



Geodesic mapping invariants: from Weyl's to current techniques

Milan Lj. Zlatanović^a, Nenad Vesić^b

^aUniversity of Niš, Faculty of Sciences and Mathematics, Niš, Serbia

^bMathematical institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia

Abstract. In this paper, we study invariants of geometric mappings in affine connection spaces, with a particular focus on geodesic mappings as a special case. We revisit the classical methodology for finding invariants and extend it to generalized affine connection spaces (with torsion). By analyzing the difference between affine connection coefficients of two spaces, we explicitly construct the Thomas projective parameter and the Weyl projective tensor, which provide fundamental invariants under geodesic mappings. We also investigate the role of torsion in these constructions.

1. Introduction

Newtonian mechanics [3] describes the motion of bodies via Newton's laws, while Einstein's theory of relativity [2] generalizes these principles. Special Relativity uses four coordinates (time is included) and describes force-free motion along straight lines in flat space-time. In General Relativity, the presence of mass and energy curves spacetime, and free motion follows geodesics. The geodesic equations involve Christoffel symbols and quadratic terms in the velocities, and due to the pseudo-Riemannian nature of the metric, the extremized interval is not necessarily minimal but corresponds to the physically natural path between two points.

In this research, we will obtain different results about transformations of manifolds in which all geodesic lines from initial manifolds transform to geodesic lines of deformed manifolds. These invariants will be obtained by commonly accepted methodology, and its recent preferment.

1.1. Geometrical Model

A vector function $M_N : \mathcal{D}_N \rightarrow \mathbb{R}^{N+k}$, $k \in \mathbb{N}$ where \mathcal{D} is an N -dimensional manifold. If $\varphi : \mathcal{D} \rightarrow \mathcal{D}'$, $\mathcal{D}' \subset \mathbb{R}^N$, is a differentiable function, the composition $M_N \circ \varphi$ is the reparametrization of manifold [4, 6].

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Email addresses: zlatmilan@yahoo.com (Milan Lj. Zlatanović), vesic.specijalac@gmail.com (Nenad Vesić)

ORCID iDs: 0000-0002-0318-1092 (Milan Lj. Zlatanović), 0000-0002-7598-9058 (Nenad Vesić)

The geometrical object $\tau_{j_1 \dots j_q}^{i_1 \dots i_p}$, expressed in the orthonormal coordinate system (O, x^1, \dots, x^N) , is a parameter of the type (p, q) if, under a reparametrization, $(O, x^1, \dots, x^N) \rightarrow (O', x'^1, \dots, x'^N)$, its values $\tau_{j'_1 \dots j'_q}^{i'_1 \dots i'_p}$ satisfy the relation

$$\tau_{j'_1 \dots j'_q}^{i'_1 \dots i'_p} = x_{i'_1}^{i_1} \dots x_{i'_p}^{i_p} x_{j'_1}^{j_1} \dots x_{j'_q}^{j_q} \tau_{j_1 \dots j_q}^{i_1 \dots i_p} + X_{j'_1 \dots j'_q}^{i'_1 \dots i'_p}, \quad (1)$$

are basics of classical physics.

A. Einstein generalized Newtonian theory of gravity by involving Special and General Relativity theories. where $x_{i'}^{i'} = \frac{\partial x'^i}{\partial x^i}$, $x_{j'}^j = \frac{\partial x^j}{\partial x'^j}$, and $X_{j'_1 \dots j'_q}^{i'_1 \dots i'_p} \neq 0$. If $X_{j'_1 \dots j'_q}^{i'_1 \dots i'_p} \equiv 0$, the geometrical object $\tau_{j_1 \dots j_q}^{i_1 \dots i_p}$ is tensor [4, 6].

Geometrical objects L_{jk}^i , which under a reparametrization satisfy the equality

$$L_{j'k'}^{i'} = x_{i'}^{i'} x_{j'}^{j'} x_{k'}^{k'} L_{jk}^i + x_{i'}^{i'} x_{j'k'}^i, \quad (2)$$

are the affine connection coefficients [4, 6].

A geodesic curve of space \mathbb{A}_N with affine connection ∇_0 is an N -dimensional vector function of parameter t whose components $x^i = x^i(t)$ satisfy the next system of differential equations

$$\frac{d^2 x^i}{dt^2} + L_{ab}^i \frac{dx^a}{dt} \frac{dx^b}{dt} = \rho \frac{dx^i}{dt}, \quad (3)$$

where ρ is a scalar function.

Geodesic lines play a fundamental role in physics. Light follows geodesic paths, and the presence of massive objects bends these paths, giving rise to gravitational lensing [1].

In this paper, our purpose is to review the invariants of transformations of affine connections [4, 6]. An N -dimensional manifold $\mathcal{M}_N = \mathcal{M}_N(x^1, \dots, x^N)$ equipped with a torsion-free affine connection ∇_0 is a symmetric affine connection space \mathbb{A}_N . The affine connection coefficients of ∇_0 are $L_{jk}^i = L_{kj}^i$. The latin indices i, j, k, \dots take values of $1, \dots, N$.

The covariant derivative of a tensor a_j^i with respect to the affine connection ∇_0 is defined as

$$a_{j|k}^i = a_{j,k}^i + L_{pk}^i a_j^p - L_{jk}^p a_p^i, \quad (4)$$

where comma denotes partial derivation, $a_{j|k}^i = \partial a_j^i / \partial x^k$.

After considering the Ricci type identity, the difference $a_{j|m|n}^i - a_{j|n|m}^i$, one concludes that the space \mathbb{A}_N has one curvature tensor [4, 6], whose components are

$$R_{jmn}^i = L_{jm,n}^i - L_{jn,m}^i + L_{jm}^p L_{pn}^i - L_{jn}^p L_{pm}^i. \quad (5)$$

The trace $R_{jm} = R_{j m}^p$ is the Ricci-tensor.

The manifold \mathcal{M}_N equipped with a non-symmetric affine connection ∇ is a generalized affine connection space \mathbb{GA}_N . The affine connection coefficients of this space are L_{jk}^i such that $L_{jk}^i \neq L_{kj}^i$ for at least one pair (j, k) of indices.

The symmetric and anti-symmetric parts of L_{jk}^i are $L_{jk}^i = \frac{1}{2}(L_{jk}^i + L_{kj}^i)$ and $L_{jk}^i = \frac{1}{2}(L_{jk}^i - L_{kj}^i) = \frac{1}{2}T_{jk}^i$, where T_{jk}^i is the torsion tensor.

Remark 1.1. For two vector fields a and b at a point P , infinitesimal parallel transport of u along v (resulting point is Q and parallel transport of v along u (resulting point is R), coincide if torsion vanishes, but torsion tensor measures its discrepancy.

The affine connection coefficients L_{jk}^i and \bar{L}_{jk}^i are not tensors, but the anti-symmetric parts L_{jk}^i are components of torsion tensor. If L_{jk}^i and \bar{L}_{jk}^i are affine connection coefficients of spaces \mathbb{A}_N and $\bar{\mathbb{A}}_N$, their difference $P_{jk}^i = \bar{L}_{jk}^i - L_{jk}^i$ is the symmetric deformation tensor. The difference $P_{jk}^i = \bar{L}_{jk}^i - L_{jk}^i$ is the deformation tensor of non-symmetric affine connection L_{jk}^i .

Four kinds of covariant derivatives with respect to non-symmetric affine connection are defined. With respect to them, a family of curvature tensors of space \mathbb{GA}_N was obtained [5, 9]

$$K_{jmn}^i = R_{jmn}^i + uL_{jmn}^i + u'L_{jmn}^i + vL_{jmn}^p L_{pn}^i + v'L_{jmn}^p L_{pn}^i + wL_{mnp}^p L_{pj}^i, \quad (6)$$

for the corresponding real constants u, u', v, v', w . Six of them are linearly independent.

Special transformations of affine connections in both symmetric and non-symmetric spaces are the geodesic mappings. N. S. Sinyukov [6], and J. Mikeš at all. [4] studied these transformations and their invariants. In [11], the concept of obtaining invariants for geodesic mappings was developed to geodesic mappings of a non-symmetric affine connection space which preserve the torsion tensor (conformal mappings). This methodology was improved in [8]. In this research, we will present invariants for geodesic mappings obtained by these two methods.

Geodesic lines are curves in symmetric affine connection spaces. In [11, 12], the interaction of the torsion tensor with the transformation of geodesic lines was analyzed. Invariants for geodesic mappings which preserve the torsion tensor (equitortion geodesic mappings) were obtained in this and many subsequent studies.

2. Invariants for geodesic mappings

Let $f : \bar{\mathbb{A}}_N \rightarrow \mathbb{A}_N$ be an equitortion mapping of symmetric affine connection space. Its basic equation is [4, 6]

$$L_{jk}^i - \bar{L}_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i, \quad (7)$$

for a 1-form ψ_j .

After contracting the last equation by i and k , one gets

$$\psi_j = \frac{1}{N+1} (L_{jp}^p - \bar{L}_{jp}^p). \quad (8)$$

After substituting the expression (8) into the basic equation (7), one gets

$$L_{jk}^i - \bar{L}_{jk}^i = \frac{1}{N+1} (\delta_j^i L_{kp}^p + \delta_k^i L_{jp}^p) - \frac{1}{N+1} (\delta_j^i \bar{L}_{kp}^p + \delta_k^i \bar{L}_{jp}^p). \quad (9)$$

The relation (9) is equivalent to the equality $T_{jk}^i = \bar{T}_{jk}^i$, for [4, 6]

$$T_{jk}^i = L_{jk}^i - \frac{1}{N+1} (\delta_j^i L_{kp}^p + \delta_k^i L_{jp}^p), \quad (10)$$

and the corresponding \bar{T}_{jk}^i . The geometrical object T_{jk}^i given by (10) is the Thomas projective parameter [4, 6, 7].

If $P_{jk}^i = L_{jk}^i - \bar{L}_{jk}^i$, the transformation of curvature tensor \bar{R}_{jmn}^i to R_{jmn}^i is

$$R_{jmn}^i = \bar{R}_{jmn}^i + P_{j[m|n}^i - P_{j]n|m}^i + P_{jm}^p P_{pn}^i - P_{jn}^p P_{pm}^i. \quad (11)$$

Because $P_{jk}^i = \delta_j^i \psi_k + \delta_k^i \psi_j$, the relation (11) transforms to

$$R_{jmn}^i = \bar{R}_{jmn}^i + \delta_j^i (\psi_{m|n} - \psi_{n|m}) + \delta_m^i (\psi_{j|n} - \psi_{j\psi_n}) - \delta_n^i (\psi_{j|m} - \psi_{j\psi_m}). \quad (12)$$

After involving the exchange $\psi_{ij} = \psi_{i||j} - \psi_i \psi_j$ in relation (12), we transform it into

$$R_{jmn}^i = \bar{R}_{jmn}^i + \delta_j^i \psi_{[mn]} + \delta_m^i \psi_{jn} - \delta_n^i \psi_{jm}. \quad (13)$$

After contracting the relation (13) by i and j , such as by i and n , one obtains $\psi_{[mn]} = \frac{1}{N+1} (R_{[mn]} - \bar{R}_{[mn]})$ such as $R_{jm} - \bar{R}_{jm} = -\psi_{[jm]} - (N-1)\psi_{jm}$. With respect to these equalities, it was obtained

$$\begin{aligned} R_{jmn}^i &= \bar{R}_{jmn}^i - \frac{1}{N+1} \delta_j^i (R_{[mn]} - \bar{R}_{[mn]}) - \frac{N}{N^2-1} \delta_{[m}^i R_{j]n} - \frac{1}{N^2-1} \delta_{[m}^i R_{n]j} \\ &+ \frac{N}{N^2-1} \delta_{[m}^i \bar{R}_{j]n} + \frac{1}{N^2-1} \delta_{[m}^i \bar{R}_{n]j}. \end{aligned} \quad (14)$$

This equality is equivalent to $W_{jmn}^i = \bar{W}_{jmn}^i$, for

$$W_{jmn}^i = R_{jmn}^i + \frac{1}{N+1} \delta_j^i R_{[mn]} + \frac{N}{N^2-1} \delta_{[m}^i R_{j]n} + \frac{1}{N^2-1} \delta_{[m}^i R_{n]j}, \quad (15)$$

and the corresponding \bar{W}_{jmn}^i . The geometrical object W_{jmn}^i is the *Weyl projective tensor* [4, 6, 10].

Let $f : \mathbb{G}\bar{\mathbb{A}}_N \rightarrow \mathbb{G}\mathbb{A}_N$ be a geodesic mapping of a non-symmetric affine connection space $\mathbb{G}\bar{\mathbb{A}}_N$. Its basic equation is [11]

$$L_{jk}^i = \bar{L}_{jk}^i + \psi_j \delta_k^i + \psi_k \delta_j^i + \xi_{jk}^i, \quad (16)$$

where ξ_{jk}^i is a tensor of the type (1,2) anti-symmetric by j and k .

By applying the same methodology used in [4, 6], adapted to the case of non-symmetric affine connections, M. Zlatanović obtained the family of invariants for the equitorsion geodesic mapping $f : \mathbb{G}\bar{\mathbb{A}}_N \rightarrow \mathbb{G}\mathbb{A}_N$ (see [11],[12])

$$\begin{aligned} \mathcal{G}W_{jmn}^i &= K_{jmn}^i + \frac{1}{N+1} \delta_j^i K_{[mn]} + \frac{N}{N^2-1} \delta_{[m}^i K_{j]n} + \frac{1}{N^2-1} \delta_{[m}^i K_{n]j} \\ &+ \frac{1}{(N+1)^2(N-1)} L_{mp}^p \left(2(N-1)u\delta_j^i L_{qn}^q + (u - Nu' - Nu - u')\delta_n^i L_{jq}^q + (N^2-1)(u+u')L_{jn}^i \right) \\ &- \frac{1}{(N+1)^2(N-1)} L_{mp}^p \left(2(N-1)u\delta_j^i L_{qn}^q + (u - Nu' - Nu - u')\delta_n^i L_{jq}^q - (N^2-1)(u+u')L_{jn}^i \right) \\ &- \frac{1}{(N+1)^2(N-1)} L_{jp}^p \left((Nu' - Nu + u' + u)\delta_{[m}^i L_{qn]}^q + (N^2-1)(u-u')L_{mn}^i \right) \\ &- \frac{1}{(N+1)^2} L_{pq}^q \left(2(N-1)u\delta_j^i L_{mn}^p + (Nu - u + Nu' + u')\delta_m^i L_{jn}^p + 2u\delta_n^i L_{jm}^p \right). \end{aligned} \quad (17)$$

where u, u', v, v', w are real constants.

3. Novel approach for obtaining invariants

The study of invariants for geometric mappings, invariants for geodesic mappings as a special case, was initially developed in [8]. The difference in affine connection coefficients $L_{jk}^i - \bar{L}_{jk}^i$ of affine connection coefficients was expressed as $L_{jk}^i - \bar{L}_{jk}^i = \omega_{jk}^i - \bar{\omega}_{jk}^i$, for geometrical objects ω_{jk}^i and $\bar{\omega}_{jk}^i$ of the type (1, 2). The basic invariants of Thomas and Weyl type for the such defined mapping are

$$\mathcal{T}_{jk}^i = \Gamma_{jk}^i - \omega_{jk}^i, \quad (18)$$

$$\mathcal{W}_{jmn}^i = R_{jmn}^i - \omega_{j|n}^i + \omega_{j|n}^i + \omega_{jm}^p \omega_{pn}^i - \omega_{jn}^p \omega_{pm}^i, \quad (19)$$

and the corresponding $\bar{\mathcal{T}}_{jk}^i$ and $\bar{\mathcal{W}}_{jmn}^i$.

In the case of geodesic mappings, $L_{jk}^i - \bar{L}_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i$, it was obtained $\psi_j = \frac{1}{N+1}(L_{jp}^p - \bar{L}_{jp}^p)$, which gives

$$L_{jk}^i - \bar{L}_{jk}^i = \frac{1}{N+1}(\delta_j^i L_{kp}^p + \delta_k^i L_{jp}^p) - \frac{1}{N+1}(\delta_j^i \bar{L}_{kp}^p + \delta_k^i \bar{L}_{jp}^p). \quad (20)$$

From the last relation, we get $\omega_{jk}^i = \frac{1}{N+1}(\delta_j^i L_{kp}^p + \delta_k^i L_{jp}^p)$, and $\bar{\omega}_{jk}^i = \frac{1}{N+1}(\delta_j^i \bar{L}_{kp}^p + \delta_k^i \bar{L}_{jp}^p)$. After substituting this special ω_{jk}^i in equations (18, 19), we obtain

$$\mathcal{G}\mathcal{T}_{jk}^i = L_{jk}^i - \frac{1}{N+1}(\delta_j^i L_{kp}^p + \delta_k^i L_{jp}^p), \quad (21)$$

$$\begin{aligned} \mathcal{G}\mathcal{W}_{jmn}^i &= R_{jmn}^i + \frac{1}{N+1}\delta_j^i R_{[mn]} - \frac{1}{(N+1)^2}\delta_m^i((N+1)L_{jp|n}^p + L_{jp}^p L_{nq}^q) \\ &\quad + \frac{1}{(N+1)^2}\delta_n^i((N+1)L_{jp|m}^p + L_{jp}^p L_{mq}^q), \end{aligned} \quad (22)$$

and the corresponding $\mathcal{G}\bar{\mathcal{T}}_{jk}^i$ and $\mathcal{G}\bar{\mathcal{W}}_{jmn}^i$.

From the equalities $\mathcal{G}\mathcal{W}_{jmn}^i = R_{jmn}^i + \frac{1}{N+1}\delta_j^i R_{[mn]} - \delta_m^i X_{jn} + \delta_n^i X_{jm}$ and $\mathcal{G}\bar{\mathcal{W}}_{jmn}^i = \bar{R}_{jmn}^i + \frac{1}{N+1}\delta_j^i \bar{R}_{[mn]} - \delta_m^i X_{jn} + \delta_n^i X_{jm}$ and $\mathcal{G}\bar{\mathcal{W}}_{jmn}^i = \bar{R}_{jmn}^i + \frac{1}{N+1}\delta_j^i \bar{R}_{[mn]} - \delta_m^i \bar{X}_{jn} + \delta_n^i \bar{X}_{jm}$, for $X_{ij} = \frac{1}{N+1}L_{ip|j}^p + \frac{1}{(N+1)^2}L_{ip}^p L_{jq}^q$ and $\bar{X}_{ij} = \frac{1}{N+1}\bar{L}_{ip|j}^p + \frac{1}{(N+1)^2}\bar{L}_{ip}^p \bar{L}_{jq}^q$, and $0 = \mathcal{G}\mathcal{W}_{ipj}^p - \mathcal{G}\bar{\mathcal{W}}_{ipj}^p$, it was obtained

$$X_{ij} - \bar{X}_{ij} = -\frac{N}{N^2-1}(R_{ij} - \bar{R}_{ij}) - \frac{1}{N^2-1}(R_{ji} - \bar{R}_{ji}). \quad (23)$$

If substitutes the expression (23) into the equality $0 = \mathcal{G}\mathcal{W}_{jmn}^i - \mathcal{G}\bar{\mathcal{W}}_{jmn}^i = (R_{jmn}^i - \bar{R}_{jmn}^i + \frac{1}{N+1}\delta_j^i(R_{[mn]} - \bar{R}_{[mn]}) - \delta_m^i(X_{jn} - \bar{X}_{jn}) + \delta_n^i(X_{jm} - \bar{X}_{jm}))$, one will obtain the equality $W_{jmn}^i = \bar{W}_{jmn}^i$, for

$$GW_{jmn}^i = R_{jmn}^i + \frac{1}{N+1}\delta_j^i R_{[mn]} + \frac{N}{N^2-1}\delta_{[m}^i R_{jn]} + \frac{1}{N^2-1}\delta_{[m}^i R_{n]j}, \quad (24)$$

and the corresponding $G\bar{W}_{jmn}^i$.

Note that covariant derivatives $|$ and \parallel are defined with respect to the affine connection space L_N and \bar{L}_N , respectively.

Remark. The antisymmetrization is indicated with square brackets, e.g. $\delta_{[m}^i R_{jn]} = \frac{1}{2}(\delta_m^i R_{jn} - \delta_n^i R_{jm})$.

4. Conclusion

In Section 3, the previous methodology was extended. By applying this methodology, we obtained the Thomas projective parameter (21) and the Weyl projective tensor (24). Both the Thomas projective parameter and the Weyl projective tensor are torsion-free invariants. However, the basic invariant $\mathcal{G}W_{jmn}^i$ (22) in a generalized affine connection space with torsion generally contains contributions from the torsion tensor and is therefore not torsion-free.

For this reason, the invariant $\mathcal{G}W_{jmn}^i$ and its trace $\mathcal{G}W_{ij} = \mathcal{G}W_{ij}^p$ can be used in physical applications, such as in cosmology or in other theories where scalar curvatures of manifolds play a role.

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