REGULAR AND T-FREDHOLM ELEMENTS IN BANACH ALGEBRAS

Dragan Djordjević

Abstract. Let \( T : A \to B \) be an algebra homomorphism of a Banach algebra \( A \) to an algebra \( B \). An element \( a \in A \) is \( T \)-Fredholm if \( T(A) \subseteq B^{-1} \) and \( a \in A \) is regular provided there is an element \( a' \in A \) such that \( a = aa'a \). We investigate regular and \( T \)-Fredholm elements in Banach algebras. As a corollary, we get a well known result [5, Theorem 3].

0. Introduction. Let \( A \) be an additive category. We say that a morphism \( a \in A \) is invertible provided there is a morphism \( a' \in A \) such that \( aa' = 1 \) and \( a'a = 1 \). The class of all invertible morphisms is denoted by \( A^{-1} \). If \( a \in A \), then a generalised inverse for \( a \) is a morphism \( a' \in A \) such that \( a = aa'a \). The regular morphisms of a category \( A \) form a class \( \hat{A} = \{ a \in A : a \in aAa \} \) [3, Definition 1.1]. If \( B \) is an additive category and \( T : A \to B \) is an additive functor, then a morphism \( a \in A \) is \( T \)-Fredholm if \( T(a) \subseteq B^{-1} \). The class of all \( T \)-Fredholm morphisms is denoted by \( \Phi_T(A) \) [3, Definition 2.1]. We shall use \( \mathbb{C} \) to denote the complex plane. For a subset \( S \) of \( \mathbb{C} \), let \( S' \) denote the set of all points of accumulation of \( S \). Let \( T : A \to B \) be an algebra homomorphism of a Banach algebra \( A \) to an algebra \( B \). For a subset \( M \) of \( A \) let \( \text{cl}M \) denote the closure of \( M \). If \( a \in A \), then \( \sigma(a) \) denotes the spectrum of \( a \). Recall [1] that an ideal \( I \) of \( A \) is inessential, if \( x \in I \Leftrightarrow \sigma(x)' \subseteq \{0\} \). We always assume that there are identities in \( A \) and \( B \) and \( T(1) = 1 \). Recall [2, 1.6] that a homomorphism \( T \) has the Riesz property if \( T^{-1}(0) \) is an inessential ideal of \( A \). Since \( A \) is a Banach algebra, then [4, 1.1]

\[(0.1) \quad A^{-1} = \hat{A} \cap \text{cl}(A^{-1}).\]

In this note we investigate regular and \( T \)-Fredholm elements in Banach algebras. As a corollary, we get a well known result [5, Theorem 3].

1. Results. Let \( T : A \to B \) be an algebra homomorphism of a Banach algebra \( A \) to an algebra \( B \).

Lemma 1. If \( T \) has the Riesz property and \( a^2 - a \in T^{-1}(0) \), then there are \( p \in \hat{A} \) and \( a', a'' \in A \) such that

\[ p = a'a = aa', \quad 1 - p = a''(1 - a) = (1 - a)a'', \quad a - p \in T^{-1}(0).\]
Proof. If \( a^2 - a \in T^{-1}(0) \), then \( \sigma(a^2 - a)^\prime \subseteq \{0\} \) and \( \sigma(a)^\prime \subseteq \{0,1\} \). There are open subsets \( U_1 \) and \( U_2 \) of \( \mathbb{C} \) such that \( 1 \in U_1, 0 \in U_2, \text{cl}U_1 \cap \text{cl}U_2 = \emptyset \) and \( \sigma(a) \subseteq U_1 \cup U_2 \). Now, if

\[
\begin{align*}
  f(z) &= \begin{cases} 1, & \text{for } z \in U_1, \\ 0, & \text{for } z \in U_2, \end{cases} \\
  h(z) &= \begin{cases} 1/z, & \text{for } z \in U_1, \\ 0, & \text{for } z \in U_2, \end{cases} \\
  f_1(z) &= \begin{cases} 0, & \text{for } z \in U_1, \\ 1, & \text{for } z \in U_2, \end{cases} \\
  h_1(z) &= \begin{cases} 0, & \text{for } z \in U_1, \\ 1/(1 - z), & \text{for } z \in U_2, \end{cases}
\end{align*}
\]

then \( f(a) = p, \ g(a) = a, \ h(a) = a^\prime, \ f_1(a) = 1 - p, \ g_1(a) = 1 - a, \ h_1(a) = a^\prime, p^2 = p \) and

\[
p = aa^\prime = a'a, \quad 1 - p = a''(1 - a) = (1 - a)a''.
\]

Hence \( a - p = a(1 - p) + (a - 1)p = (a^2 - a)(a' - a'') \in T^{-1}(0) \). \( \blacksquare \)

Lemma 2. If \( T \) has the Riesz property, then

\[
(1.1) \quad \hat{A} + T^{-1}(0) = T^{-1}(\hat{B}).
\]

Proof. Since the inclusion \( \subseteq \) in (1.1) is obvious, it is enough to prove the opposite inclusion. If \( T(a) \in \hat{B} \), then \( T(a) = T(a^2) \) and \( a^2 - a \in T^{-1}(0) \). By Lemma 1 there is a \( p \in \hat{A} \) such that \( a - p \in T^{-1}(0) \). Hence

\[
a = p + (a - p) \in \hat{A} + T^{-1}(0). \quad \blacksquare
\]

Theorem 3. If \( T(A) = B \) and \( T \) has the Riesz property, then

\[
(1.2) \quad \hat{A} + T^{-1}(0) = T^{-1}(\hat{B}).
\]

Proof. The inclusion \( \subseteq \) in (1.2) is obvious, so we have to prove the opposite inclusion. If \( T(a) \in \hat{B} \), there is \( b \in \hat{A} \) such that \( T(a)T(b)T(a) = T(a) \). For \( c = ba \), we have

\[
T(c^2 - c) = T(baba - ba) = T(b)[T(aba) - T(a)] = 0.
\]

There are \( p \in \hat{A} \) and \( a' \in A \) such that \( p = cc' = c'e \) and \( p - c \in T^{-1}(0) \). We have \( ap(c'b)ap = ap(c'cp) = ap \) and \( ap \in \hat{A} \). Thus \( ba - p \in T^{-1}(0) \) and \( T(aba - ap) = 0 \), so \( a - ap \in T^{-1}(0) \). We get

\[
a = ap + (a - ap) \in \hat{A} + T^{-1}(0) \quad \blacksquare
\]

Theorem 4. If \( T(A) = B \) and \( T \) has the Riesz property, then

\[
(1.3) \quad \hat{A} \Phi_T(A) + T^{-1}(0) = T^{-1}(\hat{B}B^{-1}).
\]

Proof. The inclusion \( \subseteq \) in (1.3) is obvious. To prove the opposite inclusion, suppose that \( T(a) \in \hat{BB}^{-1} \). By Lemma 2, there are \( b \in \hat{A}, d \in T^{-1}(0) \) and \( c \in \Phi_T(A) \) such that \( T(a) = T(b+d)T(c) \). We get \( a-bc \in T^{-1}(0) \) and \( a = bc + (a-bc) \in \hat{A} \Phi_T(A) + T^{-1}(0). \) \( \blacksquare \)
**Theorem 5.** If $B$ is a Banach algebra, $T(A) = B$, $T$ is continuous, $T$ has the Riesz property and the norm $\|\cdot\|_B$ on $B$ and the quotient norm $\|\cdot\|_q$ are equivalent, then

\begin{equation}
\hat{A} \cap \text{cl}(\Phi_T(A)) + T^{-1}(0) = T^{-1}(\hat{B} \cap \text{cl}B^{-1}).
\end{equation}

**Proof.** If $a = b + c, b \in \hat{A} \cap \text{cl}(\Phi_T(A))$ and $T(c) = 0$, then $T(a) = T(b) \in \hat{B}$. Hence there is a sequence $(b_n), b_n \in \Phi_T(A)$, such that $\lim b_n = b.$ Thus from $T(b_n) \in B^{-1}$ and $\lim T(b_n) = T(b)$ we get $T(a) \in \text{cl}B^{-1}$.

To prove the opposite inclusion in (1.4), let $T(a) \in \hat{B} \cap \text{cl}B^{-1}$. Then $a = b + c, b \in \hat{A}, T(c) = 0$ (Lemma 2) and $T(a) = T(b) \in \text{cl}B^{-1}$. There is a sequence $(b_n), b_n \in \Phi_T(A)$, such that $\|T(b - b_n)\|_B \to 0, n \to \infty$. Hence $\|b - b_n + T^{-1}(0)\|_q \to 0, n \to \infty$. Let $\epsilon > 0$ and let $n$ be a positive integer such that $1/n < \epsilon/2$ and $\|b - b_n + T^{-1}(0)\|_q < \epsilon/2$. There is $t \in T^{-1}(0)$ such that

$$\|b - b_n + t\| \leq \|b - b_n + T^{-1}\|_q + 1/n$$

and $\|b - b_n + t\| < \epsilon$. From $b_n - t \in \Phi_T(A)$ we have $b \in \text{cl}(\Phi_T(A))$. ■

Let us remark that from (0.1), Theorem 4 and Theorem 5 we have

$$\hat{A} \cap \text{cl}(\Phi_T(A)) + T^{-1}(0) = \hat{A}\Phi_T(A) + T^{-1}(0).$$

**Theorem 6.** If $B$ is a Banach algebra, $T(A) = B$ and

\begin{equation}
\hat{A} \cap \text{cl}(\Phi_T(A)) \subseteq \hat{A},
\end{equation}

then

\begin{equation}
\hat{A} \cap \text{cl}(\Phi_T(A)) = \hat{A}\Phi_T(A).
\end{equation}

**Proof.** If $a \in \hat{A} \cap \text{cl}(\Phi_T(A))$, there is $a' \in A$, such that $a = aa'a$ and $a' = a'aa'$. Let $b \in \Phi_T(A)$ and $\|b - a\||a'|| < 1$. Then

$$1 + (b - a)a' \in A^{-1}, \quad T(1 + (b - a)a') \in B^{-1}.$$

There is $b' \in \Phi_T(A)$ such that $T(b)T(b') = 1$ and $bb' = 1 + t$ for some $t \in T^{-1}(0)$. If $a'' = a' + (1 - a'a)b'(1 - a'a)$, then $a = aa''a$. Let us remark that from the proof of (0.1) we get $T(a'') \in B^{-1}$, hence $a'' \in \Phi_T(A)$. Thus $\hat{A} \cap \text{cl}(\Phi_T(A)) \subseteq \{a \in A : a \in a\Phi_T(A)a\}$. Now if $a = asa$ and $s \in \Phi_T(A)$, then there are $s_1 \in \Phi_T(A)$ and $t_1 \in T^{-1}(0)$ such that $ss_1 = 1 + t_1$. Now $a = a(ss_1 - t_1) = as(s_1 - at_1)$, $asas = as \in \hat{A}, \quad T(s_1 - at_1) = T(s_1) \in B^{-1}$ and $a \in \hat{A}\Phi_T(A)$.

To prove the opposite inclusion in (1.6), let $a = bc, b \in \hat{A}$ and $T(c) \in B^{-1}$. For $a_n = (b + (1 - b)/n)c$ we have $\lim a_n = a, (b + (1 - b)/n)(b + n(1 - b)) = 1$ and $a_n \in \Phi_T(A)$. Hence $a \in \text{cl}(\Phi_T(A))$. ■

We can’t say anything about the implication

$T$ has the Riesz property $\Rightarrow \hat{A}\Phi_T(A) \subseteq \hat{A}.$

Let $S : C \to D$ be an additive functor from an additive category $C$ to an additive category $D$. $S$ is finitely regular [3, Definition 2.4], if $S^{-1}(0) \subseteq C$. 
Lemma 7. If $S(C) = D$ and $S$ is finitely regular, then $\hat{C}\Phi_S(C) \subseteq \hat{C}$.

Proof. If $a = bc$, $b \in \hat{C}$ and $S(c) \in D^{-1}$, there is $c' \in \Phi_S(C)$ such that $S(c)S(c') = 1$. Hence $S(b)S(c)S(c')S(b)S(c) = S(b)S(c) \in \hat{D}$. By [3, Theorem 2.5], we have $a = bc \in S^{-1}(\hat{D}) = \hat{C}$. ■

Corollary 8. If $B$ is a Banach algebra, $T(A) = B$ and $T$ is finitely regular, then

$$\hat{A} \cap \text{cl}(\Phi_T(A)) = \hat{A}\Phi_T(A).$$

Proof. follows from Lemma 7 and Theorem 6. ■

We can not conclude that [5, Theorem 3] follows from our Corollary 8.

Corollary 9. If $T(A) = B$ and $T$ is finitely regular, then

$$\hat{A}\Phi_T(A) \subseteq \hat{A} \cap \text{cl}(\Phi_T(A)).$$

Proof. follows from Lemma 7 and Theorem 6. ■

Theorem 10. Suppose that the inessential ideals $I$ and $J$ of $A$ have the same sets of idempotents and $J$ is closed. Let $P_0 : A \to A/I$ and $P : A \to A/J$ respectively, be the natural homomorphisms of $A$ onto $A/I$ and $A/J$. If $P_0$ is finitely regular, then

$$\hat{A} \cap \text{cl}(\Phi_P(A)) = \hat{A}\Phi_P(A).$$

Proof. From Theorem 6 it follows $\hat{A} \cap \text{cl}(\Phi_P(A)) \subseteq \hat{A}\Phi_P(A)$. From [1, Proposition 2.2] we get $\Phi_{P_0}(A) = \Phi_P(A)$ and by Lemma 7 we have $\hat{A}\Phi_{P_0}(A) \subseteq \hat{A}$. Hence $\hat{A}\Phi_P(A) \subseteq \hat{A}$ and the proof follows from Theorem 6. ■

Now, as a Corollary, we get [5, Theorem 3]. Let $X$ be an infinite-dimensional complex Banach space. We shall use $B(X), F(X)$ and $K(X)$ respectively, to denote the set of all bounded, finite-rank and compact linear operators on $X$.

Corollary 11. If $X$ is a Banach space then $\overline{B(X)} \cap \text{cl}(\Phi(X)) = B(X) \Phi(X)$.

Proof. Set $I = F(X)$ and $J = K(X)$. It is well known that $F(X)$ and $K(X)$ have the same sets of idempotents and $F(X) \subseteq B(X)$. The proof follows by Theorem 10. ■

Acknowledgement. I am greatfull to Prof. Vladimir Rakočević for helpful suggestions and conversations.

References