Weighted partial isometries and weighted–EP elements in C^*-algebras

Dijana Mosić and Dragan S. Djordjević*

Abstract

We investigate weighted partial isometries, weighted–EP elements, weighted star-dagger, weighted normal and weighted Hermitian elements of C^*-algebras.

Key words and phrases: weighted–EP elements, Moore–Penrose inverse, group inverse, C^*-algebra.
2010 *Mathematics subject classification*: 46L05, 47A05, 15A09.

1 Introduction

In this paper we introduce and characterize weighted partial isometry, weighted star-dagger, weighted normal and weighted Hermitian element. Then, the equivalent conditions for an element of C^*-algebra to be a weighted partial isometry and weighted–EP element are investigated.

Let $\mathcal{A}$ be a unital C^*–algebra with the unit 1. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. The set of all regular elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^-$. An element $a \in \mathcal{A}$ satisfying $a^* = a$ is called *symmetric* (or *Hermitian*). An element $x \in \mathcal{A}$ is positive if $x = y^*y$ for some $y \in \mathcal{A}$. Alternatively, $x \in \mathcal{A}$ is positive if x is Hermitian and $\sigma(x) \subseteq [0, +\infty)$, where the spectrum of x is denoted by $\sigma(x)$.

An element $a \in \mathcal{A}$ is *group invertible* if there exists $a^\# \in \mathcal{A}$ such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad aa^\# = a^\#a.$$

Recall that $a^\#$ is uniquely determined by these equations. The group inverse $a^\#$ exists if and only if $a\mathcal{A} = a^2\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^2$, or if and only if $a \in a^2\mathcal{A} \cap \mathcal{A}a^2$ (see [8, 16]). We use $\mathcal{A}^\#$ to denote the set of all group invertible

*The authors are supported by the Ministry of Science, Republic of Serbia, grant no. 174007.

elements of $\mathcal{A}$. The group inverse $a^{\#}$ double commutes with a, that is, $ax = xa$ implies $a^{\#}x = xa^{\#}$ [4, 7].

An element $a^{\dagger} \in \mathcal{A}$ is the *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{A}$, if the following hold [17]:

$$aa^{\dagger}a = a, \quad a^{\dagger}aa^{\dagger} = a^{\dagger}, \quad (aa^{\dagger})^* = aa^{\dagger}, \quad (a^{\dagger}a)^* = a^{\dagger}a.$$

There is at most one $a^{\dagger}$ such that above conditions hold (see [9, 11]). The set of all Moore–Penrose invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{\dagger}$.

Theorem 1.1. [9] *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

See also [8], [11] and [16] for further interesting properties of generalized inverses in rings and C^*-algebras.

Definition 1.1. Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be invertible positive elements in $\mathcal{A}$. The element $a \in \mathcal{A}$ has the weighted MP-inverse with weights e, f if there exists $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^{\dagger}_{e,f}$ if it exists [4, 7]. The set of all weighted MP-invertible elements of $\mathcal{A}$ with weights e, f, will be denoted by $\mathcal{A}^{\dagger}_{e,f}$.

Theorem 1.2. [4] *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then the unique weighted MP-inverse $a^{\dagger}_{e,f}$ exists.*

Define the mapping $x \mapsto x^{*e,f} = e^{-1}x^*f$, for all $x \in \mathcal{A}$. Notice that $(*, e, f) : \mathcal{A} \to \mathcal{A}$ is not an involution, because in general $(xy)^{*e,f} \neq y^{*e,f}x^{*e,f}$. This fact is important for further investigation in this paper: otherwise, the weighted MP-inverse would reduce to the ordinary MP-inverse, and all other characterizations would be elementary.

The following result is frequently used in the rest of the paper.

Theorem 1.3. [14] *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be positive invertible elements of $\mathcal{A}$. For any $a \in \mathcal{A}^-$, the following is satisfied:*

(a) $(a^{\dagger}_{e,f})^{\dagger}_{f,e} = a$;

(b) $(a^{*f,e})^{\dagger}_{f,e} = (a^{\dagger}_{e,f})^{*e,f}$;

(c) $a^{*f,e} = a^{\dagger}_{e,f}aa^{*f,e} = a^{*f,e}aa^{\dagger}_{e,f}$;

(d) $a^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = a^{\dagger}_{e,f}a$;

(e) $(a^{\dagger}_{e,f})^{*e,f}a^{*f,e} = aa^{\dagger}_{e,f}$;

(f) $(a^{*f,e}a)^{\dagger}_{f,f} = a^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*e,f}$;

(g) $(aa^{*f,e})^{\dagger}_{e,e} = (a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}$;

(h) $a^{\dagger}_{e,f} = (a^{*f,e}a)^{\dagger}_{f,f}a^{*f,e} = a^{*f,e}(aa^{*f,e})^{\dagger}_{e,e}$;

(i) $(a^{*e,f})^{\dagger}_{f,e} = a(a^{*f,e}a)^{\dagger}_{f,f} = (aa^{*f,e})^{\dagger}_{e,e}a$.

Using Theorem 1.3, we can easily prove the following lemma.

Lemma 1.1. *Let $a \in \mathcal{A}^{-}$ and let e, f be invertible positive elements in $\mathcal{A}$. Then*

(i) $a^{\dagger}_{e,f}\mathcal{A} = a^{*f,e}\mathcal{A}$;

(ii) $(a^{\dagger}_{e,f})^{*e,f}\mathcal{A} = a\mathcal{A}$.

Now, we state the definition of weighted–EP elements and some characterizations of weighted–EP elements.

Definition 1.2. [14] An element $a \in \mathcal{A}$ is said to be weighted–EP with respect to two invertible positive elements $e, f \in \mathcal{A}$ (or weighted–EP w.r.t. (e,f)) if both ea and af^{-1} are EP, that is $a \in \mathcal{A}^{-}$, $ea\mathcal{A} = (ea)^{*}\mathcal{A}$ and $af^{-1}\mathcal{A} = (af^{-1})^{*}\mathcal{A}$.

Theorem 1.4. [14] *Let $\mathcal{A}$ be a unital C^{*}–algebra and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^{-}$ the following statements are equivalent:*

(i) *a is weighted–EP w.r.t. (e,f);*

(ii) $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$;

(iii) $a^{\dagger}_{e,f} = a(a^{\dagger}_{e,f})^{2} = (a^{\dagger}_{e,f})^{2}a$;

(iv) *$a \in \mathcal{A}^{\#}$ and $a^{\#} = a^{\dagger}_{e,f}$;*

(v) *$a \in \mathcal{A}^{\#}$, and both $eaa^{\#}, faa^{\#}$ are Hermitian.*

In [2], O.M. Baksalary, G.P.H. Styan and G. Trenkler used the representation of complex matrices provided in [10] to explore various classes of matrices, such as partial isometries, EP and star-dagger elements. Inspired by [2], characterizations of partial isometries, EP elements and star-dagger elements in rings with involution were investigated in [12].

In [18], Tian and Wang defined weighted–EP matrices and presented a lot of characterizations of weighted–EP matrices using various formulas for a rank of a complex matrix. In [14], weighted–EP elements of C^*-algebras were studied, extending the results from [18]. Furthermore, characterizations of weighted–EP elements in C^*-algebras in terms of factorizations were presented in [15].

Various characterizations of MP-invertible normal and Hermitian elements in rings with involution were investigated in [13]. Some of these results were proved for complex square matrices in [3], using the rank of a matrix, or in [1], using an elegant representation of square matrices as the main technique. Moreover, the operator analogues of these results were proved in [5] and [6] for linear bounded operators on Hilbert spaces, using the operator matrices as the main tool.

In this paper, we generalize the notation of partial isometries, star–dagger, normal and Hermitian elements. The paper is organized as follows. In Section 2, the definition of weighted partial isometries is introduced and a group of equivalent conditions for an element of C^*-algebra to be a weighted partial isometry and weighted–EP elements are given. In Section 3, we study weighted star-dagger and weighted-EP elements of C^*-algebra. In Section 4, some characterizations of regular weighted normal elements in C^*-algebra are presented. In Section 5, both regular and group invertible weighted Hermitian elements in C^*-algebra are investigated.

2 Characterizations of weighted partial isometries

Let $a \in \mathcal{A}^-$ and let e, f be invertible positive elements in $\mathcal{A}$. An element $a \in \mathcal{A}^-$ satisfying $a^{*f,e} = a^\dagger_{e,f}$ is called *a weighted partial isometry* with respect to elements e, f (or a weighted partial isometry w.r.t. (e,f)). An element $a \in \mathcal{A}^-$ satisfying $a^{*f,e}a^\dagger_{e,f} = a^\dagger_{e,f}a^{*f,e}$ is called *weighted star–dagger* with respect to elements e, f (or weighted star–dagger w.r.t. (e,f)). An element $a \in \mathcal{A}$ satisfying $aa^{*f,e} = a^{*f,e}a$ is called *weighted normal* with respect to elements e, f (or weighted normal w.r.t. (e,f)). An element $a \in \mathcal{A}$ satisfying $a = a^{*f,e}$ is called *weighted Hermitian* with respect to elements e, f (or weighted Hermitian w.r.t. (e,f)).

For $e = f = 1$, the following result is well known for matrices, Hilbert space operators and elements of C^*-algebras and rings with involution.

Lemma 2.1. *Let $a \in \mathcal{A}^-$ and let e, f be invertible positive elements in $\mathcal{A}$. Then a is weighted normal w.r.t. (e,f) if and only if $aa^\dagger_{e,f} = a^\dagger_{e,f}a$ and $a^{*f,e}a^\dagger_{e,f} = a^\dagger_{e,f}a^{*f,e}$.*

Proof. Assume that a is weighted normal w.r.t. (e,f). By Theorem 1.3, we get

$$a^{*f,e}a^\dagger_{e,f} = a^\dagger_{e,f}(aa^{*f,e})a^\dagger_{e,f} = a^\dagger_{e,f}a^{*f,e}aa^\dagger_{e,f} = a^\dagger_{e,f}a^{*f,e}.$$

Now,

$$\begin{aligned} aa^\dagger_{e,f} &= (a^\dagger_{e,f})^{*e,f}a^{*f,e} = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}(aa^{*f,e}) = (a^\dagger_{e,f})^{*e,f}(a^\dagger_{e,f}a^{*f,e})a \\ &= (a^\dagger_{e,f})^{*e,f}a^{*f,e}a^\dagger_{e,f}a = aa^\dagger_{e,f}a^\dagger_{e,f}a \end{aligned}$$

and

$$\begin{aligned} a^\dagger_{e,f}a &= a^{*f,e}(a^\dagger_{e,f})^{*e,f} = (a^{*f,e}a)a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f} = a(a^{*f,e}a^\dagger_{e,f})(a^\dagger_{e,f})^{*e,f} \\ &= aa^\dagger_{e,f}a^{*f,e}(a^\dagger_{e,f})^{*e,f} = aa^\dagger_{e,f}a^\dagger_{e,f}a \end{aligned}$$

imply $aa^\dagger_{e,f} = a^\dagger_{e,f}a$.

Conversely, suppose that $aa^\dagger_{e,f} = a^\dagger_{e,f}a$ and $a^{*f,e}a^\dagger_{e,f} = a^\dagger_{e,f}a^{*f,e}$. Then, we have

$$aa^{*f,e} = aa^{*f,e}(aa^\dagger_{e,f}) = a(a^{*f,e}a^\dagger_{e,f})a = (aa^\dagger_{e,f})a^{*f,e}a = a^\dagger_{e,f}aa^{*f,e}a = a^{*f,e}a.$$

Thus, a is weighted normal w.r.t. (e,f). □

By Lemma 2.1, we obtain the following result.

Lemma 2.2. *Let e, f be invertible positive elements in $\mathcal{A}$. If $a \in \mathcal{A}^-$ is weighted normal w.r.t. (e,f), then a is weighted–EP w.r.t. (e,f).*

Now, we present necessary and sufficient conditions for a regular element a of C^*-algebra to be a weighted partial isometry w.r.t. (e,f).

Theorem 2.1. *Let $a \in \mathcal{A}^-$ and let e, f be invertible positive elements in $\mathcal{A}$. Then the following statements are equivalent:*

(i) *a is a weighted partial isometry w.r.t. (e,f);*

(ii) *$aa^{*f,e} = aa^\dagger_{e,f}$;*

(iii) $a^{*f,e}a = a^{\dagger}_{e,f}a$;

(iv) $aa^{*f,e}a = a$;

(v) $a^{*f,e}aa^{*f,e} = a^{*f,e}$;

(vi) $a^{*f,e}$ *is a weighted partial isometry w.r.t.* (f,e);

(vii) $a^{\dagger}_{e,f}$ *is a weighted partial isometry w.r.t.* (f,e);

(viii) $a^{*f,e}ax = x$, *for* $x \in a^{*f,e}\mathcal{A}$;

(ix) $aa^{*f,e}x = x$, *for* $x \in a\mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii): Since a is a weighted partial isometry w.r.t. (e,f), i.e. $a^{*f,e} = a^{\dagger}_{e,f}$, then $aa^{*f,e} = aa^{\dagger}_{e,f}$. Thus, the condition (ii) is satisfied.

(ii) $\Rightarrow$ (iii): By the assumption $aa^{*f,e} = aa^{\dagger}_{e,f}$ and Theorem 1.3, we obtain

$$a^{*f,e}a = a^{\dagger}_{e,f}(aa^{*f,e})a = a^{\dagger}_{e,f}aa^{\dagger}_{e,f}a = a^{\dagger}_{e,f}a.$$

So, the condition (iii) holds.

(iii) $\Rightarrow$ (i): The equality $a^{*f,e}a = a^{\dagger}_{e,f}a$ and Theorem 1.3 imply

$$a^{*f,e} = (a^{*f,e}a)a^{\dagger}_{e,f} = a^{\dagger}_{e,f}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}.$$

Hence, the element a is a weighted partial isometry w.r.t. (e,f).

(iv) $\Rightarrow$ (ii): Multiplying $aa^{*f,e}a = a$ from the right side by $a^{\dagger}_{e,f}$, we get the condition (ii).

(ii) $\Rightarrow$ (iv): Obviously.

(v) $\Rightarrow$ (iii): Multiplying $a^{*f,e}aa^{*f,e} = a^{*f,e}$ by $(a^{\dagger}_{e,f})^{*e,f}$ from the right side, we obtain (iii).

(iii) $\Rightarrow$ (v): This part can easily be verified.

(i) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii): It follows by Theorem 1.3.

(viii) $\Rightarrow$ (i): Suppose that $a^{*f,e}ax = x$, for $x \in a^{*f,e}\mathcal{A}$. Since, by Lemma 1.1, $a^{\dagger}_{e,f} \in a^{\dagger}_{e,f}\mathcal{A} = a^{*f,e}\mathcal{A}$, then $a^{*f,e} = a^{*f,e}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}$.

(v) $\Rightarrow$ (viii): Let $a^{*f,e}aa^{*f,e} = a^{*f,e}$ and let $x \in a^{*f,e}\mathcal{A}$. Now, $x = a^{*f,e}y$, for some $y \in a^{*f,e}\mathcal{A}$ and

$$a^{*f,e}ax = a^{*f,e}aa^{*f,e}y = a^{*f,e}y = x.$$

(iv) $\Rightarrow$ (ix) $\Rightarrow$ (vii): Similarly as (v) $\Rightarrow$ (viii) $\Rightarrow$ (i).

$\square$

In the following theorem we assume that the element a is both regular and group invertible. Then, we give the conditions involving $a^{\dagger}_{e,f}$, $a^{\#}$ and $a^{*f,e}$ to ensure that a is a weighted partial isometry w.r.t. (e,f).

Theorem 2.2. *Let $a \in \mathcal{A}^{-} \cap \mathcal{A}^{\#}$ and let e, f be invertible positive elements in $\mathcal{A}$. Then a is a weighted partial isometry w.r.t. (e,f) if and only if one of the following equivalent conditions holds:*

(i) $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\#}$;

(ii) $a^{\#}a^{*f,e} = a^{\#}a^{\dagger}_{e,f}$;

(iii) $aa^{*f,e}a^{\#} = a^{\#}$;

(iv) $a^{\#}a^{*f,e}a = a^{\#}$;

(v) $a^{n}a^{*f,e} = a^{n}a^{\dagger}_{e,f}$, *for any/some integer $n \geq 2$;*

(vi) $a^{*f,e}a^{n} = a^{\dagger}_{e,f}a^{n}$, *for any/some integer $n \geq 2$;*

(vii) $a^{*f,e}(a^{\#})^{n} = a^{\dagger}_{e,f}(a^{\#})^{n}$, *for any/some integer $n \geq 2$;*

(viii) $(a^{\#})^{n}a^{*f,e} = (a^{\#})^{n}a^{\dagger}_{e,f}$, *for any/some integer $n \geq 2$;*

(ix) $aa^{*f,e}(a^{\#})^{n} = (a^{\#})^{n}$, *for any/some integer $n \geq 2$;*

(x) $(a^{\#})^{n}a^{*f,e}a = (a^{\#})^{n}$, *for any/some integer $n \geq 2$.*

Proof. If a is a weighted partial isometry w.r.t. (e,f), then $a^{*f,e} = a^{\dagger}_{e,f}$. It is not difficult to show that conditions (i)-(x) hold.

Conversely, to conclude that a is a weighted partial isometry w.r.t. (e,f), we show that either the condition $a^{*f,e} = a^{\dagger}_{e,f}$ is satisfied, or one of the preceding already established condition of this theorem or Theorem 2.1 holds.

(i) The condition $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\#}$ and Theorem 1.3 give

$$a^{*f,e} = a^{*f,e}aa^{\dagger}_{e,f} = (a^{*f,e}a^{\#})aaa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\#}aaa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}.$$

(ii) From the equality $a^{\#}a^{*f,e} = a^{\#}a^{\dagger}_{e,f}$, we have

$$a^{*f,e} = a^{\dagger}_{e,f}aa^{*f,e} = a^{\dagger}_{e,f}aa(a^{\#}a^{*f,e}) = a^{\dagger}_{e,f}aaa^{\#}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}.$$

(iii) Multiplying the hypothesis $aa^{*f,e}a^{\#} = a^{\#}$ by $a^{\dagger}_{e,f}$ from the left side, we get $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\#}$. So, the condition (i) holds.

(iv) Multiplying $a^{\#}a^{*f,e}a = a^{\#}$ by $a^{\dagger}_{e,f}$ from the right side, we obtain the condition (ii).

(v) Multiplying $a^n a^{*f,e} = a^n a^{\dagger}_{e,f}$, $n \geq 1$, from the left side by $(a^{\#})^{n-1}$, we observe that $aa^{*f,e} = aa^{\dagger}_{e,f}$. Hence, the equality (ii) of Theorem 2.1 is satisfied.

(vi) The equality $a^{*f,e}a^n = a^{\dagger}_{e,f}a^n$, $n \geq 1$, implies the condition (iii) of Theorem 2.1, in the same way as the previous part.

(vii) If we multiply the assumption $a^{*f,e}(a^{\#})^n = a^{\dagger}_{e,f}(a^{\#})^n$, $n \geq 1$, by a^{n-1} from the right side, we get $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\#}$. Therefore, the statement (i) holds.

Similarly, we can show conditions (viii)-(x). $\square$

Observe that if $e = f = 1$ in Theorem 2.1 and Theorem 2.2 we recover results from [2].

In the following result we study equivalent conditions for an element a of C^*-algebra to be a weighted partial isometry w.r.t. (e,f) and weighted–EP w.r.t. (e,f).

Theorem 2.3. *Let $a \in \mathcal{A}^- \cap \mathcal{A}^{\#}$ and let e, f be invertible positive elements in $\mathcal{A}$. Then a is a weighted partial isometry w.r.t. (e,f) and weighted–EP w.r.t. (e,f) if and only if one of the following equivalent conditions holds:*

(i) *a is a weighted partial isometry w.r.t. (e,f) and weighted normal w.r.t. (e,f);*

(ii) $a^{*f,e} = a^{\#}$;

(iii) $aa^{*f,e} = a^{\dagger}_{e,f}a$;

(iv) $a^{*f,e}a = aa^{\dagger}_{e,f}$;

(v) $aa^{*f,e} = aa^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(vi) $a^{*f,e}a = aa^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(vii) $a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(viii) $a^{\dagger}_{e,f}a^{*f,e} = a^{\#}a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(ix) $a^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(x) $a^{*f,e}a^{\dagger}_{e,f} = a^{\#}a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xi) $a^{*f,e}a^{\#} = a^{\#}a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xii) $a^{\#}a^{*f,e} = a^{\dagger}_{e,f}a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xiii) $a^{*f,e}a^{\dagger}_{e,f} = a^{\#}a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xiv) $a^{\dagger}_{e,f}a^{*f,e} = a^{\#}a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xv) $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xvi) $a^{\#}a^{*f,e} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xvii) $a^{*f,e}a^{\#} = a^{\#}a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xviii) $a^{\#}a^{*f,e} = a^{\#}a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xix) $aa^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}a$;

(xx) $aa^{*f,e}a^{\dagger}_{e,f} = a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xxi) $a^{\dagger}_{e,f}a^{*f,e}a = a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xxii) $aa^{*f,e}a^{\#} = a^{\dagger}_{e,f}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xxiii) $a^{\#}a^{*f,e}a = a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xxiv) $aa^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f} = a^{*f,e}a^{\dagger}_{e,f}a$;

(xxv) $a^{*f,e}a^2 = a$ *and* $a = aaa^{\dagger}_{e,f}$;

(xxvi) $a^2a^{*f,e} = a$ *and* $a = a^{\dagger}_{e,f}aa$;

(xxvii) $aa^{\dagger}_{e,f}a^{*f,e} = a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xxviii) $a^{*f,e}a^{\dagger}_{e,f}a = a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$.

Proof. Suppose that a is a weighted partial isometry w.r.t. (e,f) and weighted–EP w.r.t. (e,f), then $a^{*f,e} = a^{\dagger}_{e,f} = a^{\#}$. It is not difficult to verify that conditions (i)-(xvi) hold.

Conversely, we will prove that a is a weighted partial isometry w.r.t. (e,f) and weighted–EP w.r.t. (e,f), or we will show that the element a satisfies one of the preceding already established conditions of this theorem.

(i) If a is a weighted partial isometry w.r.t. (e,f) and weighted normal w.r.t. (e,f), then a is a weighted partial isometry w.r.t. (e,f) and weighted–EP w.r.t. (e,f), by Lemma 2.1.

(ii) The hypothesis $a^{*f,e} = a^{\#}$ gives $aa^{*f,e} = a^{*f,e}a$, that is, the element a is weighted normal w.r.t. (e,f). By Lemma 2.1, a is weighted–EP and $a^{\dagger}_{e,f} = a^{\#} = a^{*f,e}$. So, a is a weighted partial isometry w.r.t. (e,f).

(iii) Using the equality $aa^{*f,e} = a^{\dagger}_{e,f}a$, we get

$$\begin{aligned} aa^{\dagger}_{e,f} &= aa^{\dagger}_{e,f}a = (aa^{*f,e})(a^{\dagger}_{e,f})^{*e,f} = a^{\dagger}_{e,f}a(a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f} \\ &= a^{\dagger}_{e,f}a(a^{\dagger}_{e,f}a)(a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}aaa^{*f,e}(a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f} \\ &= a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}, \end{aligned}$$

and

$$a^{\dagger}_{e,f}a = aa^{*f,e} = (aa^{*f,e})aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}.$$

Hence, $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$, i.e a is weighted–EP w.r.t. (e,f). By (iii), $aa^{\dagger}_{e,f} = aa^{*f,e}$, and, by the condition (ii) of Theorem 2.1, a is a weighted partial isometry w.r.t. (e,f).

(iv) This part can be proved similarly as part (iii).

(v) Since $aa^{*f,e} = aa^{\#}$, then $eaa^{\#} = eaf^{-1}a^{*}e$ is Hermitian. The condition $a = a^{\dagger}_{e,f}aa$ gives that $faa^{\#} = fa^{\dagger}_{e,f}a$ is Hermitian. Now, by Theorem 1.4, a is weighted–EP w.r.t. (e,f). Thus, $aa^{*f,e} = aa^{\dagger}_{e,f}$ and, by Theorem 2.1(ii), a is a weighted partial isometry w.r.t. (e,f).

(vi) The equalities $a^{*f,e}a = aa^{\#}$ and $a = aaa^{\dagger}_{e,f}$ imply that $faa^{\#} = a^{*}ea$ and $ea^{\#}a = eaa^{\dagger}_{e,f}$ are Hermitian. So, a is weighted–EP w.r.t. (e,f), by Theorem 1.4, and $a^{*f,e}a = aa^{\dagger}_{e,f}$, i.e. (iv) holds.

(vii) From the equality $a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\#}$, we obtain

$$\begin{aligned} aa^{\dagger}_{e,f} &= a^{2}a^{\#}a^{\dagger}_{e,f} = a^{2}a^{\dagger}_{e,f}aa^{\#}a^{\dagger}_{e,f} = a^{2}(a^{\dagger}_{e,f}a^{\#})aa^{\dagger}_{e,f} = a^{2}a^{*f,e}a^{\dagger}_{e,f}aa^{\dagger}_{e,f} \\ &= a^{2}(a^{*f,e}a^{\dagger}_{e,f}) = a^{2}a^{\dagger}_{e,f}a^{\#} = a^{2}a^{\dagger}_{e,f}a(a^{\#})^{2} = aa^{\#}. \end{aligned}$$

The assumption $a = a^{\dagger}_{e,f}aa$ implies $aa^{\#} = a^{\dagger}_{e,f}a$. Since $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$, we deduce that a is weighted–EP w.r.t. (e,f) and $a^{\#} = a^{\dagger}_{e,f}$. Therefore, by (vii), $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\#}$, that is, the condition (i) of Theorem 2.2 holds. So, a is a weighted partial isometry w.r.t. (e,f).

The conditions (viii)-(x) follow similarly as previous part.

(xi) Suppose that $a^{*f,e}a^{\#} = a^{\#}a^{\dagger}_{e,f}$ and $a = aaa^{\dagger}_{e,f}$. Then

$$a^{*f,e}a = (a^{*f,e}a^{\#})a^2 = a^{\#}a^{\dagger}_{e,f}a^2 = (a^{\#})^2aa^{\dagger}_{e,f}a^2 = a^{\#}a.$$

Hence, the condition (vi) is satisfied.

(xii) Analogy as part (xi).

(xiii) The equality $a^{*f,e}a^{\dagger}_{e,f} = a^{\#}a^{\#}$ implies

$$a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a(a^{*f,e}a^{\dagger}_{e,f}) = a^{\dagger}_{e,f}aa^{\#}a^{\#} = a^{\dagger}_{e,f}a^{\#}.$$

So, the condition (vii) holds.

(xiv) Similarly as (xiii).

(xv) Applying the condition $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$, we get

$$a^{\#} = (aa^{\dagger}_{e,f})a(a^{\#})^2 = (a^{\dagger}_{e,f})^{*e,f}(a^{*f,e}a^{\#}) = (a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}a^{\dagger}_{e,f},$$

which yields

$$aa^{\#} = a(a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}a^{\dagger}_{e,f} = a((a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}a^{\dagger}_{e,f})aa^{\dagger}_{e,f} = aa^{\#}aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}.$$

From $a = a^{\dagger}_{e,f}aa$, it follows $aa^{\#} = a^{\dagger}_{e,f}a$. Thus, $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$ and a is weighted–EP w.r.t. (e,f). By $a^{\dagger}_{e,f} = a^{\#}$ and $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$, we conclude that $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{\#}$. The condition (i) of Theorem 2.2 implies that a is a weighted partial isometry w.r.t. (e,f).

(xvi) In the same way as part (xv).

(xvii) Since $a^{*f,e}a^{\#} = a^{\#}a^{\#}$, then the condition (vi) is satisfied:

$$a^{*f,e}a = (a^{*f,e}a^{\#})aa = a^{\#}a^{\#}aa = a^{\#}a.$$

(xviii) In the same way as the previous part.

(xix) Assume that $aa^{*e,f}a^{\dagger}_{e,f} = a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}a$. Now, we get

$$a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}(aa^{*f,e}a^{\dagger}_{e,f}) = a^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}. \tag{1}$$

The equalities (xiii) and (1) imply $aa^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}a$. By Theorem 1.4, we deduce that a is weighted–EP w.r.t. (e,f). From (1) and $a^{\dagger}_{e,f} = a^{\#}$, we observe that $a^{*f,e}a^{\#} = a^{\#}a^{\#}$ and $a = aaa^{\dagger}_{e,f}$, i.e. (xvii) is satisfied.

(xx) Multiplying $aa^{*f,e}a^{\dagger}_{e,f} = a^{\#}$ by $a^{\dagger}_{e,f}$ from the left side, we get

$$a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\#}.$$

Thus, the condition (vii) holds.

(xxi) In the same way as the previous part, it follows (viii).

(xxii) Multiplying $aa^{*f,e}a^{\#} = a^{\dagger}_{e,f}$ by $a^{\dagger}_{e,f}$ from the left side, we show that (xv) holds.

(xxiii) Multiplying $a^{\dagger}_{e,f} = a^{\#}a^{*f,e}a$ by $a^{\dagger}_{e,f}$ from the right side, we get (xvi).

(xxiv) If we multiply first $aa^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}$ from the left side by $a^{\dagger}_{e,f}$ and then $a^{\dagger}_{e,f} = a^{*f,e}a^{\dagger}_{e,f}a$ from the right side by $a^{\dagger}_{e,f}$, we see that $a^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{*f,e}a^{\dagger}_{e,f}$. Further, by (xxiv), we have $aa^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}a$ implying, by Theorem 1.4, a is weighted–EP w.r.t. (e,f). Therefore, from $aa^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}$ and $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$, we obtain $a^{*f,e} = a^{\dagger}_{e,f}$, i.e. a is a weighted partial isometry w.r.t. (e,f).

(xxv) Multiplying $a^{*f,e}a^2 = a$ by $a^{\#}$ from the right side, we get $a^{*f,e}a = aa^{\#}$. Thus, the condition (vi) is satisfied.

(xxvi) This part implies (v) in the similar way as (xxv) $\Rightarrow$ (vi).

(xxvii) Multiplying $aa^{\dagger}_{e,f}a^{*f,e} = a^{\#}$ by $a^{\dagger}_{e,f}$ from the left side, we obtain

$$a^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}a^{\#}.$$

Hence, a satisfies the condition (ix).

(xxviii) Since $a^{*f,e}a^{\dagger}_{e,f}a = a^{\#}$ implies $a^{*f,e}a^{\dagger}_{e,f} = a^{\#}a^{\dagger}_{e,f}$, the condition (x) holds.

□

3 Weighted–EP and weighted star-dagger elements

We begin this section with sufficient conditions for a regular element a in C^*-algebra to be weighted star–dagger w.r.t. (e,f).

Theorem 3.1. *Let $a \in \mathcal{A}^-$ and let e, f be invertible positive elements in $\mathcal{A}$. Then each of the following conditions is sufficient for a to be weighted star–dagger w.r.t. (e,f):*

(i) $a^{*f,e} = a^{*f,e}a^{\dagger}_{e,f}$;

(ii) $a^{*f,e} = a^{\dagger}_{e,f}a^{*f,e}$;

(iii) $a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$;

(iv) $a^{*f,e} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$;

(v) $a^{\dagger}_{e,f} = a^{*f,e}a^{*f,e}$;

(vi) $a = (a^{\dagger}_{e,f})^{*e,e}a$;

(vii) $a = a(a^{\dagger}_{e,f})^{*f,f}$.

Proof. (i) Applying the condition $a^{*f,e} = a^{*f,e}a^{\dagger}_{e,f}$, we obtain

$$aa^{\dagger}_{e,f} = (a^{\dagger}_{e,f})^{*e,f}a^{*f,e} = (a^{\dagger}_{e,f})^{*e,f}a^{*f,e}a^{\dagger}_{e,f} = aa^{\dagger}_{e,f}a^{\dagger}_{e,f}$$

and

$$\begin{aligned} a^{*f,e}a^{\dagger}_{e,f} &= a^{*f,e} = a^{\dagger}_{e,f}aa^{*f,e} = a^{\dagger}_{e,f}(aa^{\dagger}_{e,f})aa^{*f,e} \\ &= a^{\dagger}_{e,f}aa^{\dagger}_{e,f}a^{\dagger}_{e,f}aa^{*f,e} = a^{\dagger}_{e,f}a^{*f,e}. \end{aligned}$$

(ii) This part can be proved in the same way as part (i).

(iii) Using the assumption $a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$, we obtain

$$a^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}a(a^{\dagger}_{e,f}a^{\dagger}_{e,f}) = a^{*f,e}aa^{\dagger}_{e,f} = a^{*f,e},$$

i.e. the condition (i) holds. Hence, $a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}$.

(iv) Since $a^{*f,e} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}$, then

$$a^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}a(a^{\dagger}_{e,f}a^{\dagger}_{e,f}) = a^{*f,e}aa^{*f,e} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}aa^{*f,e} = a^{\dagger}_{e,f}a^{*f,e}.$$

(v) The hypothesis $a^{\dagger}_{e,f} = a^{*f,e}a^{*f,e}$ gives

$$a^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}a^{*f,e}a^{*f,e} = a^{\dagger}_{e,f}a^{*f,e}.$$

(vi) Applying the involution to $a = (a^{\dagger}_{e,f})^{*e,e}a$, we observe that $a^* = a^*ea^{\dagger}_{e,f}e^{-1}$ which gives

$$a^{*f,e} = f^{-1}a^*ea^{\dagger}_{e,f}e^{-1}e = a^{*f,e}a^{\dagger}_{e,f}.$$

So, the condition (i) is satisfied and a is weighted star–dagger w.r.t. (e,f).

(vii) It follows similarly as the previous part. □

In addition, we prove the following theorem related to weighted–EP elements in a C^*-algebra.

Theorem 3.2. *Let $a, b \in \mathcal{A}$ and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^{-} \cap \mathcal{A}^{\#}$, the following statements are equivalent:*

(i) *$aba = a$ and a is weighted–EP w.r.t. (e,f);*

(ii) *$a^{\dagger}_{e,f} = a^{\dagger}_{e,f}ba = aba^{\dagger}_{e,f}$;*

(iii) *$a^{*f,e} = a^{*f,e}ba = aba^{*f,e}$.*

Proof. (i) $\Rightarrow$ (ii): If $aba = a$ and a is weighted–EP w.r.t. (e,f). We get

$$a^{\dagger}_{e,f} = a^{\#} = (a^{\#})^2 a = (a^{\#})^2 aba = a^{\#}ba = a^{\dagger}_{e,f}ba.$$

Similarly, we can show $a^{\dagger}_{e,f} = aba^{\dagger}_{e,f}$. Thus, the condition (ii) is satisfied.

(ii) $\Rightarrow$ (i): By $a^{\dagger}_{e,f} = a^{\dagger}_{e,f}ba = aba^{\dagger}_{e,f}$, we get

$$aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}ba = a(a^{\dagger}_{e,f}ba)aa^{\#} = aa^{\dagger}_{e,f}aa^{\#} = aa^{\#}$$

and

$$a^{\dagger}_{e,f}a = aba^{\dagger}_{e,f}a = a^{\#}a(aba^{\dagger}_{e,f})a = a^{\#}aa^{\dagger}_{e,f}a = a^{\#}a.$$

Therefore, $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$ and, by Theorem 1.4, a is weighted–EP w.r.t. (e,f). From $a^{\#} = a^{\dagger}_{e,f}$ and (ii), we have $a^{\#} = a^{\#}ba$ and consequently $a = a^2a^{\#} = a^2a^{\#}ba = aba$. So, the condition (i) is satisfied.

(ii) $\Rightarrow$ (iii): The equality $a^{\dagger}_{e,f} = a^{\dagger}_{e,f}ba$ implies

$$a^{*f,e} = a^{*f,e}aa^{\dagger}_{e,f} = a^{*f,e}aa^{\dagger}_{e,f}ba = a^{*f,e}ba.$$

In the same way, the conditon $a^{\dagger}_{e,f} = aba^{\dagger}_{e,f}$ gives $a^{*f,e} = aba^{*f,e}$. Thus, (iii) holds.

(iii) $\Rightarrow$ (ii): Multiplying $a^{*f,e} = a^{*f,e}ba$ from the left side by $(a^{\dagger}_{e,f})^{*e,f}$, we have $aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}ba$. Then

$$a^{\dagger}_{e,f} = a^{\dagger}_{e,f}(aa^{\dagger}_{e,f}) = a^{\dagger}_{e,f}aa^{\dagger}_{e,f}ba = a^{\dagger}_{e,f}ba.$$

Analogously, we can verify the second equality. Hence, the condition (ii) holds.

□

4 Weighted normal elements

First, we formulate the following result which can be proved directly by the definition of the weighted MP-inverse and Theorem 1.3.

Lemma 4.1. *If $a \in \mathcal{A}^-$, and if e, f are invertible positive elements in $\mathcal{A}$, then $aa^{*e,f}a \in \mathcal{A}^-$ and $(aa^{*f,e}a)^\dagger_{e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}$.*

Now, we state characterizations of regular weighted normal elements in a C^*-algebra.

Theorem 4.1. *Let $a \in \mathcal{A}^-$ and let e, f be invertible positive elements in $\mathcal{A}$. The following statements are equivalent:*

(i) *a is weighted normal w.r.t. (e,f);*

(ii) *$a(aa^{*f,e}a)^\dagger_{e,f} = (aa^{*f,e}a)^\dagger_{e,f}a$;*

(iii) *$a^\dagger_{e,f}(a + a^{*f,e}) = (a + a^{*f,e})a^\dagger_{e,f}$.*

Proof. (i) ⇒ (ii): Suppose that a is weighted normal w.r.t. (e,f). By Lemma 2.1, we get $aa^\dagger_{e,f} = a^\dagger_{e,f}a$ and $a^{*f,e}a^\dagger_{e,f} = a^\dagger_{e,f}a^{*f,e}$ implying

$$\begin{aligned}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f} &= (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}(aa^\dagger_{e,f}) = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}a^\dagger_{e,f}a \\ &= (a^\dagger_{e,f})^{*e,f}(a^\dagger_{e,f}a^{*f,e})(a^\dagger_{e,f})^{*e,f} = (a^\dagger_{e,f})^{*e,f}a^{*f,e}a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f} \\ &= (aa^\dagger_{e,f})a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f} = a^\dagger_{e,f}aa^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}.\end{aligned}$$

By this equality and Theorem 1.3, we obtain

$$(2)\quad aa^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f} = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}a.$$

Since $(aa^{*f,e}a)^\dagger_{e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}$, by Lemma 4.1, then, from (2), we have $a(aa^{*f,e}a)^\dagger_{e,f} = (aa^{*f,e}a)^\dagger_{e,f}a$. Thus, the condition (ii) is satisfied.

(ii) ⇒ (iii): The equality $a(aa^{*f,e}a)^\dagger_{e,f} = (aa^{*f,e}a)^\dagger_{e,f}a$, by Lemma 4.1, can be written as

$$aa^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}a,$$

which is equivalent to, by Theorem 1.3,

$$(3)\qquad (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}.$$

Multiplying (3) by $a^{*f,e}$ from the left and from the right side, we obtain

$$a^{\dagger}_{e,f}a^{*f,e} = a^{*f,e}a^{\dagger}_{e,f}. \tag{4}$$

From (3) and (4), we obtain

$$\begin{aligned} aa^{\dagger}_{e,f} &= aa^{*f,e}((a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}) = a(a^{*f,e}a^{\dagger}_{e,f})(a^{\dagger}_{e,f})^{*e,f} \\ &= aa^{\dagger}_{e,f}a^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = aa^{\dagger}_{e,f}a^{\dagger}_{e,f}a, \end{aligned}$$

and

$$\begin{aligned} a^{\dagger}_{e,f}a &= (a^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*e,f})a^{*f,e}a = (a^{\dagger}_{e,f})^{*e,f}(a^{\dagger}_{e,f}a^{*f,e})a \\ &= (a^{\dagger}_{e,f})^{*e,f}a^{*f,e}a^{\dagger}_{e,f}a = aa^{\dagger}_{e,f}a^{\dagger}_{e,f}a, \end{aligned}$$

which gives $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$. By this equality and (4), we observe that the condition (iii) holds.

(iii) $\Rightarrow$ (i): The assumption $a^{\dagger}_{e,f}(a + a^{*f,e}) = (a + a^{*f,e})a^{\dagger}_{e,f}$ is equivalent to

$$a^{\dagger}_{e,f}a + a^{\dagger}_{e,f}a^{*f,e} = aa^{\dagger}_{e,f} + a^{*f,e}a^{\dagger}_{e,f}. \tag{5}$$

Multiplying (5) by a from the left and from the right side, we get

$$aa^{\dagger}_{e,f}a^{*f,e}a = aa^{*f,e}a^{\dagger}_{e,f}a.$$

Multiplying this equality by $a^{\dagger}_{e,f}$ from the left and from the right side, we obtain, by Theorem 1.3,

$$a^{\dagger}_{e,f}a^{*f,e} = a^{*f,e}a^{\dagger}_{e,f}. \tag{6}$$

The equality (5) and (6) imply

$$a^{\dagger}_{e,f}a = aa^{\dagger}_{e,f}. \tag{7}$$

Therefore, by (7), (6) and Lemma 2.1, a is weighted normal w.r.t. (e,f). $\square$

In the following result, we establish necessary and sufficient conditions for an element a of a C^*-algebra to be weighted normal.

Theorem 4.2. *Let $a \in \mathcal{A}^-$, and let e, f be invertible positive elements in $\mathcal{A}$. Then a is weighted normal w.r.t. (e,f) if and only if $a \in \mathcal{A}^{\#}$ and one of the following equivalent conditions holds:*

(i) $aa^{*f,e}a^{\#} = a^{\#}aa^{*f,e}$ *and* $a = a^{\dagger}_{e,f}aa$;

(ii) $aa^{\#}a^{*f,e} = a^{\#}a^{*f,e}a$ *and* $a = a^{\dagger}_{e,f}aa$;

(iii) $a^{*f,e}aa^{\#} = a^{\#}a^{*f,e}a$ *and* $a = aaa^{\dagger}_{e,f}$;

(iv) $aa^{*f,e}a^{\#} = a^{*f,e}a^{\#}a$ *and* $a = aaa^{\dagger}_{e,f}$;

(v) $aaa^{*f,e} = aa^{*f,e}a$ *and* $a = a^{\dagger}_{e,f}aa$;

(vi) $aa^{*f,e}a = a^{*f,e}aa$ *and* $a = aaa^{\dagger}_{e,f}$;

(vii) $a^{*f,e}a^{\#} = a^{\#}a^{*f,e}$;

(viii) $a^{*f,e}a^{\dagger}_{e,f} = a^{\#}a^{*f,e}$ *and* $a = aaa^{\dagger}_{e,f}$;

(ix) $a^{*f,e}a^{\#} = a^{\dagger}_{e,f}a^{*f,e}$ *and* $a = a^{\dagger}_{e,f}aa$;

(x) $aa^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xi) $a^{\dagger}_{e,f}a^{*f,e}a = a^{*}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xii) $aa^{*f,e}a^{\#} = a^{*}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xiii) $a^{\#}a^{*f,e}a = a^{*f,e}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xiv) $a^{*f,e}a^{\#}a^{\#} = a^{\#}a^{*f,e}a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xv) $a^{\#}a^{*f,e}a^{\#} = a^{\#}a^{\#}a^{*f,e}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xvi) $a^{*f,e}a^{*f,e}a^{\#} = a^{*f,e}a^{\#}a^{*f,e}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xvii) $a^{*f,e}a^{\#}a^{*f,e} = a^{\#}a^{*f,e}a^{*f,e}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xviii) $a^{*f,e}a^{\dagger}_{e,f}a^{\#} = a^{\#}a^{*f,e}a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xix) $a^{*f,e}a^{\#}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}a^{\#}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xx) $a^{\dagger}_{e,f}a^{*f,e}a^{\#} = a^{\#}a^{\dagger}_{e,f}a^{*f,e}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xxi) $a^{\dagger}_{e,f}a^{\#}a^{*f,e} = a^{\#}a^{*f,e}a^{\dagger}_{e,f}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xxii) *There exists some* $x \in \mathcal{R}$ *such that* $ax = a^{*f,e}$ *and* $(a^{\dagger}_{e,f})^{*e,f}x = a^{\dagger}$ *and* $a = aaa^{\dagger}_{e,f}$.

Proof. If a is weighted normal w.r.t. (e,f), then it commutes with $a^{\dagger}_{e,f}$ and $a^{*f,e}$ and $a^{\#} = a^{\dagger}_{e,f}$. It is not difficult to verify that conditions (i)-(xxii) hold.

Conversely, we assume that $a \in \mathcal{A}^{\#}$. To conclude that a is normal, we show that the condition $aa^{*f,e} = a^{*f,e}a$ is satisfied, or that the element is subject to one of the preceding already established conditions of this theorem.

(i) The hypothesis $aa^{*f,e}a^{\#} = a^{\#}aa^{*f,e}$ implies

$$\begin{aligned} a^{\dagger}_{e,f}aaa^{*f,e} &= a^{\dagger}_{e,f}aaa^{*f,e}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aaa^{*f,e}aa^{\#}aa^{\dagger}_{e,f} \\ &= a^{\dagger}_{e,f}a(aa^{*f,e}a^{\#})aaa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aa^{\#}aa^{*f,e}aaa^{\dagger}_{e,f} \\ &= a^{*f,e}aaa^{\dagger}_{e,f} \end{aligned}$$

and

$$\begin{aligned} a^{*f,e} &= a^{\dagger}_{e,f}a(a^{\#}aa^{*f,e}) = (a^{\dagger}_{e,f}aaa^{*f,e})a^{\#} = a^{*f,e}aaa^{\dagger}_{e,f}a^{\#} \\ &= a^{*f,e}aaa^{\dagger}_{e,f}a(a^{\#})^{2} = a^{*f,e}aa^{\#}. \end{aligned} \tag{8}$$

Further, from (8) and the assumption $a = a^{\dagger}_{e,f}aa$,

$$\begin{aligned} aa^{\dagger}_{e,f} &= (a^{\dagger}_{e,f})^{*e,f}a^{*f,e} = (a^{\dagger}_{e,f})^{*e,f}a^{*f,e}aa^{\#} = aa^{\dagger}_{e,f}aa^{\#} \\ &= aa^{\#} = a^{\dagger}_{e,f}aaa^{\#} = a^{\dagger}_{e,f}a. \end{aligned}$$

By Theorem 1.4, we deduce that a is weighted–EP w.r.t. (e,f). Since $a^{\#} = a^{\dagger}_{e,f}$ and $aa^{*f,e}a^{\#} = a^{\#}aa^{*f,e}$, then

$$a^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}a^{\#} = a^{\dagger}_{e,f}(aa^{*f,e}a^{\#}) = a^{\#}a^{\#}aa^{*f,e} = a^{\#}a^{*f,e} = a^{\dagger}_{e,f}a^{*f,e}.$$

So, a is weighted normal w.r.t. (e,f), by Lemma 2.1.

(ii) Applying the equality $aa^{\#}a^{*f,e} = a^{\#}a^{*f,e}a$, we obtain

$$\begin{aligned} aa^{*f,e}a^{\#} &= a(aa^{\#}a^{*f,e})a^{\#} = (aa^{\#}a^{*f,e})aa^{\#} = a^{\#}a^{*f,e}aaa^{\#} \\ &= (a^{\#}a^{*f,e}a) = aa^{\#}a^{*f,e} = a^{\#}aa^{*f,e}. \end{aligned}$$

Therefore, the condition (i) holds.

(iii) Assume that $a^{*f,e}aa^{\#} = a^{\#}a^{*f,e}a$ and $a = aaa^{\dagger}_{e,f}$. Then we get

$$\begin{aligned} a^{\dagger}_{e,f}aaa^{*f,e} &= a^{\dagger}_{e,f}aaa^{*f,e}aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aa(a^{*f,e}aa^{\#})aa^{\dagger}_{e,f} \\ &= a^{\dagger}_{e,f}aaa^{\#}a^{*f,e}aaa^{\dagger}_{e,f} = a^{*f,e}aaa^{\dagger}_{e,f}, \end{aligned}$$

and

$$\begin{aligned} a^{*f,e} &= (a^{*f,e}aa^{\#})aa^{\dagger}_{e,f} = a^{\#}(a^{*f,e}aaa^{\dagger}_{e,f}) = a^{\#}a^{\dagger}_{e,f}aaa^{*f,e} \\ &= (a^{\#})^2aa^{\dagger}_{e,f}aaa^{*f,e} = a^{\#}aa^{*f,e}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} a^{\dagger}_{e,f}a &= a^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = a^{\#}aa^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = a^{\#}aa^{\dagger}_{e,f}a \\ &= a^{\#}a = a^{\#}aaa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}, \end{aligned}$$

which gives that a is weighted–EP w.r.t. (e,f), by Theorem 1.4. From $a^{\#} = a^{\dagger}_{e,f}$ and $a^{*f,e}aa^{\#} = a^{\#}a^{*f,e}a$, we have

$$a^{*f,e}a = a^{\dagger}_{e,f}aa^{*f,e}a = a(a^{\#}a^{*f,e}a) = a(a^{*f,e}aa^{\#}) = aa^{*f,e}.$$

Hence, a is weighted normal w.r.t. (e,f).

(iv) By the equality $aa^{*f,e}a^{\#} = a^{*f,e}a^{\#}a$, we obtain

$$\begin{aligned} a^{*f,e}aa^{\#} &= a^{*f,e}a^{\#}a = aa^{*f,e}a^{\#} = a^{\#}a(aa^{*f,e}a^{\#}) \\ &= a^{\#}(aa^{*f,e}a^{\#})a = a^{\#}a^{*f,e}a^{\#}aa = a^{\#}a^{*f,e}a. \end{aligned}$$

Thus, the condition (iii) holds.

(v) Suppose that $aaa^{*f,e} = aa^{*f,e}a$ and $a = a^{\dagger}_{e,f}aa$. Now, we see that

$$aa^{\#}a^{*f,e} = (a^{\#})^2(aaa^{*f,e}) = (a^{\#})^2aa^{*f,e}a = a^{\#}a^{*f,e}a$$

and (ii) is satisfied.

(vi) Using the assumption $aa^{*f,e}a = a^{*f,e}aa$, we get

$$aa^{*f,e}a^{\#} = (aa^{*f,e}a)(a^{\#})^2 = a^{*f,e}aa(a^{\#})^2 = a^{*f,e}aa^{\#} = a^{*f,e}a^{\#}a,$$

i.e. the equality (iv) holds.

(vii) The equality $a^{*f,e}a^{\#} = a^{\#}a^{*f,e}$, by double commutativity of group inverse and $(a^{\#})^{\#} = a$, implies $a^{*f,e}a = aa^{*f,e}$.

(viii) If $a^{*f,e}a^{\dagger}_{e,f} = a^{\#}a^{*f,e}$ and $a = aaa^{\dagger}_{e,f}$, then

$$a^{\#}a^{*f,e} = a^{*f,e}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a(a^{*f,e}a^{\dagger}_{e,f}) = a^{\dagger}_{e,f}aa^{\#}a^{*f,e},$$

which yields

$$\begin{aligned} aa^\dagger_{e,f} &= a^\#(aaa^\dagger_{e,f}) = a^\# a = (a^\#)^2 aa^\dagger_{e,f}a^2 = (a^\# a^{*f,e})(a^\dagger_{e,f})^{e,f}a \\ &= a^\dagger_{e,f}aa^\# a^{*f,e}(a^\dagger_{e,f})^{e,f}a = a^\dagger_{e,f}a^\# aa^\dagger_{e,f}a^2 = a^\dagger_{e,f}a. \end{aligned}$$

By Theorem 1.4, a is weighted–EP w.r.t. (e,f) and, by the equality (viii), $a^{*f,e}a^\# = a^{*f,e}a^\dagger_{e,f} = a^\# a^{*f,e}$. So, the condition (vii) holds.

(ix) This part can be proved similarly as part (viii).

(x) From the hypothesis $aa^{*f,e}a^\dagger_{e,f} = a^{*f,e}$, we observe that

$$a^{*f,e}a^\dagger_{e,f} = aa^{*f,e}a^\dagger_{e,f}a^\dagger_{e,f} = a^\# a(aa^{*f,e}a^\dagger_{e,f})a^\dagger_{e,f} = a^\#(aa^{*f,e}a^\dagger_{e,f}) = a^\# a^{*f,e}.$$

Therefore, the condition (viii) holds.

(xi) This condition implies condition (ix), in the same way as (x) $\Rightarrow$ (viii).

(xii) Multiplying $aa^{*f,e}a^\# = a^*$ by $a^\dagger_{e,f}$ from the left side we get $a^{*f,e}a^\# = a^\dagger_{e,f}a^{*f,e}$. Hence, the condition (ix) holds.

(xiii) Similarly to the previous part.

(xiv) Multiplying the equality $a^{*f,e}a^\# a^\# = a^\# a^{*f,e}a^\#$ from the right side by a^2, we obtain $a^{*f,e}aa^\# = a^\# a^{*f,e}a$. Thus, the condition (iii) is satisfied.

(xv) The condition (xv) implies (i) in the similar way as (xiv) $\Rightarrow$ (iii).

(xvi) Multiplying $a^{*f,e}a^{*f,e}a^\# = a^{*f,e}a^\# a^{*f,e}$ from the right side by $aa^\#$, we get $a^{*f,e}a^{*f,e}a^\# = a^{*f,e}a^\# a^{*f,e}aa^\#$. So, $a^{*f,e}a^\# a^{*f,e} = a^{*f,e}a^\# a^{*f,e}aa^\#$. Multiplying this equality from the left side by $(a^\dagger_{e,f})^{*e,f}$, we get

$$aa^\dagger_{e,f}a^\# a^{*f,e} = aa^\dagger_{e,f}a^\# a^{*f,e}aa^\#.$$

Now, $aa^\dagger_{e,f}a(a^\#)^2a^{*f,e} = aa^\dagger_{e,f}a(a^\#)^2a^{*f,e}aa^\#$ gives $a^\# a^{*f,e} = a^\# a^{*f,e}aa^\#$. Then

$$\begin{aligned} aa^\dagger_{e,f} &= (a^\dagger_{e,f})^{*e,f}a^{*f,e} = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}aa^{*f,e} = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}a^2(a^\# a^{*e,f}) \\ &= (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}a^2a^\# a^{*e,f}aa^\# = aa^\dagger_{e,f}aa^\# = aa^\#. \end{aligned}$$

Consequently, by the assumption $a = a^\dagger_{e,f}aa$, $aa^\dagger_{e,f} = a^\dagger_{e,f}a$. Hence,

$$\begin{aligned} a^{*f,e}a^\# &= (a^\dagger_{e,f}a)a^{*f,e}a^\# = aa^\dagger_{e,f}a^{*f,e}a^\# = (a^\dagger_{e,f})^{*e,f}(a^{*f,e}a^{*f,e}a^\#) \\ &= (a^\dagger_{e,f})^{*e,f}a^{*f,e}a^\# a^{*f,e} = aa^\dagger_{e,f}a(a^\#)^2a^{*f,e} = a^\# a^{*f,e}, \end{aligned}$$

i.e. the condition (vii) holds.

(xvii) Similarly as part (xvi).

(xviii) Assume that $a^{*f,e}a^{\dagger}_{e,f}a^{\#} = a^{\#}a^{*f,e}a^{\dagger}_{e,f}$ and $a = aaa^{\dagger}_{e,f}$. Then $a^{\#}a = aa^{\dagger}_{e,f}$ gives

$$a^{*f,e}a^{\dagger}_{e,f}a^{\#} = a^{*f,e}(aa^{\dagger}_{e,f})a^{\dagger}_{e,f}a^{\#} = a^{*f,e}a^{\#}aa^{\dagger}_{e,f}a(a^{\#})^2 = a^{*f,e}a^{\#}a^{\#} \tag{9}$$

and

$$\begin{aligned} a^{\#}a^{*f,e}a^{\dagger}_{e,f} &= a^{\#}a^{*f,e}(aa^{\dagger}_{e,f})a^{\dagger}_{e,f} = a^{\#}a^{*f,e}a^{\#}(aa^{\dagger}_{e,f}) \\ &= a^{\#}a^{*f,e}a^{\#}aa^{\#} = a^{\#}a^{*f,e}a^{\#}. \end{aligned} \tag{10}$$

Since the left side of equalities (9) and (12) are equal, we observe that $a^{*f,e}a^{\#}a^{\#} = a^{\#}a^{*f,e}a^{\#}$ and (xiv) is satisfied.

(xix) Using $a^{*f,e}a^{\#}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{*f,e}a^{\#}$, we have

$$\begin{aligned} a^{*f,e}a^{\#}a^{\#} &= a^{*f,e}(a^{\#})^2aa^{\dagger}_{e,f}aa^{\#} = (a^{*f,e}a^{\#}a^{\dagger}_{e,f})aa^{\#} = a^{\dagger}_{e,f}a^{*f,e}a^{\#}aa^{\#} \\ &= a^{\dagger}_{e,f}a^{*f,e}a^{\#} = a^{*f,e}a^{\#}a^{\dagger}_{e,f}. \end{aligned}$$

Multiplying this equality from the left side by $(a^{\dagger}_{e,f})^{*e,f}$, we show that $aa^{\dagger}_{e,f}a(a^{\#})^2a^{\#} = aa^{\dagger}_{e,f}a(a^{\#})^2a^{\dagger}_{e,f}$, that is, $a^{\#}a^{\#} = a^{\#}a^{\dagger}_{e,f}$. Now,

$$a^{*f,e}aa^{\#} = a^{*f,e}a^2(a^{\#}a^{\#}) = a^{*f,e}a^2a^{\#}a^{\dagger}_{e,f} = a^{*f,e}aa^{\dagger}_{e,f} = a^{*f,e}. \tag{11}$$

From (xix) and (11), we have

$$\begin{aligned} a^{*f,e}a^{\#} &= a^{*f,e}(a^{\#})^2aa^{\dagger}_{e,f}a = (a^{*f,e}a^{\#}a^{\dagger}_{e,f})a \\ &= a^{\dagger}_{e,f}a^{*f,e}a^{\#}a = a^{\dagger}_{e,f}(a^{*f,e}aa^{\#}) = a^{\dagger}_{e,f}a^{*f,e}. \end{aligned}$$

Thus, the condition (ix) holds.

(xx) It follows the condition (xv) in the similar way as in part (xviii).

(xxi) Analogy as (xix).

(xxii) Since there exists some $x \in \mathcal{R}$ such that $ax = a^{*f,e}$ and $(a^{\dagger}_{e,f})^{*e,f}x = a^{\dagger}_{e,f}$, then

$$a^{\dagger}_{e,f} = (a^{\dagger}_{e,f})^{*e,f}x = (a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}(ax) = (a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}a^{*f,e},$$

which yields

$$a^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}(a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}aa^{\dagger}_{e,f}a^{*f,e} = a^{\dagger}_{e,f}a^{*f,e}. \tag{12}$$

Now, we get

$$a^{\dagger}_{e,f} = (a^{\dagger}_{e,f})^{*e,f}(a^{\dagger}_{e,f}a^{*f,e}) = (a^{\dagger}_{e,f})^{*e,f}a^{*f,e}a^{\dagger}_{e,f} = aa^{\dagger}_{e,f}a^{\dagger}_{e,f}$$

and

$$\begin{aligned} a^{\dagger}_{e,f}a &= a^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = a^{\dagger}_{e,f}aa^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = aa^{\dagger}_{e,f}a^{\dagger}_{e,f}aa^{*f,e}(a^{\dagger}_{e,f})^{*e,f} \\ &= a^{\#}a(aa^{\dagger}_{e,f}a^{\dagger}_{e,f})aa^{*f,e}(a^{\dagger}_{e,f})^{*e,f} = a^{\#}aa^{\dagger}_{e,f}aa^{\dagger}_{e,f}a = a^{\#}a. \end{aligned}$$

The assumption $a = aaa^{\dagger}_{e,f}$ gives $a^{\dagger}_{e,f}a = a^{\#}a = a^{\#}aaa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}$. By this equality, (12) and Lemma 2.1, we conclude that a is weighted normal w.r.t. $(e,\ f)$. □

5 Weighted Hermitian elements

In this section, we study equivalent conditions for an element of a C^*-algebra to be weighted Hermitian.

Lemma 5.1. *Let $a \in \mathcal{A}$ and let e, f be invertible positive elements in $\mathcal{A}$. Then a is weighted Hermitian w.r.t. (e,f) if and only if a is weighted Hermitian w.r.t. (f,e).*

Proof. By definition, a is weighted Hermitian w.r.t. (e,f) if and only if $a = a^{*f,e}$ which is equivalent to $fae^{-1} = a^*$. Applying the involution, we see that this equality is equivalent to $a^{*e,f} = e^{-1}a^*f = a$, that is a is weighted Hermitian w.r.t. (f,e).

□

Theorem 5.1. *Let $a \in \mathcal{A}^-$ and let e, f be invertible positive elements in $\mathcal{A}$. Then a is weighted Hermitian w.r.t. (e,f) if and only if $a \in \mathcal{A}^{\#}$ and one of the following equivalent conditions holds:*

(i) $aa^{\#} = a^{*f,e}a^{\dagger}_{e,f}$;

(ii) $aa^{\#} = a^{\dagger}_{e,f}a^{*f,e}$;

(iii) $a^{\dagger}_{e,f}a = a^{\#}a^{*f,e}$;

(iv) $aa^{\dagger}_{e,f} = a^{*f,e}a^{\#}$;

(v) $a^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{\#}$;

(vi) $a^{\#}a^{*f,e}a^{\#} = a^{\dagger}_{e,f}$;

(vii) $aa^{\#} = a^{*f,e}a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(viii) $a^{*f,e}aa^{\#} = a$ *and* $a = aaa^{\dagger}_{e,f}$;

(ix) $a^{*f,e}a^{*f,e}a^{\#} = a^{*f,e}$ *and* $a = a^{\dagger}_{e,f}aa$;

(x) $a^{*f,e}a^{\dagger}_{e,f}a^{\#} = a^{\dagger}_{e,f}$ *and* $a = a^{\dagger}_{e,f}aa$;

(xi) $a^{*f,e}a^{\dagger}_{e,f}a^{\#} = a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xii) $aa^{*f,e}a^{\dagger}_{e,f} = a$ *and* $a = a^{\dagger}_{e,f}aa$;

(xiii) $a^{*f,e}a^{\#}a^{\#} = a^{\#}$ *and* $a = aaa^{\dagger}_{e,f}$;

(xiv) $aa = a^{*f,e}a$ *and* $a = aaa^{\dagger}_{e,f}$;

(xv) $aa^{\dagger}_{e,f} = a^{*f,e}a^{\dagger}_{e,f}$ *and* $a = a^{\dagger}_{e,f}aa$.

Proof. If a is weighted Hermitian w.r.t. (e,f), then it commutes with its weighted Moore–Penrose inverse and $a^{\#} = a^{\dagger}_{e,f}$. It is not difficult to verify that conditions (i)-(xii) hold.

Conversely, we assume that $a \in \mathcal{A}^{\#}$, and show that a satisfies the equality $a = a^{*f,e}$ or one of the preceding, already established condition of this theorem.

(i) The assumption $aa^{\#} = a^{*f,e}a^{\dagger}_{e,f}$ implies

$$\begin{aligned} a^{\dagger}_{e,f}a &= a^{\dagger}_{e,f}a(aa^{\#}) = a^{\dagger}_{e,f}aa^{*f,e}a^{\dagger}_{e,f} = a^{*f,e}a^{\dagger}_{e,f} \\ &= (a^{*f,e}a^{\dagger}_{e,f})aa^{\dagger}_{e,f} = aa^{\#}aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}, \end{aligned}$$

which yields

$$a^{*f,e} = a^{*f,e}(aa^{\dagger}_{e,f}) = (a^{*f,e}a^{\dagger}_{e,f})a = aa^{\#}a = a.$$

Hence, we conclude that a is weighted Hermitian w.r.t. (e,f).

(ii) We can verify this part in the similar way as the condition (i).

(iii) Using the equality $a^{\dagger}_{e,f}a = a^{\#}a^{*f,e}$, we get

$$a^{\dagger}_{e,f}aa = a^{\dagger}_{e,f}aa(a^{\dagger}_{e,f}a) = a^{\dagger}_{e,f}aaa^{\#}a^{*f,e} = a^{\dagger}_{e,f}aa^{*f,e} = a^{*f,e}.$$

Now,

$$a^{*f,e} = (a^{\dagger}_{e,f}a)a = a^{\#}a^{*f,e}a = a^{\#}a(a^{\#}a^{*f,e})a = a^{\#}aa^{\dagger}_{e,f}aa = a.$$

(iv) Similarly as part (iii).

(v) Since $a^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{\#}$, then

$$aa^{\#} = aa^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f} = a(a^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f})aa^{\dagger}_{e,f} = aa^{\#}aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}$$

and

$$a^{\#}a = a^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f}a = a^{\dagger}_{e,f}a(a^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f})a = a^{\dagger}_{e,f}aa^{\#}a = a^{\dagger}_{e,f}a$$

implying $a^{\dagger}_{e,f}a = a^{\dagger}_{e,f}a$. Therefore,

$$a^{*f,e} = a^{*f,e}aa^{\dagger}_{e,f} = (a^{*f,e}a^{\dagger}_{e,f}a^{\dagger}_{e,f})aa = a^{\#}aa = a.$$

(vi) From the hypothesis $a^{\#}a^{*f,e}a^{\#} = a^{\dagger}_{e,f}$, we have

$$\begin{aligned} a^{\#}a^{*f,e} &= a^{\#}a^{*f,e}aa^{\dagger}_{e,f} = (a^{\#}a^{*f,e}a^{\#})aaa^{\dagger}_{e,f} = a^{\dagger}_{e,f}aaa^{\dagger}_{e,f} \\ &= a^{\dagger}_{e,f}aaa^{\#}a^{*f,e}a^{\#} = a^{*f,e}a^{\#}, \end{aligned}$$

which gives that a is weighted normal w.r.t. (e,f), by Theorem 4.2. Further, a is weighted–EP w.r.t. (e,f), by Lemma 2.1, and

$$a = aa^{\dagger}_{e,f}a = aa^{\#}a^{*f,e}a^{\#}a = a^{\#}aa^{*f,e}aa^{\#} = a^{\dagger}_{e,f}aa^{*f,e}aa^{\dagger}_{e,f} = a^{*f,e}.$$

(vii) Suppose that $aa^{\#} = a^{*f,e}a^{\#}$ and and $a = aaa^{\dagger}_{e,f}$. Then

$$a = (aa^{\#})a = a^{*f,e}a^{\#}a = a^{\dagger}_{e,f}a(a^{*f,e}a^{\#})a = a^{\dagger}_{e,f}aaa^{\#}a = a^{\dagger}_{e,f}aa$$

and

$$aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}(aaa^{\dagger}_{e,f}) = a^{\dagger}_{e,f}a,$$

i.e. a is weighted–EP w.r.t. (e,f). Because $a^{\#} = a^{\dagger}_{e,f}$, we obtain

$$a = (aa^{\#})a = a^{*f,e}a^{\#}a = a^{*f,e}aa^{\dagger}_{e,f} = a^{*f,e}.$$

(viii) Multiplying $a^{*f,e}aa^{\#} = a$ by $a^{\#}$ from the right side, we observe that a satisfies (vii).

(ix) By the condition $a^{*f,e}a^{*f,e}a^{\#}=a^{*f,e}$, we obtain

$$aa^{\dagger}_{e,f}=(a^{\dagger}_{e,f})^{*e,f}a^{*f,e}=(a^{\dagger}_{e,f})^{*e,f}a^{*f,e}a^{*f,e}a^{\#}=aa^{\dagger}_{e,f}a^{*f,e}a^{\#}$$

and

$$aa^{\#}=(aa^{\dagger}_{e,f})aa^{\#}=aa^{\dagger}_{e,f}a^{*f,e}a^{\#}aa^{\#}=aa^{\dagger}_{e,f}a^{*f,e}a^{\#}.$$

So, $aa^{\dagger}_{e,f}=aa^{\#}$. The assumption $a=a^{\dagger}_{e,f}aa$ implies $aa^{\dagger}_{e,f}=aa^{\#}=a^{\dagger}_{e,f}aaa^{\#}=a^{\dagger}_{e,f}a$. Now

$$a=(aa^{\#})a=(aa^{\dagger}_{e,f})a^{*f,e}(a^{\#}a)=a^{\dagger}_{e,f}aa^{*f,e}aa^{\dagger}_{e,f}=a^{*f,e}.$$

(x) Assume that $a^{*f,e}a^{\dagger}_{e,f}a^{\#}=a^{\dagger}_{e,f}$ and $a=a^{\dagger}_{e,f}aa$. From $aa^{\#}=a^{\dagger}_{e,f}a$ and

$$aa^{\#}=aa^{\dagger}_{e,f}aa^{\#}=aa^{*f,e}a^{\dagger}_{e,f}a^{\#}aa^{\#}=a(a^{*f,e}a^{\dagger}_{e,f}a^{\#})=aa^{\dagger}_{e,f},$$

we deduce that $a^{\dagger}_{e,f}a=aa^{\dagger}_{e,f}$. Hence,

$$a^{*f,e}=a^{*f,e}aa^{\dagger}_{e,f}=a^{*f,e}a^{\dagger}_{e,f}a=(a^{*f,e}a^{\dagger}_{e,f}a^{\#})a^2=a^{\dagger}_{e,f}a^2=a.$$

(xi) Applying $a^{*f,e}a^{\dagger}_{e,f}a^{\#}=a^{\#}$, we get

$$aa^{\dagger}_{e,f}=a^{\#}aaa^{\dagger}_{e,f}=a^{*f,e}a^{\dagger}_{e,f}a^{\#}aaa^{\dagger}_{e,f}=a^{*f,e}a^{\dagger}_{e,f}aa^{\dagger}_{e,f}=a^{*f,e}a^{\dagger}_{e,f}.$$

Consequently,

$$a=(aa^{\dagger}_{e,f})a=a^{*f,e}a^{\dagger}_{e,f}a=a^{\dagger}_{e,f}a(a^{*f,e}a^{\dagger}_{e,f})a=a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}a=a^{\dagger}_{e,f}aa$$

and, by the condition $a=aaa^{\dagger}_{e,f}$, we conclude $a^{\dagger}_{e,f}a=a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}=aa^{\dagger}_{e,f}$. Since $a^{*f,e}a^{\dagger}_{e,f}a^{\#}=a^{\#}$ and $a^{\dagger}_{e,f}=a^{\#}$, then $a^{*f,e}a^{\#}a^{\#}=a^{\#}$ and

$$a^{*f,e}a^{\#}=(a^{*f,e}a^{\#}a^{\#})a=a^{\#}a.$$

Thus, (vii) holds.

(xii) If $aa^{*f,e}a^{\dagger}_{e,f}=a$ and $a=a^{\dagger}_{e,f}aa$, then

$$a^{\dagger}_{e,f}a=a^{\dagger}_{e,f}aa^{*f,e}a^{\dagger}_{e,f}=a^{\dagger}_{e,f}(aa^{*f,e}a^{\dagger}_{e,f})aa^{\dagger}_{e,f}=a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}=aa^{\dagger}_{e,f}.$$

Therefore, a is weighted–EP w.r.t. (e,f), by Theorem 1.4, and $a^{\dagger}_{e,f}=a^{\#}$. Now, we get condition (i):

$$aa^{\#}=a^{\dagger}_{e,f}a=a^{\dagger}_{e,f}aa^{*f,e}a^{\dagger}_{e,f}=a^{*f,e}a^{\dagger}_{e,f}.$$

(xiii) Multiplying $a^{*f,e}a^{\#}a^{\#} = a^{\#}$ by a^2 from the right side, we show that (viii) is satisfied.

(xiv) The equalities $aa = a^{*f,e}a$ and $a = aaa^{\dagger}_{e,f}$ give

$$a^{*f,e} = (a^{*f,e}a)a^{\dagger}_{e,f} = aaa^{\dagger}_{e,f} = a.$$

(xv) From the condition $aa^{\dagger}_{e,f} = a^{*f,e}a^{\dagger}_{e,f}$ and $a = a^{\dagger}_{e,f}aa$, we observe

$$a = (aa^{\dagger}_{e,f})a = a^{*f,e}a^{\dagger}_{e,f}a = f^{-1}(ea^{\dagger}_{e,f}aa)^* = f^{-1}(ea)^* = a^{*f,e}.$$

$\square$

6 Conclusions

In this paper, we study weighted partial isometries, weighted–EP, weighted star–dagger, weighted normal and weighted Hermitian elements of C^*-algebras. As a consequence, for $e = f = 1$, we obtain some well known characterizations of partial isometries, EP, star–dagger, normal and Hermitian elements. The identity $(ab)^* = b^*a^*$ are important when we proved the equivalent statements characterizing the condition of being a partial isometry, EP, star–dagger, normal and Hermitian element in a ring with involution $\mathcal{R}$ in [12, 13]. Since $(*, e, f) : \mathcal{A} \to \mathcal{A}$ is not in general an involution, in most statements an additional condition needs to be consider.

References

[1] O.M. Baksalary, G. Trenkler, *Characterizations of EP, normal and Hermitian matrices*, Linear and Multilinear Algebra 56 (2008), 299–304.

[2] O. M. Baksalary, G.P.H. Styan, G. Trenkler, *On a matrix decomposition of Hartwig and Spindelböck*, Linear Algebra Appl. 430(10) (2009), 2798–2812.

[3] S. Cheng, Y. Tian, *Two sets of new characterizations for normal and EP matrices*, Linear Algebra Appl. 375 (2003), 181–195.

[4] D. Cvetković, D.S. Djordjević, J.J. Koliha, *Moore–Penrose inverse in rings with involution*, Linear Algebra Appl. 426 (2007), 371–381.

[5] D.S. Djordjević, *Characterization of normal, hyponormal and EP operators*, J. Math. Anal. Appl. 329 (2) (2007), 1181-1190.

[6] D S. Djordjević, J.J. Koliha, *Characterizing hermitian, normal and EP operators*, Filomat 21:1 (2007), 39–54.

[7] D.S. Djordjević, V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

[8] M.P. Drazin, *Pseudoinverse in associative rings and semigroups*, Amer. Math. Monthly 65 (1958), 506-514.

[9] R.E. Harte, M. Mbekhta, *On generalized inverses in C*-algebras*, Studia Math. 103 (1992), 71–77.

[10] R.E. Hartwig, K. Spindelböck. *Matrices for which* A^* *and* $A^\dagger$ *commute*, Linear Multilinear Algebra 14 (1984), 241–256.

[11] J.J. Koliha, *The Drazin and Moore–Penrose inverse in C*-algebras*, Math. Proc. Royal Irish Acad. 99A (1999), 17–27.

[12] D. Mosić, D.S. Djordjević, *Partial isometries and EP elements in rings with involution*, Electronic J. Linear Algebra 18 (2009), 761–772.

[13] D. Mosić, D. S. Djordjević, *Moore-Penrose-invertible normal and Hermitian elements in rings*, Linear Algebra Appl. 431 (5-7) (2009), 732–745.

[14] D. Mosić, D.S. Djordjević, *Weighted–EP elements in* C^**-algebras*, Electronic J. Linear Algebra 22 (2011), 912–930.

[15] D. Mosić, D.S. Djordjević, *Factorization of weighted-EP elements in* C^**-algebras*, Applied Math. Comput. 218 (9) (2012), 5383–5390.

[16] P. Patrício, A. Veloso da Costa *On the Drazin index of regular elements*, Cent. Eur. J. Math. 7(2) (2009), 200-205.

[17] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

[18] Y. Tian, H. Wang, *Characterizations of EP matrices and weighted–EP matices*, Linear Algebra Appl. 434(5) (2011), 1295–1318.

Address:
Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia
E-mail:
Dijana Mosić: `dijana@pmf.ni.ac.rs`
Dragan S. Djordjević: `dragan@pmf.ni.ac.rs`