligned} J_p(n,k,\varepsilon_2) &= \int \cdots \int_{[\varepsilon_1,1+\varepsilon_1]^n} \left(\frac{(t_1 + (\varepsilon_2-\varepsilon_1))^p + \cdots + (t_n + (\varepsilon_2-\varepsilon_1))^p}{n}\right)^{\frac{k}{p}} dt_1 \ldots dt_n \\ &\ge \int \cdots \int_{[\varepsilon_1,1+\varepsilon_1]^n} \left(\frac{t_1^p + \cdots + t_n^p}{n}\right)^{\frac{k}{p}} dt_1 \ldots dt_n = J_p(n,k,\varepsilon_1) \end{aligned}$$

since the functions $x \to x^p$ and $x \to x^{k/p}$ are both decreasing when $p < 0$. For $\varepsilon \le x_i \le 1+\varepsilon$ it is easy to see that $\varepsilon^k \le (M_{n,p}(x_1,\ldots,x_n))^k \le (1+\varepsilon)^k$, so

$$\varepsilon^k \le J_p(n,k,\varepsilon) \le (1+\varepsilon)^k. \tag{5}$$

Also $J_p(n,k,\varepsilon)$ is increasing in ε, so the limit

$$J_p(n,k,0) := \lim_{\varepsilon\to 0+} J_p(n,k,\varepsilon) = \int\cdots\int_{[0,1]^n} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{k}{p}} dx_1 \ldots dx_n$$

exists,

$$0 \le J_p(n,k,0) \le 1, \tag{6}$$

by (5) and

$$J_p(n,k,0) \le J_p(n,k,\varepsilon), \ \forall n \in \mathbb{N}. \tag{7}$$

From (6) it follows that the sequence $(J_p(n,k,0))_n$ has an accumulation point. Let a be an arbitrary one. From (7) and (4) we conclude that

$$a \le \lim_{n\to\infty} J_p(n,k,\varepsilon) = (a_p(\varepsilon))^{k/p}.$$

This is valid for every $\varepsilon > 0$ so

$$0 \le a \le \lim_{\varepsilon\to 0+} (a_p(\varepsilon))^{k/p}. \tag{8}$$

It is easy to show that

$$\lim_{\varepsilon\to 0+} (a_p(\varepsilon))^{k/p} = \begin{cases} \frac{1}{(p+1)^{k/p}}, & -1 < p < 0 \\ 0, & -\infty < p \le -1 \end{cases}.$$

Hence, $a = 0$ in the case $-\infty < p \le -1$. Therefore, for $-\infty < p \le -1$, the sequence $(J_p(n,k,0))_n$ is convergent and

$$\lim_n \int\cdots\int_{[0,1]^n} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{k}{p}} dx_1 \ldots dx_n = 0. \tag{9}$$

Suppose now that $-1 < p < 0$. Form (8) we have

$$a \le \frac{1}{(p+1)^{k/p}}.$$

On the other hand, the function $x \to x^{k/p}$, $(1+\varepsilon)^p \le x \le \varepsilon^p$ is convex, the set $[\varepsilon, 1+\varepsilon]^n$ is of the measure equal to one and the function $(x_1, \ldots, x_n) \to (x_1^p + \cdots + x_n^p)/n$, $\varepsilon \le x_i \le 1+\varepsilon$ is continuous, so by Jensen's inequality we obtain

$$\begin{aligned} J_p(n,k,\varepsilon) &\ge \left(\int\cdots\int_{[\varepsilon,1+\varepsilon]^n} \frac{x_1^p + \cdots + x_n^p}{n} dx_1 \ldots dx_n\right)^{\frac{k}{p}} \\ &= \left(\int\cdots\int_{[\varepsilon,1+\varepsilon]^n} x_1^p \, dx_1 \ldots dx_n\right)^{\frac{k}{p}} = \left(a_p(\varepsilon)\right)^{k/p}. \end{aligned}$$

This is valid for every $\varepsilon > 0$, so

$$J_p(n,k,0) \ge \lim_{\varepsilon\to 0+} \left(a_p(\varepsilon)\right)^{k/p} = \frac{1}{(p+1)^{k/p}},$$

and hence $a \geq 1/(p+1)^{k/p}$. It follows that $a = 1/(p+1)^{k/p}$. Therefore, for $-1 < p < 0$, $(J_p(n,k,0))_n$ is convergent and

$$\lim_n \int \cdots \int_{[0,1]^n} \left(\frac{x_1^p + \cdots + x_n^p}{n} \right)^{\frac{k}{p}} dx_1 \ldots dx_n = \frac{1}{(p+1)^{k/p}}. \tag{10}$$

In the same way as before, let $K = I = [0,1]$, $\varphi_n(x_1, \ldots, x_n) = M_{n,p}(x_1, \ldots, x_n) \in [0,1]$, $x_i \in [0,1]$ and

$$c = \begin{cases} 0, & -\infty < p \leq -1 \\ 1/(p+1)^{1/p}, & -1 < p < 0 \end{cases}.$$

The formula (2) follows from (9), (10) and Corollary 2.2. □

We conclude this note with some remarks. We have proved the nontrivial extension of formula (2) from the cases $p = 1$ and $p = 0$ to the general case $p \in [-\infty, \infty]$. We gave a proof based on elementary facts. Theorem 2.3 also holds in the case when f is a complex-valued continuous function on $[0,1]$. Indeed, suppose that $f(x) = u(x) + iv(x)$. Then we can apply Theorem 2.3 on functions $u, v \in C([0,1])$. The claim now follows from the linearity of the map

$$f \to \lim_{n\to\infty} \int \cdots \int_{[0,1]^n} f\left(M_{n,p}(x_1, \ldots, x_n)\right) dx_1 \ldots dx_n, \quad f \in C([0,1]).$$

Note that

$$\lim_{p\to -1+} \frac{1}{(p+1)^{1/p}} = 0 \text{ and } \lim_{p\to 0} \frac{1}{(p+1)^{1/p}} = \frac{1}{e}.$$

Let $h(p) = (1+p)^{1/p}$. Then

$$\lim_{p\to\infty} \ln h(p) = \lim_{p\to\infty} \frac{\ln(1+p)}{p} = \lim_{p\to\infty} \frac{1/(1+p)}{1} = 0,$$

so

$$\lim_{p\to\infty} \frac{1}{(p+1)^{1/p}} = 1.$$

It follows that for $f \in C([0,1])$, the function

$$p \to \lim_{n\to\infty} \int \cdots \int_{[0,1]^n} f\left(M_{n,p}(x_1, \ldots, x_n)\right) dx_1 \ldots dx_n, \quad -\infty \leq p \leq \infty$$

is continuous.

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References

[1] G. J. Székely, *Contests in higher mathematics*, Springer, New York, 1996.
[2] R. Meise, D. Vogt, *Introduction to functional analysis*, Clarendon Press, 1997.
[3] M. Arsenović, M. Dostanić, D. Jocić, *Teorija mere, funkcionalna analiza, teorija operatora*, Zavod za udžbenike, 2012.