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On linear preservers of submajorization on $\ell^p(I)^+$, where $p \in (1, \infty)$

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Abstract. We study the structure of linear operators that preserve submajorization on the positive cone $\ell^p(I)^+$, where $p \in (1, \infty)$ and I is an arbitrary nonempty set. Using a constructive approach, we show that the set of all linear preservers is norm-closed.

1. Introduction

The theory of majorization has proved to be a powerful framework for deriving and generalizing various classes of mathematical inequalities [1, 6, 12]. Owing to its versatility, it has found deep applications across numerous scientific disciplines [5], most notably in quantum mechanics [13, 14].

In the last decade, substantial progress has been achieved in extending classical majorization concepts to more general settings such as sequence spaces [8, 16] and discrete Lebesgue spaces [2, 3, 9, 11]. Parallel to these developments, the study of linear operators that preserve different types of majorization relations has become an active topic of research [2, 7, 10, 11].

In particular, the notion of submajorization on $\ell^p(I)$ and its linear preservers has been analyzed in [11]. This relation is characterized via increasable doubly substochastic operators, introduced and discussed in [4, 11]. The aim of this paper is to present a constructive proof that the set of all linear preservers $\mathcal{P}_s(\ell^p(I)^+)$ of submajorization ($\prec_s$) is norm-closed in the set of all bounded linear operators on $\ell^p(I)$, where I is an arbitrary nonempty set and $p \in (1, \infty)$.

When I is finite, the desired result follows directly from the compactness of the set $iDSS(\ell^p(I))$. However, in the infinite case, Example 2.3 demonstrates that $iDSS(\ell^p(I))$ fails to be compact. In this case, closedness of $\mathcal{P}_s(\ell^p(I)^+)$ will be proved in Theorem 2.6 using the auxiliary Lemma 2.5. Although an analogue of Theorem 2.6 appears as a corollary in [11], the aim of this work is to give a constructive proof of the theorem.

Notations and preliminaries

Throughout this paper, unless explicitly stated otherwise, I will denote an arbitrary nonempty set and $p \in (1, \infty)$. The Banach space $\ell^p(I)$ consists of all functions $f : I \longrightarrow \mathbb{R}$ such that $\sum_{i \in I} |f(i)|^p < \infty$, equipped with

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the standard p-norm. Its positive cone is defined as

$$\ell^p(I)^+ := \{ f \in \ell^p(I) : f(i) \geq 0 \text{ for every } i \in I \}.$$

We recall that each function $f \in \ell^p(I)$ may be represented in the following form $f = \sum_{i \in I} f(i) e_i$ using Kronecker delta functions δ_{ij}, where $e_i(j) = \delta_{ij}$, $i \in I$ and $e_i : I \longrightarrow \mathbb{R}$.

We will consider bounded linear operators acting on discrete Lebesgue spaces $\ell^p(I)$. If $A : \ell^p(I) \to \ell^p(I)$ is bounded, it can be represented by a (possibly infinite) matrix $[a_{ij}]_{i,j \in I}$ depending on the cardinality of I. Setting $a_{ij} = \langle Ae_j, e_i \rangle$ for all $i, j \in I$, where the dual pairing $\langle \cdot, \cdot \rangle : \ell^p(I) \times \ell^q(I) \longrightarrow \mathbb{R}$ is given by

$$\langle f, g \rangle = \sum_{i \in I} f(i) g(i),$$

we obtain the matrix form of A:

$$Af(i) = \sum_{j \in I} a_{ij} f(j), \quad \forall i \in I, \quad \text{or equivalently,} \quad Af = \sum_{i \in I} \Big(\sum_{j \in I} a_{ij} f(j) \Big) e_i.$$

Definition 1.1. *[11, Definition 3.1][2, Definition 2.1][9, Definition 3.1] Let $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. The operator A is called:*

- positive *if $Af \in \ell^p(I)^+$ for each $f \in \ell^p(I)^+$;*
- doubly stochastic *if A is positive and satisfies*

$$\forall i \in I \ \sum_{j \in I} \langle Ae_j, e_i \rangle = 1, \quad \forall j \in I \ \sum_{i \in I} \langle Ae_j, e_i \rangle = 1;$$

- doubly substochastic *if A is positive and*

$$\forall i \in I \ \sum_{j \in I} \langle Ae_j, e_i \rangle \leq 1, \quad \forall j \in I \ \sum_{i \in I} \langle Ae_j, e_i \rangle \leq 1;$$

- increasable doubly substochastic *if there exists a doubly stochastic operator $A_1 : \ell^p(I) \longrightarrow \ell^p(I)$ such that*

$$\forall i, j \in I : \quad \langle Ae_j, e_i \rangle \leq \langle A_1 e_j, e_i \rangle;$$

- a permutation *if there exists a bijection $\theta : I \longrightarrow I$ satisfying $Ae_j = e_{\theta(j)}$ for all $j \in I$.*

The sets of all doubly substochastic, increasable doubly substochastic, and permutation operators on $\ell^p(I)$ will be denoted by $DSS(\ell^p(I))$, $iDSS(\ell^p(I))$, and $P(\ell^p(I))$, respectively.

It is straightforward that

$$iDSS(\ell^p(I)) \subseteq DSS(\ell^p(I)).$$

For finite index sets I, the equality $iDSS(\ell^p(I)) = DSS(\ell^p(I))$ follows from the classical result of von Neumann [15]. When I is infinite, the inclusion is proper: $iDSS(\ell^p(I)) \subsetneq DSS(\ell^p(I))$. Typical examples showing this strictness are the left and right shift operators, which are doubly substochastic but not increasable doubly substochastic.

Definition 1.2. *[11, Definition 3.6] For two functions $f, g \in \ell^p(I)^+$, we say that f is* submajorized *by g if there exists an increasable doubly substochastic operator $D \in iDSS(\ell^p(I))$ such that $f = Dg$. We denote this relation by $f \prec_s g$.*

Definition 1.3. *A bounded linear operator $T : \ell^p(I) \to \ell^p(I)$ is called a* preserver of submajorization *on $\ell^p(I)^+$ if*

$$f \prec_s g \;\Rightarrow\; Tf \prec_s Tg, \quad f, g \in \ell^p(I)^+.$$

The set of all such linear preservers is denoted by $\mathcal{P}_s(\ell^p(I)^+)$.

Theorem 1.4. *[11, Corollary 3.8] Let $f, g \in \ell^p(I)^+$. The following are equivalent:*

i) $f \prec_s g$ and $g \prec_s f$;

ii) There exists $P \in P(\ell^p(I))$ such that $g = Pf$.

Next, for an infinite index set I, we consider the map $P_\theta : \ell^p(I) \to \ell^p(I)$ defined by

$$P_\theta(f) := \sum_{k\in I} f(k) e_{\theta(k)}, \quad f \in \ell^p(I),$$

where $\theta : I \to I$ is an injective map. Clearly, P_θ is a bounded linear operator on $\ell^p(I)$ with $\|P_\theta\| = 1$. If θ is bijective, then P_θ is a permutation.

For infinite I, the structure of linear preservers of submajorization ($\prec_s$) was characterized in [11, Corollary 4.5] as follows.

Theorem 1.5. *[11, Corollary 4.5] Let I be an infinite set and let $T : \ell^p(I) \to \ell^p(I)$ be a bounded linear operator. The following statements are equivalent:*

i) $T \in \mathcal{P}_s(\ell^p(I)^+)$;

ii) $Te_j \prec_s Te_k$ and $Te_k \prec_s Te_j$ for all $k, j \in I$, and for each $i \in I$ there is at most one $j \in I$ such that $\langle Te_j, e_i\rangle > 0$;

iii) $T = \sum_{k\in I_0} \lambda_k P_{\theta_k}$, where $(\lambda_k)_{k\in I_0} \in \ell^p(I_0)^+$, $I_0 \subset I$ is at most countable, and

$$\theta_k \in \Theta := \{\theta_k : I \xrightarrow{1-1} I \mid k \in I_0,\ \theta_i(I) \cap \theta_j(I) = \emptyset,\ i \neq j\}.$$

2. Closedness of the set of all linear preservers of submajorization

In this section, the goal is to show that the set of all linear preservers of submajorization on $\ell^p(I)^+$ is norm-closed within the set of all bounded linear operators on $\ell^p(I)$. In the finite-dimensional setting, the proof follows from the compactness of $iDSS(\ell^p(I))$. When the index set I is infinite, Example 2.3 demonstrates that $iDSS(\ell^p(I))$ fails to be compact. Hence, in that case, the closedness of $\mathcal{P}_s(\ell^p(I)^+)$ will be established using Theorem 2.6.

From now on, we assume that $p \in (1, \infty)$.

Lemma 2.1. *Let I be an arbitrary nonempty finite set. Then the set $iDSS(\ell^p(I))$ is bounded and norm-closed in the set of all bounded linear operators on $\ell^p(I)$. Moreover, $iDSS(\ell^p(I))$ is compact.*

Proof. Since $iDSS(\ell^p(I)) = DSS(\ell^p(I))$ whenever I is finite, the statement follows from the compactness of $DSS(\ell^p(I))$, proved in [10, Corollary 4.1]. For completeness, we present a direct constructive argument.

By [9, Lemma 3.3], every doubly substochastic operator has norm at most 1. Because every increasable doubly substochastic operator is also doubly substochastic, it follows immediately that $iDSS(\ell^p(I))$ is bounded.

Let $(D_k)_{k\in\mathbb{N}}$ be a sequence in $iDSS(\ell^p(I))$ converging in norm to some bounded linear operator $D : \ell^p(I) \to \ell^p(I)$. For fixed $i_0, j_0 \in I$, we have

$$|\langle D_k e_{j_0} - De_{j_0}, e_{i_0}\rangle|^p \leq \sum_{i\in I} |\langle D_k e_{j_0} - De_{j_0}, e_i\rangle|^p \leq \|D_k e_{j_0} - De_{j_0}\|^p \longrightarrow 0,$$

as $k \to \infty$. Hence,

$$\lim_{k\to\infty} \langle D_k e_j, e_i\rangle = \langle De_j, e_i\rangle, \quad \forall\, i, j \in I.$$

Since I is finite, we can interchange limit and summation directly. Thus,

$$\sum_{i\in I} \langle De_j, e_i\rangle = \sum_{i\in I} \lim_{k\to\infty} \langle D_k e_j, e_i\rangle = \lim_{k\to\infty} \sum_{i\in I} \langle D_k e_j, e_i\rangle \leq 1, \quad \forall j \in I,$$

and, analogously, $\sum_{j\in I} \langle De_j, e_i\rangle \leq 1$ for all $i \in I$. Hence, $D \in DSS(\ell^p(I)) = iDSS(\ell^p(I))$, which shows that the set is norm-closed. □

Alternatively, the compactness of $iDSS(\ell^p(I))$ can be observed more directly: since the unit ball in $\mathcal{B}(\ell^p(I))$ is compact whenever I is finite, and $iDSS(\ell^p(I))$ is a closed subset thereof, the result follows immediately.

Theorem 2.2. *Let I be a finite set. Then $\mathcal{P}_s(\ell^p(I)^+)$ is a norm-closed subset of the set of all bounded linear operators on $\ell^p(I)$.*

Proof. Let $(T_k)_{k\in\mathbb{N}}$, $T_k \in \mathcal{P}_s(\ell^p(I)^+)$ be a sequence converging in norm to a bounded linear operator $T : \ell^p(I) \to \ell^p(I)$. Fix $f, g \in \ell^p(I)^+$ with $f \prec_s g$. Then $T_k f \prec_s T_k g$ for every k, so there exist operators $D_k \in iDSS(\ell^p(I))$ satisfying

$$D_k T_k g = T_k f.$$

By Lemma 2.1, there exists a subsequence $(D_{k_j})_{j\in\mathbb{N}}$ and an operator $D \in iDSS(\ell^p(I))$ such that $\lim_{j\to\infty} D_{k_j} = D$. Taking limits, we obtain

$$Tf = \lim_{j\to\infty} T_{k_j} f = \lim_{j\to\infty} D_{k_j} T_{k_j} g = DTg.$$

Hence, $Tf \prec_s Tg$, which shows that $T \in \mathcal{P}_s(\ell^p(I)^+)$. □

When I is infinite, the set $iDSS(\ell^p(I))$ is no longer compact.

Example 2.3. Consider a sequence of operators $(D_n)_{n\in\mathbb{N}}$ on $\ell^p(I)$ defined by

$$\langle D_n e_j, e_i\rangle = \begin{cases} 1, & \text{if } i = j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Each D_n acts as a projection onto the one-dimensional subspace generated by e_n, that is,

$$D_n(f_1, f_2, f_3, \dots) = (0, 0, \dots, f_n, 0, 0, \dots),$$

where the only nonzero coordinate is in the n-th position.

In matrix form, whenever $I = \mathbb{N}$, the operator D_n can be represented schematically as

$$D_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{bmatrix} \text{ (the 1 lies in the } n\text{-th row and } n\text{-th column).}$$

Clearly, $D_n \in iDSS(\ell^p(I))$ for every $n \in \mathbb{N}$. However, $(D_n)_{n\in\mathbb{N}}$ is not norm-convergent, and no subsequence of it converges in the operator norm. Consequently, $iDSS(\ell^p(I))$ cannot be compact when I is infinite.

We note that the set $iDSS(\ell^p(I))$ is not norm-closed.

Example 2.4. Consider a sequence of operators $(D_n)_{n\in\mathbb{N}}$ on $\ell^p(\mathbb{N})$ with matrix representations $D_n = [d^n_{ij}]_{i,j\in\mathbb{N}}$, defined by

$$d^n_{ij} = \begin{cases} \frac{1}{n}, & \text{if } i = 1 \text{ and } j \in \{1, \dots, n\}, \\ 0, & \text{if } i = 1 \text{ and } j \notin \{1, \dots, n\}, \\ 1 - \frac{1}{n}, & \text{if } i = k \text{ for some } k \geq 2 \text{ and } j = k - 1, \\ \frac{1}{n}, & \text{if } i = k \text{ for some } k \geq 2 \text{ and } j = n + k - 1. \end{cases}$$

The corresponding infinite matrices have the form

$$D_n = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1-\frac{1}{n} & 0 & 0 & \cdots & 0 & \frac{1}{n} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1-\frac{1}{n} & 0 & \cdots & 0 & 0 & \frac{1}{n} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1-\frac{1}{n} & \cdots & 0 & 0 & 0 & \frac{1}{n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \frac{1}{n} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 1-\frac{1}{n} & 0 & 0 & 0 & 0 & \frac{1}{n} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1-\frac{1}{n} & 0 & 0 & 0 & 0 & \frac{1}{n} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Clearly, each D_n is an increasable doubly substochastic operator (in fact doubly stochastic). It is straightforward to verify that $(D_n)_{n\in\mathbb{N}}$ converges in norm to the right shift operator

$$D_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which is not an increasable doubly substochastic operator.

More generally, the same conclusion holds if we consider a sequence of increasable doubly substochastic operators $(D_n)_{n\in\mathbb{N}}$ on $\ell^p(I)$, where I is an arbitrary infinite set, with matrix representations $D_n = [d^n_{ij}]_{i,j\in I}$ defined by

$$d^n_{ij} = \begin{cases} \frac{1}{n}, & \text{if } i = i_1 \text{ and } j \in \{i_1,\dots,i_n\},\\ 0, & \text{if } i = i_1 \text{ and } j \notin \{i_1,\dots,i_n\},\\ 1-\frac{1}{n}, & \text{if } i = i_k \text{ for some } k\geq 2 \text{ and } j = i_{k-1},\\ \frac{1}{n}, & \text{if } i = i_k \text{ for some } k\geq 2 \text{ and } j = i_{n+k-1},\\ 1, & \text{if } i \notin J \text{ and } j = i,\\ 0, & \text{if } i \notin J \text{ and } j \neq i,\end{cases}$$

where $J = \{i_k \mid k \in \mathbb{N}\}$ is a countably infinite subset of I. Again, $(D_n)_{n\in\mathbb{N}}$ converges in norm to the right shift operator on $\ell^p(I)$, which is not increasable doubly substochastic. Therefore, the set $iDSS(\ell^p(I))$ is not closed in the operator norm.

In the finite case, we relied on the fact that every sequence of increasable doubly substochastic operators admits a convergent subsequence whose limit remains in $iDSS(\ell^p(I))$. Example 2.3 demonstrates that this property no longer holds when I is infinite. Hence, the argument used for compactness in the finite-dimensional setting does not extend to this case. Nevertheless, we shall now establish that $\mathcal{P}_s(\ell^p(I)^+)$ remains norm-closed even for infinite index sets. In order to prove this, we need the following lemma.

Lemma 2.5. *Let I be an infinite set and let $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be two sequences in $\ell^p(I)^+$ such that*

$$\lim_{k\to\infty} f_k = f \in \ell^p(I)^+ \quad \text{and} \quad \lim_{k\to\infty} g_k = g \in \ell^p(I)^+.$$

Assume $f_k \prec_s g_k$ and $g_k \prec_s f_k$ for every $k \in \mathbb{N}$. Then there exists a bijection

$$\Omega^+ : I_f^+ \longrightarrow I_g^+$$

such that

$$f(i) = g\big(\Omega^+(i)\big), \qquad \forall i \in I_f^+.$$

Moreover, if $I^+_{f_k} \cap I^+_{g_k} = \emptyset$ for every $k \in \mathbb{N}$, then

$$f \prec_s g \quad \text{and} \quad g \prec_s f.$$

Proof. Since $f, g \in \ell^p(I)$ their supports $I_f^+ = \{i \in I : f(i) > 0\}$ and I_g^+ are at most countable, and consequently the sets of their values $f(I_f^+)$ and $g(I_g^+)$ are at most countable. Therefore we may choose a strictly decreasing sequence of positive reals

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots \longrightarrow 0$$

such that

$$\alpha_k \notin f(I_f^+) \cup g(I_g^+), \qquad \forall k \in \mathbb{N}_0. \tag{1}$$

For $m \in \mathbb{N}$ define the level sets

$$I_f^m := \{\, i \in I_f^+ : \ \alpha_m < f(i) < \alpha_{m-1} \,\}, \qquad I_g^m := \{\, i \in I_g^+ : \ \alpha_m < g(i) < \alpha_{m-1} \,\}.$$

By construction, each I_f^m and I_g^m is finite and pairwise disjoint with respect to m, and

$$I_f^+ = \bigcup_{m\in\mathbb{N}} I_f^m, \qquad I_g^+ = \bigcup_{m\in\mathbb{N}} I_g^m,$$

so they form partitions of I_f^+ and I_g^+, respectively. Fix $m \in \mathbb{N}$. Because I_f^m is finite, there exists $\varepsilon > 0$ such that

$$f(i) \in (\alpha_m + \varepsilon, \alpha_{m-1} - \varepsilon) \subset (\alpha_m, \alpha_{m-1}), \qquad \forall i \in I_f^m.$$

Since $f_k \to f$ in $\ell^p(I)$ we have $\|f_k - f\| \to 0$, hence there is N_1 with

$$|f_k(i) - f(i)| < \varepsilon, \qquad \forall k > N_1, \ \forall i \in I_f^m.$$

Therefore, $I_f^m \subseteq I_{f_k}^m$ for all $k > N_1$.

Next, because the values of f on different level-sets are separated by the gaps chosen in (1), there exists $r > 0$ such that the enlarged interval $(\alpha_m - r, \alpha_{m-1} + r)$ does not meet the values of f coming from other levels:

$$(\alpha_m - r, \alpha_{m-1} + r) \cap \bigcup_{j \neq m} f(I_f^j) = \emptyset.$$

Since $f_k \to f$ uniformly on the finite set I_f^+ (in the sense $|f_k(i) - f(i)| \leq \|f_k - f\|$) there is N_2 with

$$|f_k(i) - f(i)| < r, \qquad \forall k > N_2, \ \forall i \in I_f^+$$

which imply

$$(\alpha_m, \alpha_{m-1}) \cap \bigcup_{j \neq m} f_k(I_f^j) = \emptyset \ \text{ and } \ I_{f_k}^m \subseteq I_f^m.$$

Combining these two facts yields that for every $k > \max\{N_1, N_2\}$ one has

$$I_{f_k}^m = I_f^m. \tag{2}$$

The same argument applied to the sequence (g_k) provides $\widetilde{N}$ so that for all $k > \widetilde{N}$ we have $I_{g_k}^m = I_g^m$. Put $N_m := \max\{N_1, N_2, \widetilde{N}\}$ so (2) and its g-analogue hold for all $k > N_m$.

Choose an arbitrary $\epsilon > 0$. By the convergence of (f_k) and (g_k) to f and g, respectively, there exists $k_0 > N_m$ such that for every $i \in I$ we have

$$|f(i) - f_k(i)| < \tfrac{\epsilon}{2} \quad \text{and} \quad |g(i) - g_k(i)| < \tfrac{\epsilon}{2}, \qquad k > k_0.$$

By the assumption $f_k \prec_s g_k$ and $g_k \prec_s f_k$, Theorem 1.4 provides a bijection

$$\omega_k : I_{f_k}^+ \longrightarrow I_{g_k}^+,$$

satisfying $f_k(i) = g_k(\omega_k(i))$ for all $i \in I_{f_k}^+$. Restricting ω_k to the level $I_f^m = I_{f_k}^m$ yields a bijection

$$\widetilde{\omega}^m := \omega_k|_{I_f^m} : I_f^m \longrightarrow I_g^m.$$

Fix $i \in I_f^m$. Using $f_k(i) = g_k(\widetilde{\omega}^m(i))$ and the convergence of f_k and g_k, we obtain

$$|f(i) - g(\widetilde{\omega}^m(i))| \leq |f(i) - f_k(i)| + |g_k(\widetilde{\omega}^m(i)) - g(\widetilde{\omega}^m(i))| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$f(i) = g(\widetilde{\omega}^m(i)), \qquad \forall i \in I_f^m.$$

Consequently, for each $m \in \mathbb{N}$ we have bijections

$$\widetilde{\omega}^m : I_f^m \longrightarrow I_g^m$$

satisfying $f(i) = g(\widetilde{\omega}^m(i))$ for all $i \in I_f^m$. Assembling these levelwise bijections, we define

$$\Omega^+ : I_f^+ \longrightarrow I_g^+, \qquad \Omega^+(i) := \widetilde{\omega}^m(i) \quad \text{whenever } i \in I_f^m.$$

By construction, Ω^+ is a bijection and $f(i) = g(\Omega^+(i))$ for every $i \in I_f^+$.

Finally, assume $I_{f_k}^+ \cap I_{g_k}^+ = \emptyset$ for all k. We claim $I_f^+ \cap I_g^+ = \emptyset$. Indeed, if $i_0 \in I_f^+ \cap I_g^+$ then for large k we would have $i_0 \in I_{f_k}^+ \cap I_{g_k}^+$ by convergence, which is a contradiction. Hence $I_f^0 := I \setminus I_f^+$ and $I_g^0 := I \setminus I_g^+$ have the same cardinality, so there exists a bijection

$$\Omega^0 : I_f^0 \longrightarrow I_g^0.$$

Merging Ω^+ and Ω^0 we obtain a bijection $\Omega : I \to I$ such that

$$f(i) = g\big(\Omega(i)\big), \qquad \forall i \in I.$$

Thus $f = Pg$ for the permutation P induced by Ω, and by Theorem 1.4 we conclude $f \prec_s g$ and $g \prec_s f$. □

Theorem 2.6. *Let I be an infinite set. Then $\mathcal{P}_s(\ell^p(I)^+)$ is norm-closed in the set of all bounded linear operators on $\ell^p(I)$.*

Proof. Let $(T_k)_{k\in\mathbb{N}}$, $T_k \in \mathcal{P}_s(\ell^p(I)^+)$ be a sequence converging in operator norm to $T \in \mathcal{B}(\ell^p(I))$. For any $i, j \in I$ we have $e_i \prec_s e_j$ and $e_j \prec_s e_i$, hence $T_k e_i \prec_s T_k e_j$ and $T_k e_j \prec_s T_k e_i$ for every k. Passing to the limit and applying Lemma 2.5 (to the sequences $T_k e_i$ and $T_k e_j$) we deduce

$$Te_i \prec_s Te_j \quad \text{and} \quad Te_j \prec_s Te_i, \qquad \forall i, j \in I.$$

Next we show that for each $r \in I$ there is at most one $s \in I$ with $\langle Te_s, e_r \rangle > 0$. Otherwise, suppose there exist $j_1 \neq j_2$ with

$$\langle Te_{j_1}, e_r \rangle > 0 \quad \text{and} \quad \langle Te_{j_2}, e_r \rangle > 0.$$

By norm convergence $T_k \to T$ we would have for sufficiently large k also $\langle T_k e_{j_1}, e_r \rangle > 0$ and $\langle T_k e_{j_2}, e_r \rangle > 0$, contradicting the characterization of preservers in Theorem 1.5, since each $T_k \in \mathcal{P}_s(\ell^p(I)^+)$. Therefore for every r there is at most one such s.

Finally, Theorem 1.5 implies that $T \in \mathcal{P}_s(\ell^p(I)^+)$, i.e. the set of preservers is norm-closed. □

References

[1] T. Ando, *Majorization, doubly stochastic matrices, and comparison of eigenvalues*, Linear Algebra Appl. 118 (1989) 163-248.

[2] F. Bahrami, A. Bayati, S. M. Manjegani, *Linear preservers of majorization on $\ell^p(I)$*, Linear Algebra Appl. 436 (2012) 3177-3195.

[3] A. B. Eshkaftaki, *Doubly (sub)stochastic operators on ℓ^p spaces*, J. Math. Anal. Appl. 498(1) (2021), article:124923.

[4] A. B. Eshkaftaki, *Increasable doubly substochastic matrices with application to infinite linear equations*, Linear Multilinear Algebra 70(20) (2021) 5902–5912

[5] G. Gour, D. Jennings, F. Buscemi, R. Duan and I. Marvian, *Quantum majorization and a complete set of entropic conditions for quantum thermodynamics*, Nature Communications 9 5352 (2018).

[6] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, second ed., Cambridge University Press, London and New York, 1952.

[7] A. M. Hasani, M. A. Vali, *Linear maps which preserve or strongly preserve weak majorization*, J. Inequal. Appl. 2007:082910 (2008).

[8] V. Kaftal, G. Weiss, *An infinite dimensional Schur-Horn Theorem and majorization theory*, J. Funct. Anal. 259 (2010) 3115-3162.

[9] M. Ljubenović, *Weak majorization and doubly substochastic operators on $\ell^p(I)$*, Linear Algebra Appl. 486 (2015) 295-316.

[10] M. Ljubenović, D. S. Djordjević, *Linear preservers of weak majorization on $\ell^p(I)^+$, when $p \in (1, \infty)$*, Linear Algebra Appl. 497 (2016) 181-198.

[11] M.Z. Ljubenović, D.S. Rakić, *Submajorization on $\ell^p(I)^+$ determined by increasable doubly substochastic operators and its linear preservers*, Banach J. Math. Anal. 15 60 (2021).

[12] A. W. Marshall, I. Olkin, B.C. Arnold, *Inequalities: Theory of majorization and its applications*, second ed., Springer, 2011.

[13] S.M. Manjegani, S. Moein, *Quasi doubly stochastic operator on ℓ^1 and Nielsen's theorem*, J. Math. Phys. 60 103508 (2019).

[14] M. A. Nielsen, *An introduction of majorization and its applications to quantum mechanics*, Lecture Notes, Department of Physics, University of Queensland, Australia; 2002. Available at http://michaelnielsen.org/blog/talks/2002/maj/book.ps.

[15] J. von Neumann, *A certain zero-sums two-person game equivalent to the optimal assignment problem*, Contributions to the Theory of Games 2 (1953) 5-12.

[16] R. Pereira, S. Plosker, *Extending a characterisation of majorization to infinite dimensions*, Linear Algebra Appl. 468 (2015) 80-86.