PERTURBATIONS, QUASINILPOTENT EQUIVALENCE AND COMMUNICATING OPERATORS

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Abstract

Quasinilpotent equivalent elements of a unital Banach algebra have the same left and right spectrum; if the Banach algebra is the algebra of operators on a Banach space, then they (also) have the same approximate point and surjectivity spectrum. We apply the notion of quasinilpotent equivalent to investigate perturbation of left/right and upper/lower Fredholm (and Weyl, Browder) spectrum by polynomially Riesz operators. The resulting theorems lead to an improvement of some recently obtained results.

1. Introduction

A Banach space operator (i.e. a bounded linear transformation) $A \in BX$ is *Riesz*, $A \in R(\mathcal{X})$, if its non–zero spectral points are finite rank poles of the resolvent (of A); $A \in BX$ is *polynomially Riesz*, $A \in \text{Poly}^{-1}(R(\mathcal{X}))$, if there exists a non–trivial polynomial $p(.)$ such that $p(A) \in R(\mathcal{X})$. Perturbation of Banach space operators by Riesz operators, more generally by polynomially Riesz operators, has been considered by a number of authors. Thus it is known that the 'Browder spectrum $\sigma_b(A)$ of an operator $A \in BX$ is the largest distinguished part of the spectrum $\sigma(A)$ of A which is stable under perturbation by commuting Riesz operators' [15]. Jeribi and Moalla [11] have considered perturbation by polynomially compact operators, and

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Baklouti [8] in his consideration of perturbation by polynomially Fredholm perturbation operators has introduced the idea of 'communicating operators'. Here, if we let Ptrb(Φ) denote the perturbation class of Fredholm operators, then for an operator A such that $p(A) \in$ Ptrb(Φ) for some non–trivial polynomial $p(.)$ the operators A and B communicate if there exists a continuous function $\varphi : [0,1] \to \mathbf{C}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi([0,1])\pi_A^{-1}(0)$ has no points in common with the Fredholm spectrum of B. (Here, π_A denotes the minimal polynomial such that $\pi_A(A) \in$ Ptrb(Φ).) The concept of communicating operators has been exploited by Živković-Zlatanović *et al.* [19, 21] to obtain enhanced results on the perturbation of one sided Fredholm (Banach algebra) elements by algebraically Riesz operators.

Given a unital Banach algebra $\mathcal{A}$, and elements $a, b \in \mathcal{A}$, the elements a, b are *quasinilpotent equivalent* if

$$d(a,b) = \max\{\lim_{n\to\infty} ||\delta^n_{a,b}(1)||^{\frac{1}{n}}, \lim_{n\to\infty} ||\delta^n_{b,a}(1)||^{\frac{1}{n}}\} = 0,$$

where $\delta_{a,b} \in B(\mathcal{A})$ is the generalized derivation $\delta_{a,b}(x) = ax - xb$ for all $x \in \mathcal{A}$. Quasinilpotent operators have the same left, and right, spectrum (hence, also the same spectrum). More is true in the case in which $\mathcal{A}$ is the algebra BX: Quasinilpotent operators in BX have the same (left, right,) spectrum, the same approximate point spectrum and the same surjectivity spectrum. Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are two unital Banach algebras, and $T : \mathcal{A} \to \mathcal{B}$ is a homomorphism mapping left (resp., right) Fredholm elements of $\mathcal{A}$ onto left (resp., right) invertible elements of $\mathcal{B}$. Then $d(Ta, Tb) = 0$ for some $a, b \in \mathcal{A}$ implies a and b have the same left (resp., right) Fredholm spectrum. Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are algebras of operators, and T maps upper (resp., lower) Fredholm operators onto bounded below (resp., surjective) operators, then $d(TA, TB) = 0$ for some $A, B \in \mathcal{A}$ implies A and B have the same upper (resp., lower) Fredholm spectrum.

In this paper we apply the idea of quasinilpotent equivalence to obtain results on perturbations of (one–sided) Fredholm, Weyl and Browder elements by polynomially Riesz (or, holomorphically Riesz) elements of a Banach algebra. It is proved that if $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras, $a, b \in \mathcal{A}$, $T : \mathcal{A} \to \mathcal{B}$ is a homomorphism, $T(ab - ba)$ is in the *radical of* $\mathcal{B}$ and $p(Tb)$ is quasinilpotent in $\mathcal{B}$ for some polynomial p, then either of the conditions $p(Ta)$ is left invertible and $p^{-1}(0) \cap \sigma_l(Ta) = \emptyset$ (resp., right invertible and $p^{-1}(0) \cap \sigma_r(Ta) = \emptyset$) in $\mathcal{B}$ implies $T(a - b)$ is left invertible (resp., right invertible, invertible) in $\mathcal{B}$ (cf. [18]). Working in the algebra BX, let $\sigma_f^\times(A)$ stand for one of the upper/lower and right/left Fredholm spectrum of the operator A. Recall that given operators $A, B \in BX$, the operator A *is in* $\times$*–communication with* B *if there exists a continuous function* $\varphi : [0,1] \to \mathbf{C}$ *such that*

$$\varphi(0) = 0, \varphi(1) = 1 \text{ and } \mu\varphi([0,1]) \notin \sigma_f^\times(A)$$

for all $\mu \in \pi_B^{-1}(0)$ [8]. Here $\pi_B(z) = \prod_{i=1}^m (z - \mu_i)$ is the *minimal polynomial* such that $\pi_B(B)$ is a Riesz operator. Let $\Phi^\times(\mathcal{X})$ denote the class of $\times$–Fredholm operators in BX (where $\times$ stands for one of upper, lower, right, left or (simply) Fredholm). Suppose that $\pi_B(B) = \prod_{i=1}^{m}(B-\mu_i)$ is Riesz. We prove: (a) If $AB - BA \in \mathrm{Ptrb}(\Phi^{\times}(\mathcal{X}))$, then, for every scalar λ such that $\pi_B(A,\lambda) = \prod_{i=1}^{m}(A-\lambda\mu_i) \in \Phi^{\times}(\mathcal{X})$, $A - \lambda B \in \Phi^{\times}(\mathcal{X})$. If, in addition, $\lambda = \lambda(t)$ is a continuous function from a connected subset $\mathbf{I}$ of the reals (into the set of complex numbers) such that $\lambda(t_1) = 0$ and $\lambda(t_2) = 1$ for some $t_1 < t_2 \in \mathbf{I}$, then $\mathrm{ind}(A) = \mathrm{ind}(A-B) = \mathrm{ind}(A-\lambda(t)B)$ for all $t \in [t_1, t_2]$. (b) Suppose that $AB - BA = 0$. If $\pi_B(A,\lambda) = \prod_{i=1}^{m}(A-\lambda\mu_i) \in \Phi^{\times}(\mathcal{X})$ for some scalar λ, then $\pi_B(A,\lambda) \in \times$–Browder implies $A - \lambda B \in \times$–Browder; if, in addition, $\lambda = \lambda(t)$ is the continuous function of (a) and A has SVEP at 0 whenever $\times$–Fredholm is left or upper Fredholm (resp., A^* has SVEP at 0 whenever $\times$–Fredholm is right or lower Fredholm; both A and A^* have SVEP at 0 whenever $\times$–Fredholm is simply Fredholm), then $A - B \in \times$–Browder. It is seen that this is sufficient to obtain, and sometimes enhance, the results of [8; 11; 19; 21]. We consider an application to paranormal, in particular normal, Banach space operators.

2. Some terminology and notation

An operator $A \in BX$ has SVEP (= *the single-valued extension property*) at a point $\lambda_0 \in \mathbf{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. (Here we have shortened $A - \lambda I$ to $A - \lambda$.) Evidently, every A has SVEP at points in the resolvent $\rho(A) = \mathbf{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \boldsymbol{C}$.* The *ascent of* A, $\mathrm{asc}(A)$ (resp. *descent of* A, $\mathrm{dsc}(A)$), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n\mathcal{X} = A^{n+1}\mathcal{X}$): If no such integer exists, then $\mathrm{asc}(A)$, resp. $\mathrm{dsc}(A)$, $= \infty$. It is well known that $\mathrm{asc}(A) < \infty$ implies A has SVEP at 0, $\mathrm{dsc}(A) < \infty$ implies A^* (= the dual operator) has SVEP at 0, finite ascent and descent for an operator implies their equality, and that a point $\lambda \in \sigma(A)$ is a pole (of the resolvent) of A if and only if $\mathrm{asc}(A-\lambda) = \mathrm{dsc}(A-\lambda) < \infty$ (see [1; 13; 14]).

Quasinilpotent equivalence preserves SVEP, i.e., if $A, B \in BX$ are quasinilpotent equivalent then A has SVEP (everywhere) implies B has SVEP [13]. Perturbation of an operator by a commuting operator does not, in general, preserve SVEP (even if the commuting operator has SVEP). However, if the commuting operator is a quasinilpotent, then (the quasinilpotent equivalence of the operator and its perturbation ensures that) the operator has SVEP (everywhere), which implies SVEP (everywhere) for the perturbed operator. Does this property hold for localized SVEP? The following recent result of Aiena and Müller [3, theorem 0.3] answers this question in the affirmative for commuting Riesz perturbations.

Proposition 2.1. *If $A \in BX$ and $R \in BX$ is a Riesz operator which commutes with A, then A has SVEP at λ implies $A + R$ has SVEP at λ.*

The left spectrum $\sigma_l(a)$, the right spectrum $\sigma_r(a)$ and the spectrum $\sigma(a)$ of an element $a \in \mathcal{A}$, $\mathcal{A}$ a unital Banach algebra, are the sets $\sigma_l(a) = \{\lambda \in \mathbf{C} : a - \lambda$ is not left invertible$\} = \{\lambda \in \mathbf{C} : a - \lambda \notin \mathcal{A}^{-1}_{left}\}$, $\sigma_r(a) = \{\lambda \in \mathbf{C} : a - \lambda$ is not right invertible$\} = \{\lambda \in \mathbf{C} : a - \lambda \notin \mathcal{A}^{-1}_{right}\}$ and $\sigma(a) = \sigma_l(a) \cup \sigma_r(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin \mathcal{A}^{-1}\}$; if $\mathcal{A}$ is the algebra BX, then the approximate points spectrum $\sigma_a(A)$ and the surjectivity spectrum $\sigma_s(A)$ of an $A \in BX$ are the sets $\sigma_a(A) = \{\lambda \in \mathbf{C} : A - \lambda$ is not bounded below$\}$ and $\sigma_s(A) = \{\lambda \in \mathbf{C} : A - \lambda$ is not surjective$\}$. In the following we shall use the notation $\sigma_\times(.)$ to denote (either) one of these spectra. An operator $A \in BX)$ is *upper semi–Fredholm* (resp., *lower semi–Fredholm*) if $A\mathcal{X}$ is closed, and the deficiency index $\alpha(A) = \dim A^{-1}(0) < \infty$ (resp., $\beta(A) = \dim(\mathcal{X}/A\mathcal{X}) < \infty$); A is semi–Fredholm if it is either upper or lower semi–Fredholm, and A is Fredholm if it is both upper and lower semi–Fredholm. The index of a semi–Fredholm operator is the integer $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. The operator A is *left* (resp., *right*) *Fredholm* if it is upper semi-Fredholm and $A\mathcal{X}$ (resp., lower semi–Fredholm and $A^{-1}(0)$) is complemented in $\mathcal{X}$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: the upper (lower) Fredholm spectrum $\sigma_f^{upper}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not upper semi–Fredholm$\}$ (resp., $\sigma_f^{lower}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not lower semi–Fredholm$\}$), left (right) Fredholm spectrum $\sigma_f^{left}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not left semi–Fredholm$\}$ (resp., $\sigma_f^{right}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not right semi–Fredholm$\}$), and Fredholm spectrum $\sigma_f(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not Fredhom$\}$. An operator $A \in BX$ is upper Browder (resp., lower Browder, left Browder, right Browder, (simply) Browder) if it is upper Fredholm with $\mathrm{asc}(A) < \infty$ (resp., lower Fredholm with $\mathrm{dsc}(A) < \infty$, left Fredholm with $\mathrm{asc}(A) < \infty$, right Fredholm with $\mathrm{dsc}(A) < \infty$, Fredholm with $\mathrm{asc}(A) = \mathrm{dsc}(A) < \infty$). The Browder spectrum $\sigma_b(A)$ of an operator $A \in BX$ is the set $\sigma_b(A) = \{\lambda \in \mathbf{C} : A - \lambda$ is not Browder$\}$.

The quasinilpotent part $H_0(A)$ of an operator $A \in BX$ is the (generally) non–closed set

$$H_0(A) = \{x \in \mathcal{X} : \lim_{n\to\infty} ||A^n x||^{\frac{1}{n}} = 0\}.$$

Evidently, $A^{-n}(0) \subseteq H_0(A)$ for all integers $n \geq 1$. It is fairly straightforward to see, [2, theorem 2.3], that if $A \in \Phi^{upper}(\mathcal{X})$ (or, $A \in \Phi^{left}(\mathcal{X})$) and $\mathrm{asc}(A) < \infty$, then there exists an integer $n \geq 1$ such that $H_0(A) = A^{-n}(0)$. Suppose that $A, B \in BX$ are commuting operators. Then $AB \in \Phi^\times(\mathcal{X})$ if and only if $A, B \in \Phi^\times(\mathcal{X})$. Furthermore, if also A and B have finite ascent, then there exist integers p_1 and p_2 such that $\dim(H_0(A)) = \dim(A^{-p_1}(0)) < \infty$ and $\dim(H_0(B)) = \dim(B^{-p_2}(0)) < \infty$; consequently, with $p = \max(p_1, p_2)$, $\dim((AB)^{-p}(0)) \leq \dim(H_0(AB)) \leq \dim(H_0(A)) + \dim(H_0(B)) < \infty$, which implies $\mathrm{asc}(AB) < \infty$ and hence that AB is upper–Browder. A similar argument holds for other cases (use duality), and one has: If $A, B \in BX$ are commuting $\times$–Browder operators, then AB is $\times$–Browder operator. The converse holds:

Proposition 2.2. (cf. *[9, theorem 7.9.2]) If $A, B \in BX$ are commuting operators, then A, B are $\times$–Browder if and only if AB is $\times$–Browder.*

PROOF. A, B being in $\Phi^{\times}(\mathcal{X})$ whenever $AB \in \Phi^{\times}(\mathcal{X})$, the proof of the converse statement for the case in which $\times$ is left or upper is a direct consequence of the fact that the following inequalities imply finite ascent:

$$\dim(A^{-p}(0)) \leq \dim(H_0(A)) \leq \dim(H_0(AB)) < \infty, \text{ and}$$
$$\dim(B^{-p}(0)) \leq \dim(H_0(B)) \leq \dim(H_0(AB)) < \infty$$

for all integers $p \geq 1$. Working with the dual, the other case is similarly proved. ■

Any other notation/terminology will be defined in the sequel on a first use basis.

3. Quasinilpotent equivalence, homomorphisms and holomorphically Riesz operators

3.1. Quasinilpotent equivalence.
Let $\mathcal{A}$ be a unital Banach algebra (with unit 1), and let $\delta_{a,b}$, a and $b \in \mathcal{A}$, denote the generalized derivation $\delta_{a,b}(x) = L_a(x) - R_b(x) = ax - xb$ for all $x \in \mathcal{A}$ (where L_a and R_a denote the operators $L_a(x) = ax$ and $R_a(x) = xa$ of left and (respectively) right multiplication by a). The function

$$d(a,b) = \max\{\lim_{n\to\infty} ||\delta^n_{a,b}(1)||^{\frac{1}{n}}, \lim_{n\to\infty} ||\delta^n_{b,a}(1)||^{\frac{1}{n}}\}$$

defines a semi–metric on $\mathcal{A}$ such that

$$d(a,b) = d(L_a, L_b) = d(R_a, R_b) \text{ and } d(L_a, R_b) \leq d(L_a, R_a) + d(R_a, R_b).$$

Recall from [17] (and [16]) that if $d(a,b) = 0$, then $\sigma(a) = \sigma(b)$. More is true. Suppose $a \in \mathcal{A}$ is left invertible by $a_\ell \in \mathcal{A}$. If a and b commute, then (a straightforward argument shows that) $d(a,b) = 0 \Longleftrightarrow d(a_\ell b, 1) = 0$. In the general case, i.e. if a and b do not commute, then let $\mathcal{B}$ denote a maximal commutative subalgebra of the Banach algebra $B(\mathcal{A})$ of bounded linear operators on $\mathcal{A}$ containing L_a, R_b and the identity operator I $(= L_1 = R_1)$. Then $\sigma(L_{a_\ell} R_b, \mathcal{B}) = \sigma(L_{a_\ell} R_b, B(\mathcal{A})) = \sigma(a_\ell)\sigma(b)$. Since

$$\begin{aligned}
& \delta^n_{I, L_{a_\ell} R_b}(1) = \delta^{n-1}_{I, L_{a_\ell} R_b}\{(I - L_{a_\ell} R_b)(1)\} = (I - L_{a_\ell} R_b)^n(1) \\
= \; & L^n_{a_\ell}\{(L_a - R_b)^n(1)\} = L^n_{a_\ell}\delta^n_{a,b}(1) = (-1)^n L^n_{a_\ell}\{\delta^n_{b,a}(1)\} \\
= \; & (-1)^n L^n_{a_\ell}\{(R_b - L_a)^n(1)\} = (-1)^n (L_{a_\ell} R_b - I)^n(1) = (-1)^n \delta^n_{L_{a_\ell} R_b, I}(1),
\end{aligned}$$

it follows that if $d(a,b) = 0$, then $d(L_a, 1) = 0$. Hence, in either of the cases, $\sigma(a_\ell b) = \{1\}$, which then implies that b is left invertible. Since

$$d(a,b) = 0 \Longleftrightarrow d(a - \lambda, b - \lambda) = 0$$

for all complex λ, $d(a,b)=0$ also implies $\sigma_l(a)=\sigma_l(b)$. A similar argument works for the right spectrum, and we conclude that $d(a,b)=0$ implies

$$\sigma_\times(a)=\sigma_\times(b), \quad \sigma_\times=\sigma \text{ or } \sigma_l \text{ or } \sigma_r.$$

Considering L_a and R_a (etc.) as Banach space operators (acting on the Banach space BX), the equality $d(a,b)=0$ implies also that

$$\sigma_\times(L_a)=\sigma_\times(L_b) \text{ and } \sigma_\times(R_a)=\sigma_\times(R_b),$$

where $\sigma_\times=\sigma$ or σ_a or σ_s [13, proposition 3.4.11]. Since $\sigma(L_a)=\sigma(R_a)=\sigma(a)$, $\sigma_a(L_a)=\sigma_a(a)=\sigma_s(R_a)$ and $\sigma_s(L_a)=\sigma_s(a)=\sigma_a(R_a)$ whenever $\mathcal{A}=BX$, we have that if $a,b\in BX$ and $d(a,b)=0$ then $\sigma_\times(a)=\sigma_\times(b)$ for $\sigma_\times=\sigma$ or σ_a or σ_s. Summarising:

Theorem 3.1. *If $\mathcal{A}$ is a unital Banach algebra, a and $b\in\mathcal{A}$ and $d(a,b)=0$, then $\sigma_\times(a)=\sigma_\times(b)$, where $\sigma_\times=\sigma$ or σ_l or σ_r. Furthermore, if $\mathcal{A}=BX$, then also $\sigma_\times(a)=\sigma_\times(b)$ for $\sigma_\times=\sigma_a$ and σ_s.*

3.2. Homomorphisms and holomorphically Riesz operators.
Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras, and let $T:\mathcal{A}\to\mathcal{B}$ be a homomorphism:

$$T(ab)=T(a)T(b), a \text{ and } b\in\mathcal{A}, \text{ and } T(1)=1.$$

We say that *an element $a\in\mathcal{A}$ is $\times$–Fredholm (more precisely, T-$\times$–Fredholm), $a\in\Phi^\times$, if $T(a)\in\mathcal{B}_\times^{-1}$*. Here $\mathcal{B}_\times^{-1}$ stands for one of $\mathcal{B}_l^{-1}$ (= the left invertible elements of $\mathcal{B}$), $\mathcal{B}_r^{-1}$ (= the right invertible elements of $\mathcal{B}$) and $\mathcal{B}^{-1}$ (= the invertible elements of $\mathcal{B}$), and (correspondingly) $\Phi^\times$ stands for one of $\Phi^l=\{a\in\mathcal{A}: a \text{ is left Fredholm}\}$, $\Phi^r=\{a\in\mathcal{A}: a \text{ is right Fredholm}\}$ and $\Phi=\{a\in\mathcal{A}: a \text{ is Fredholm}\}$. *An element $a\in\mathcal{A}$ is T–Riesz if $Ta\in\mathrm{QN}(\mathcal{B})$, equivalently $a\in T^{-1}\mathrm{QN}(\mathcal{B})$*, where $\mathrm{QN}(\mathcal{B})=\{b\in\mathcal{B}: b \text{ is quasinilpotent}\}=\{b\in\mathcal{B}:\sigma(b,\mathcal{B})=\{0\}\}$.

Let $H(\sigma(a))$ denote the class of functions f which are holomorphic in a neighbourhood of $\sigma(a)$. We say that $a\in\mathcal{A}$ *is holomorphically Riesz if there exists an $f\in H(\sigma(a))$ such that $f(a)\in T^{-1}\mathrm{QN}(\mathcal{B})$ (equivalently, $a\in\mathrm{Holo}^{-1}T^{-1}\mathrm{QN}(\mathcal{B})$).* If $f(a)\in T^{-1}\mathrm{QN}(\mathcal{B})$ for a function f which is holomorphic on $\Omega\supset\sigma(a)$ and non–constant on open connected subsets of Ω (alternatively, if Ω is connected), then f has at most a finite number of zeros λ_i such that $f(Ta)=\prod_{i=1}^n(Ta-\lambda_i)^{\alpha_i}g_1(Ta)$, where the holomorphic function $g_1(z)\neq 0$ (on any neighbourhood of $\sigma(a)$). Let $\pi_a(z)$ denote the monic irreducible polynomial with roots λ_i, $1\le i\le n$, each repeated according to its multiplicity α_i. Then $f(z)=\pi_a(z)g(z)$, where $g(z)\neq 0$ on any neighbourhood of $\sigma(a)$ and $\pi_a(a)\in T^{-1}\mathrm{QN}(\mathcal{B})$. We have (see also [8; 12; 18]:

Lemma 3.2. *Let $a\in\mathcal{A}$, and let Ω be an open subset of the complex plane such that $\sigma(a)\subseteq\Omega$. If Ω is connected (more generally, if f is non–constant on connected* *components of* Ω*), then* $f(a) \in T^{-1}\mathrm{QN}(\mathcal{B})$ *if and only if there exists a minimal polynomial* π_a *such that* $\pi_a(a) \in T^{-1}\mathrm{QN}(\mathcal{B})$*.*

The following theorem is proved for polynomially Riesz (Banach algebra) elements, using a different argument, in [18, theorems 11.2 and 12.3] (see also [19, theorems 2.1, 2.3]). Let $\mathrm{Rad}(\mathcal{B}) = \{b \in \mathcal{B} : 1 - \mathcal{B}b \subset \mathcal{B}^{-1}\} = \{b \in \mathcal{B} : 1 - b\mathcal{B} \subset \mathcal{B}^{-1}\}$ denote the *radical of* $\mathcal{B}$.

Theorem 3.3. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are unital Banach algebras,* $a, b \in \mathcal{A}$*,* $T : \mathcal{A} \to \mathcal{B}$ *is a homomorphism,* $T(ab - ba) \in \mathrm{Rad}(\mathcal{B})$ *and* $f(Tb) \in \mathrm{QN}(\mathcal{B})$ *for some* $f \in H(\sigma(b))$*.*

(i) *If* $f(Ta) \in (\mathcal{B}_\times^{-1})$*, then* $T(a - b) \in (\mathcal{B}_\times^{-1})$ *(equivalently,* $a - b \in \Phi^\times$*).*
(ii) *If* $f^{-1}(0) \cap \sigma_\times(Ta) = \emptyset$*, then* $T(a - b) \in (\mathcal{B}_\times^{-1})$ *(equivalently,* $a - b \in \Phi^\times$*).*

PROOF. It suffices to prove the theorem for the case $T(ab - ba) = 0$; in the general case, $T(ab - ba) \in \mathrm{Rad}(\mathcal{B})$, one transfers the argument to the quotient algebra $\mathcal{B}/\mathrm{Rad}(\mathcal{B})$ (see Remark 3.4 below). Observe that $T(ab - ba) \in \mathrm{Rad}(\mathcal{B})$ implies $T(f(a)f(b) - f(b)f(a)) \in \mathrm{Rad}(\mathcal{B})$.

(i) The hypothesis $T(ab) = T(ba)$ implies $T(f(a)f(b)) = T(f(b)f(a))$. Set $Tf(b) = f(Tb) = d$ and $Tf(a) = f(Ta) = e$. Then $de - ed = 0$ and

$$\delta^n_{e-d,e}(1) = (-1)^n d^n = (-1)^n \delta^n_{e.e-d}(1) \Longrightarrow d(e - d, e) = 0.$$

Hence

$$\begin{aligned} & \sigma_\times(e - d) = \sigma_\times(Tf(a) - Tf(b))) = \sigma_\times\{(Ta - Tb)g(Ta, Tb)\} \\ = \quad & \sigma_\times(Tf(a)) \end{aligned}$$

for some analytic function g. This, together with $Tf(a) \in \mathcal{B}_\times^{-1}$, implies $T(a - b) \in \mathcal{B}_\times^{-1}$. Equivalently, $a - b \in \Phi^\times$.

(ii) The hypothesis $f^{-1}(0) \cap \sigma_\times(Ta) = \emptyset$ implies that $Tf(a) \in \mathcal{B}_\times^{-1}$. The proof now follows from the argument above. ■

Remark 3.4. (i) If (in the theorem above) we let $\Pi_p : \mathcal{B} \to \mathcal{B}/\mathrm{Rad}(\mathcal{B})$ denote the natural (projection) homomorphism, then $TaTb - TbTa \in \mathrm{Rad}(\mathcal{B})$ and $\Pi_p f(Tb) \in \mathrm{QN}(\mathcal{B}/\mathrm{Rad}(\mathcal{B}))$ imply $d(\Pi_p(f(Ta) - f(Tb)), \Pi_p f(Ta)) = 0$. Hence $\sigma(f(Ta), \mathcal{B}) = \sigma(\Pi_p f(Ta), \mathcal{B}/\mathrm{Rad}(\mathcal{B})) = \sigma(\Pi_p(f(Ta) - f(Tb)), \mathcal{B}/\mathrm{Rad}(\mathcal{B})) \subset \sigma(\Pi_p(f(Ta)), \mathcal{B}/\mathrm{Rad}(\mathcal{B})) - \sigma(\Pi_p(f(Tb)), \mathcal{B}/\mathrm{Rad}(\mathcal{B})) = \sigma(\Pi_p(f(Ta)), \mathcal{B}/\mathrm{Rad}(\mathcal{B})) = \sigma(f(Ta), \mathcal{B})$.

4. Algebra BX: Perturbations

Let $\Phi^\times(\mathcal{X})$ denote operators $A \in BX$ which are $\times$–Fredholm, where $\times$–Fredholm stands for one of left–Fredholm, right-Fredholm, upper–Fredholm, lower–Fredholm and (simply) Fredholm. Then $\sigma_f^\times(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi^\times(\mathcal{X})\}$ is the '$\times$–

essential (Fredholm) spectrum of $A \in BX$'. Let $H_c(\sigma(A)) = \{f \in H(\sigma(A)) : f$ is non–constant on the connected components of $\sigma(A)\}$. Recall from Lemma 3.2 that if an $A \in BX$ is holomorphically Riesz for some $f \in H_c(\sigma(A))$, $A \in (\mathrm{Holo})^1_{\mathrm{c}}(R(\mathcal{X}))$ then there exists a monic irreducible polynomial $\pi_A(A) = \prod_{i=1}^m (A - \mu_i)$, the minimal polynomial of A, such that $\pi_A(A) \in R(\mathcal{X})$.

Let $\mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ denote the closed two sided ideal,

$$\mathrm{Ptrb}(\Phi^\times(\mathcal{X})) = \{A \in BX : A + B \in \Phi^\times(\mathcal{X}) \text{ for every } B \in \Phi^\times(\mathcal{X})\},$$

of the perturbation class of $\Phi^\times(\mathcal{X})$. Then $\mathrm{Ptrb}(\Phi^{right}(\mathcal{X})) = \mathrm{Ptrb}(\Phi^{left}(\mathcal{X})) = \mathrm{Ptrb}(\Phi(\mathcal{X}))$ and $\mathrm{Ptrb}(\Phi^{upper}(\mathcal{X})) \cup \mathrm{Ptrb}(\Phi^{lower}(\mathcal{X})) \subseteq \mathrm{Ptrb}(\Phi(\mathcal{X}))$.

If Π denotes the Calkin homomorphism $\Pi : BX \to BX/K(\mathcal{X})$, $K(\mathcal{X})$ the ideal of compact operators on $\mathcal{X}$, then an $A \in BX$ is in $\Phi^{left}(\mathcal{X})$ (similarly, $\Phi^{right}(\mathcal{X})$, $\Phi(\mathcal{X})$) if it has a left–invertible (resp., right invertible, invertible) image in $BX/K(\mathcal{X})$ [6]. In the following we shall be interested in a further couple of homomorphisms of the algebra BX: the homomorphism $\Pi_p : BX \to BX/\mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ (effected by the natural projection of BX into the quotient algebra $BX/\mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$) and the homomorphism $\Pi_q : BX \to B(\mathcal{X}_q)$, $\mathcal{X}_q = \ell^\infty(\mathcal{X})/m(\mathcal{X})$, effecting the 'essential enlargement $A \to \Pi_q(A) = A_q$' (of [5] and [14, theorems 17.6 and 17.9]). It is clear from the definition that $AB - BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ for some $A, B \in BX$ if and only if $\Pi_p(AB - BA) = \Pi_p(A)\Pi_p(B) - \Pi_p(B)\Pi_p(A) = 0$. Furthermore, if a function f is holomorphic in a neighbourhood of $\sigma(A) \cup \sigma(B)$, then $AB - BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ implies $f(A)f(B) - f(B)f(A) \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$, and hence $\Pi_p(f(A)f(B) - f(B)f(A)) = 0$.

Theorem 4.1. *Suppose that $A, B \in BX$, where $B \in \mathrm{Poly}^{-1}(R(\mathcal{X}))$ (equivalently, $B \in (\mathrm{Holo})_c^{-1}(R(\mathcal{X}))$) with minimal polynomial $\pi_B(z) = \prod_{i=1}^m (z - \mu_i)$.*

(a) *If $AB - BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$, then $A - \lambda B \in \Phi^\times(\mathcal{X})$ for every scalar λ such that $\pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda\mu_i) \in \Phi^\times(\mathcal{X})$. If, additionally, (i) $\lambda = \lambda(t)$ is a continuous function from a connected subset $\boldsymbol{I}$ of the reals (into the set of complex numbers) such that $\lambda(t_1) = 0$ and $\lambda(t_2) = 1$ for some $t_1 < t_2 \in \boldsymbol{I}$, then $ind(A) = ind(A - B) = ind(A - \lambda(t)B)$ for all $t \in [t_1, t_2]$; (ii) $AB - BA = 0$ and $\pi_B(A, \lambda) \in \times$–Browder, then $A - \lambda B \in \times$–Browder.*

(b) *Suppose now that $AB - BA = 0$ and $\pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda(t)\mu_i) \in \Phi^\times(\mathcal{X})$, where $\lambda(t)$ is the continuous function of (a) above. If A has SVEP at 0 whenever $\times$–Fredholm is left or upper Fredholm (resp., A^* has SVEP at 0 whenever $\times$–Fredholm is right or lower Fredholm; both A and A^* have SVEP at 0 whenever $\times$–Fredholm is simply Fredholm), then $A - B \in \times$–Browder.*

Proof. Let T denote the homomorphism $\Pi_p : BX \to BX/\mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ in the proof of (a) (below), and let T denote the Calkin homomorphism Π if $\times =$ either left or right, respectively the homomorphism Π_q if $\times =$ either upper or lower, in the proof of (b).

(a) Let $\prod_{i=1}^m (A - \lambda\mu_i) = E$, $\lambda^m \pi_B(B) = F$, and $E - F = D$. Then the hypothesis $AB - BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ implies $T(EF - FE) = \Pi_p(EF - FE) = 0$, and

$$\begin{aligned} &\delta^n_{T(D),T(E)}(I) = \delta^{n-1}_{T(D),T(E)}((-1)T(F)) \\ = \quad &\cdots = (-1)^n T(F)^n = \cdots = (-1)^n \delta^n_{T(E),T(D)}(I). \end{aligned}$$

Hence, since TF is quasinilpotent,

$$d(T(D), T(E)) = 0,$$

which (since $E \in \Phi^\times(\mathcal{X})$) implies that $D \in \Phi^\times(\mathcal{X})$. Since $D = (A - \lambda B)g_\lambda(A, B) =)g_\lambda(A, B)(A - \lambda B)$ for some polynomial g_λ, it follows that $A - \lambda B \in \Phi^\times(\mathcal{X}))$. To complete the proof of (i), we note that both A and $A - B$ are $\times$–Fredholm. Since every locally constant function on a connected set is constant, we have from the local constancy of the index function that $\mathrm{ind}(A) = \mathrm{ind}(A - B) = \mathrm{ind}(A - \lambda(t)B)$ for all $t \in [t_1, t_2]$.

To prove (ii), suppose that $\pi_B(A, \lambda) \in \times$–Browder. Then $\pi_B(A, \lambda)$ has SVEP at 0 whenever $\times$–Browder is left or upper Browder, $(\pi_B(A, \lambda))^*$ has SVEP at 0 whenever $\times$–Browder is right or lower Browder, and both $\pi_B(A, \lambda)$ and $(\pi_B(A, \lambda))^*$ have SVEP at 0 whenever $\times$–Browder is Browder. We consider the first case: The other cases are similarly proved. It is clear from the proof of (i) above that $D = E - F = \pi_B(A, \lambda) - \lambda^m \pi_B(B) \in \Phi^\times(\mathcal{X})$. The commutativity of A and B implies the commutativity of E and F. Suppose now that E (is $\times$–Browder, and so) has SVEP at 0. Since $F \in R(\mathcal{X})$ and E, F commute, it follows from an application of 2.1 that D has SVEP at 0. Hence $D \in \times$–Browder. But then $D = (A - \lambda B)g_\lambda(A, B) = g_\lambda(A, B)(A - \lambda B)$ (for some function g_λ) implies (by Proposition 2.2) that $A - \lambda B \in \times$–Browder.

(b) Suppose now that A, B commute, $\pi_B(A, \lambda) \in \Phi^\times(\mathcal{X})$ and $\lambda = \lambda(t)$ is the continuous function of part (a). Then the argument above implies that $A - \lambda(t)B \in \Phi^\times(\mathcal{X})$ and $\mathrm{ind}(A - \lambda(t)B) = \mathrm{ind}(A)$ for all $t \in [t_1, t_2]$. Suppose further that A has SVEP at 0. (As before, we consider the case $\times =$ left or upper only. The proof for the case in which A^* has SVEP follows from a duality argument: Observe that $d(D, E) = 0 \Longleftrightarrow d(D^*, E^*) = 0$.) Then $A \in \Phi^\times(\mathcal{X})$ has finite ascent $\mathrm{asc}(A) < \infty$ [1, theorem 3.16], and hence $A^{-\infty}(0) \cap A^\infty(\mathcal{X}) = \{0\} = \overline{A^{-\infty}(0)} \cap A^\infty(\mathcal{X})$ [1, lemma 3.2]. The commutativity of A and B combined with the continuity of $\lambda(t) : [t_1, t_2] \to \mathbf{C}$ implies that $\lambda(t) \mapsto \overline{(A - \lambda(t)B)^{-\infty}(0)} \cap (A - \lambda(t)B)^\infty(\mathcal{X})$ is locally constant, hence constant on (the convex set) $[t_1, t_2]$. Consequently, $\overline{A^{-\infty}(0)} \cap A^\infty(\mathcal{X}) =$

$\{0\} = \overline{(A-B)^{-\infty}(0)} \cap (A-B)^{\infty}(\mathcal{X}) = (A-B)^{-\infty}(0) \cap (A-B)^{\infty}(\mathcal{X})$, which implies that $A-B$ has SVEP at 0 [1, corollary 2.26]. Since already $A-B \in \Phi^{\times}(\mathcal{X})$, $A-B$ is $\times$–Browder [1, theorem 3.44]. ■

A closer examination of the argument of the proof of theorem 4.1 shows that for operators B such that $\pi_B(B) \in R(\mathcal{X})$ either of the hypotheses $AB-BA \in \mathrm{Ptrb}(\Phi^{\times}(\mathcal{X}))$ and $AB-BA=0$ implies $\pi_B(A) \in \Phi^{\times}(\mathcal{X})$ if and only if $\pi_B(A) - \pi_B(B) \in \Phi^{\times}(\mathcal{X})$; furthermore, if $AB-BA=0$ and $\pi_B(A)$ has SVEP at 0 when $\times =$ left or upper (resp., $\pi_B(A^*)$ has SVEP at 0 when $\times =$ right or lower, both $\pi_B(A)$ and $\pi_B(A^*)$ have SVEP at 0 when $\Phi^{\times}(\mathcal{X}) = \Phi(\mathcal{X})$), then $\pi_B(A) - \pi_B(B)$ is $\times$–Browder whenever $\pi_B(A) \in \Phi^{\times}(\mathcal{X})$.

Remark 4.2. (i) It is immediate from the proof above that if the hypotheses of Theorem 4.1(a)(i) are satisfied, then A is $\times$–Weyl implies $A-B$ is $\times$–Weyl with the same index. (Recall that A is: left (resp., upper) Weyl if $A \in \Phi^{left}(\mathcal{X})$ (resp., $A \in \Phi^{upper}(\mathcal{X})$) and $\mathrm{ind}(A) \leq 0$, right (resp., lower) Weyl if $A \in \Phi^{right}(\mathcal{X})$ (resp., $A \in \Phi^{lower}(\mathcal{X})$) and $\mathrm{ind}(A) \geq 0$, and Weyl if $A \in \Phi(\mathcal{X})$ and $\mathrm{ind}(A) = 0$.) As observed in the proof of (b) above, if A and B are as in the statement of Theorem 4.1(b), then SVEP at 0 transfers from A to $A-B$. Consequently, if A in Theorem 4.1(b) is lower (or right) Weyl and has SVEP at 0, then $A-B$ is lower (resp., right) Weyl with SVEP at 0, consequently Weyl with SVEP at 0 and hence Browder.
(ii) Theorem 4.1 subsumes a number of extant results, amongst them a number of the results from [19, section 3] on *communicating operators* (see Corollary 4.3 below). We note here that if an $A \in BX$ is an *almost bounded below* (or, *almost surjective*) operator of [19, corollary 3.1], then A (resp., A^*) has SVEP at 0, and hence if (also) $A \in \Phi^{left}(\mathcal{X})$ (resp., $A \in \Phi^{right}(\mathcal{X})$), then $\mathrm{asc}(A) < \infty$ (resp., $\mathrm{dsc}(A) < \infty$).

Let $A, B \in BX$ such that $B \in \mathrm{Poly}^{-1}(R(\mathcal{X}))$ (with minimal polynomial π_B). Following [8, 11] and [19, 21], we say that A *is in $\times$–communication with B if there exists a continuous function $\varphi : [0,1] \to \mathbf{C}$ such that*

$$\varphi(0) = 0, \varphi(1) = 1 \text{ and } \mu\varphi([0,1]) \notin \sigma_f^{\times}(A)$$

for all $\mu \in \pi_B^{-1}(0)$. (Thus, A is in left–communication with B if $\varphi(t)\pi_B^{-1}(0) \notin \sigma_f^{l}(A)$ for all $0 \leq t \leq 1$.) Clearly, if A is in left–communication (resp., right–communication or simply communication) with B, and $\pi_B(z) = \prod_{i=1}^{m}(z-\mu_i)$, then $\prod_{i=1}^{m}(A - \varphi(t)\mu_i) \in \Phi^{\times}(\mathcal{X})$, $\Pi_p(\prod_{i=1}^{m}(A-\varphi(t)\mu_i))$ is left (resp., right or simply) invertible for all $0 \leq t \leq 1$ and $\Pi_p(\pi_B(B))$ is quasinilpotent in the quotient algebra. The homomorphism Π_q maps upper–Fredholm (resp., lower–Fredholm) operators onto bounded below (resp,. surjective) operators [14]. The following corollary of Theorem 4.1 appears in [19].

Corollary 4.3. *Suppose that $A, B \in BX$, $B \in \mathrm{Poly}^{-1}(R(\mathcal{X}))$ with minimal polynomial $\pi_B(z) = \prod_{i=1}^{m}(z-\mu_i)$, and A is in $\times$–communication with B.*

(i) *If* $AB-BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$*, then* $A-B \in \Phi^\times(\mathcal{X})$ *and* $ind(A-B) = ind(A)$.
(ii) *If* $AB - BA = 0$*, and* $asc(A) < \infty$ *(resp.,* $dsc(A) < \infty$*), then* $A - B \in \times-$*Browder for* $\times =$ *left or upper (resp.,* $\times =$ *right or lower).*

Proof. The hypothesis $\mathrm{asc}(A) < \infty$ (resp., $\mathrm{dsc}(A) < \infty$) implies A (resp., A^*) has SVEP at 0, and the hypothesis A is in $\times$–communication with B implies $\varphi(t)\pi_B^{-1}(0) \notin \sigma_f^\times(A)$ for all $0 \le t \le 1$. Theorem 4.1 applies. ∎

We remark here that the conclusion $A - B \in \Phi^\times(\mathcal{X})$ in (i) of the Corollary above does not require the full force of the hypothesis A *is in* $\times$*–communication with* B: the mere fact that $\pi_B(A) \in \Phi^\times(\mathcal{X})$ would do. Again, the hypothesis that $\pi_B(A) \in \Phi^\times(\mathcal{X})$ has SVEP at 0 if $\times =$ left or upper and $\pi_B(A^*)$ has SVEP at 0 if $\times =$ is right or lower, then $AB - BA = 0$ in (ii) of the Corollary is sufficient to guarantee $A - B \in \times$–Browder (once again the full force of the hypothesis A *is in* $\times$*–communication with* B is not required).

Theorem 4.1 has an extension to operators $A, B \in B(\mathcal{X})$ such that $f(B) \in R(\mathcal{X})$ for some $f \in H(\sigma(A) \cup \sigma(B))$. Recall from Proposition 2.2 that if $C, D \in B(\mathcal{X})$ are commuting operators, then CD is $\times$–Browder if and only if C and D are $\times$–Browder.

Theorem 4.4. *([21, theorem 2.1]) Let* $A, B \in B(\mathcal{X})$ *be such that* $f(B) \in R(\mathcal{X})$ *for some* $f \in H(\sigma(A) \cup \sigma(B))$.
(i) *If* $AB - BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ *and* $f(A) \in \Phi^\times(\mathcal{X})$*, then* $A - B \in \Phi^\times(\mathcal{X})$.
(ii) *If* $AB - BA = 0$ *and* $f(A) \in \times$*–Browder, then* $A - B \in \times$*–Browder for* $\times =$ *left or upper (resp.,* $\times =$ *right or lower).*

Proof.
(i) The hypothesis $AB - BA \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X}))$ implies

$$f(A)f(B) - f(B)f(A) \in \mathrm{Ptrb}(\Phi^\times(\mathcal{X})).$$

Choosing $\lambda = 1$ in Theorem 4.1, the hypothesis $f(B) \in R(\mathcal{X})$ then implies that

$$d(\Pi_p(f(A) - f(B)), \Pi_p f(A)) = 0.$$

Thus $f(A) \in \Phi^\times(\mathcal{X})$ implies $f(A) - f(B) = (A-B)g(A, B) \in \Phi^\times(\mathcal{X})$; hence $A - B \in \Phi^\times(\mathcal{X})$.

(ii) In this case $f(A)f(B) - f(B)f(A) = 0$ and $f(B) \in R(\mathcal{X})$ imply $f(A) - tf(B)$ and $f(A)$ have the same $\times$–Fredholm spectrum for all $0 \le t \le 1$. The hypothesis $f(A) \in \times$–Browder implies that either $f(A)$ (if $\times$ is either left or upper) or $f(A^*)$ (if $\times$ is either right or lower)has SVEP at 0. Since SVEP at 0 transfers from $f(A)$ to $f(A) - tf(B)$ (resp., from $f(A^*)$ to $f(A^*) - tf(B^*)$) for all $0 \le t \le 1$, see [3] or argue as in the proof of Theorem 4.1(ii), $f(A) -$

$f(B) \in \Phi^{\times}(\mathcal{X})$ (resp., $f(A^*) - f(B^*) \in \Phi^{\times}(\mathcal{X})$) implies $f(A) - f(B) \in \times-$Browder. Consequently, $A - B \in \times$–Browder. ■

The hypotheses of Theorem 4.4, or something akin to the hypotheses of Theorem 4.4, may be achieved in a number of ways (see theorems 2.2, 2.3 and 2.4 of [21]): We mention here just one such case.

Corollary 4.5. *Let $A, B \in BX$ with $f(B) \in R(\mathcal{X})$ for some $f \in H_c(\sigma(B))$ and $g(A) \in R(\mathcal{X})$ for some $g \in H_c(\sigma(A))$.*

(i) *If $AB - BA \in \mathrm{Ptrb}(\Phi^{\times}(\mathcal{X}))$ and $f^{-1}(0) \cap g^{-1}(0) = \emptyset$, then $A - B$ is Browder.*
(ii) *If $AB - BA = 0$ and $f^{-1}(0) \cap g^{-1}(0) = \emptyset$, then $A - B$ is Browder, $A - \mu$ is Browder at every $\mu \in f^{-1}(0)$ and $B - \lambda$ is Browder at every $\lambda \in g^{-1}(0)$.*

Proof. From $f(B) \in R(\mathcal{X})$ for some $f \in H_c(\sigma(B))$, $g(A) \in R(\mathcal{X})$ for some $g \in H_c(\sigma(A))$ and $f^{-1}(0) \cap g^{-1}(0) = \emptyset$ it follows that there exist (minimal) polynomials π_A and π_B such that $\pi_A(A), \pi_B(B) \in R(\mathcal{X})$ and $\pi_A^{-1}(0) \cap \pi_B^{-1}(0) = \emptyset$. According to [20, theorem 2.1] we have $\sigma_b(A) = \pi_A^{-1}(0)$ and $\sigma_b(B) = \pi_B^{-1}(0)$. From the spectral mapping theorem for Browder spectrum it follows that $\sigma_b(\pi_B(A)) = \pi_B(\sigma_b(A)) = \pi_B(\pi_A^{-1}(0))$, and since $\pi_B(\pi_A^{-1}(0))$ does not contain 0, we get that $\pi_B(A)$ is Browder. Similarly, $\pi_A(B)$ is Browder. (The fact that $\pi_B(A)$ and $\pi_A(B)$ are Browder implies without any further hypotheses that $A - \mu$ is Browder at every $\mu \in f^{-1}(0)$ and $B - \lambda$ is Browder at every $\lambda \in g^{-1}(0)$.) Furthermore, the operators $A, B, \pi_B(A), \pi_B(B)$ (and their adjoints) have SVEP (everywhere). If $AB - BA \in \mathrm{Ptrb}(\Phi^{\times}(\mathcal{X})$, then (as in the proof of Theorem 4.1) $\pi_B(A) - \pi_B(B) \in \Phi^{\times}(\mathcal{X})$. Hence, since SVEP at 0 transfers from $\pi_B(A)$ and its adjoint to $\pi_B(A) - \pi_B(B)$ and its adjoint (respectively), $\pi_B(A) - \pi_B(B)$ is Browder, and this in turn implies (see Proposition 2.2) that $A - B$ is Browder. The proof of (ii) is now evident. ■

Corollary 4.5 strengthens the conclusions of [21, theorem 2.4].

5. An application to normal operators

An operator $A \in BX$ is *normal* if $A = H + iK$ for some commuting Hermitian operators $H, K \in BX$. Trivially, $A \in BX$ normal implies $A - \lambda$ normal for all complex λ. Normal operators A satisfy the (James–Birkhoff) orthogonality property

$$A^{-1}(0) \perp \overline{A(\mathcal{X})} : ||x|| \leq ||x + Ay||, \text{all } \ x \in A^{-1}(0) \ \text{ and } \ y \in \mathcal{X}$$

[4, page 25]. Hence, for normal $A \in BX$, $A^{-1}(0) \cap A(\mathcal{X}) = \{0\}$, consequently $\mathrm{asc}(A) \leq 1$. Since the adjoint operator A^* of a normal operator $A \in BX$ is again normal, both A and A^* have SVEP (everywhere) for a normal operator $A \in BX$. Thus, for normal $A \in BX$, $\sigma(A) = \sigma(A^*) = \sigma_a(A)$, A is semi–Fredholm if and only if it is Fredholm (indeed, Weyl, even Browder), A is bounded below implies A is invertible, and if $A(\mathcal{X})$ is closed then $\mathcal{X} = A^{-1}(0) \oplus A(\mathcal{X})$ (implies $\mathrm{dsc}(A) < \infty$). The following theorem is a generalisation of [19, theorem 3.8].

Theorem 5.1. *Let $B \in \mathrm{Poly}^{-1}(R(\mathcal{X}))$, with minimal polynomial $\pi_B(z) = \prod_{i=1}^{m}(z - \mu_i)$, and let $A \in BX$ be a normal operator.*

(i) *If If $\pi_B(A)$ is semi–Fredholm and $AB - BA = 0$, then $A - B$ is Browder.*
(ii) *If $\pi_B(A,t) = \prod_{i=1}^{m}(A - t\mu_i)$ is semi–Fredholm for all $t \in [0,1]$ and $AB - BA \in \mathrm{Ptrb}(\Phi(\mathcal{X}))$, then $A - B \in \Phi(\mathcal{X})$ and ind$(A - B) = 0$ (so that $A - B$ is Weyl).*

PROOF.
(i) The hypotheses $\pi_B(A)$ is semi–Fredholm (hence, Fredholm), $AB - BA = 0$ (hence, $\pi_B(A)\pi_B(B) - \pi_B(B)\pi_B(A) = 0$) and $\pi_B(B) \in R(\mathcal{X})$ imply $\pi_B(A) - \pi_B(B) \in \Phi(\mathcal{X})$. Since both normal operator A and its adjoint A^* have SVEP (everywhere), $\pi_B(A)$ and $\pi_B(A^*)$ have SVEP at 0. Hence $\pi_B(A) - \pi_B(B)$ and $\pi_B(A^*) - \pi_B(B^*) = (\pi_B(A) - \pi_B(B))^*$ have SVEP at 0. Consequently, $\pi_B(A) - \pi_B(B)$ is Browder [1, theorem 3.52]. Hence $A - B$ is Browder.

(ii) Choose $\lambda = \lambda(t) = t$, $t \in [0,1]$, in the proof of Theorem 4.1, and conclude from $\pi_B(A,t) - \pi_B(B)$ is semi–Fredhom that $A - B$ is semi–Fredhom with ind$(A - B)$ = ind(A). Since both A and A^* have SVEP at 0, ind$(A) = 0$. ■

Theorem 5.1 has an extension to paranormal operators $A \in BX$: $||Ax||^2 \leq ||A^2x||$ for all unit vectors $x \in \mathcal{X}$ [10, section 54]. Since paranormal operators have (finite) ascent ≤ 1, hence SVEP at 0, an upper (or left) semi–Fredholm paranormal operator is upper (resp., left) Browder with index ≤ 0 and a lower (resp., right) semi–Fredholm paranormal operator with index ≥ 0 is Browder.

Corollary 5.2. *Let $B \in \mathrm{Poly}^{-1}(R(\mathcal{X}))$ (with minimal polynomial π_B). Suppose that $A \in BX$ is a paranormal operator such that $A - \mu$ is $\times$–Fredholm for all $\mu \in t\pi_B^{-1}(0)$; $t \in [0,1]$. If:*

(i) *$AB - BA \in \mathrm{Ptrb}(\Phi(\mathcal{X}))$, then $A - B \in \Phi^{\times}(\mathcal{X})$ and ind$(A - B) =$ ind(A).*

(ii) *$AB - BA = 0$, then A and $A - B$ are $\times$–Browder. Furthermore, if $\pi_B(A,t) = \prod_{i=1}^{m}(A - t\mu_i)$ is lower or right semi–Fredholm and ind$(\pi_B(A,t)) \geq 0$, $t \in [0,1]$, then $A - B$ is Browder.*

PROOF. The proof of (i) is clear from that of Theorem 4.1. (Observe that if $\Phi^{\times}(\mathcal{X})$ = right or lower Fredholm, then, see the argument below, $A - B$ is Fredholm with ind$(A - B) \leq 0$.)

To complete the proof of (ii), one observes that a paranormal operator has SVEP at every point μ such that $A - \mu \in \Phi^{\times}(\mathcal{X})$ [7, corollary 2.10]. This, if $A - \mu$ is lower (or, right) semi–Fredholm implies ind$(A - \mu) \leq 0$, which then forces $A - \mu$ to be Fredholm. Thus if $\pi_B(A,t)$ is lower or right semi–Fredholm, then $\pi_B(A,t) \in \Phi(\mathcal{X})$ with ind$(\pi_B(A,t)) \leq 0$. If we now also have that ind$(\pi_B(A,t)) \geq 0$, then $\pi_B(A,t)$

is Weyl with SVEP (at 0). Consequently, $\pi_B(A,t) - t^m \pi_B(B)$ is Browder and this implies $A - B$ is Browder. ■

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