UNIFIED APPROACH TO THE REVERSE ORDER RULE FOR GENERALIZED INVERSES

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Abstract. In this paper we consider the reverse order rule of the form $(AB)^{(2)}_{K,L} = B^{(2)}_{T,S}A^{(2)}_{M,N}$ for outer generalized inverses with prescribed range and kernel. As corollaries, we get generalizations of the well-known results of Bouldin (SIAM J. Appl. Math. **25** (1973), 489–495; in "Recent Applications of Generalized Inverses, vol 66. Pitman Ser. Res. Notes in Math, (1982), 233–248) and Izumino (Tohoku Math. J. **34** (1982), 43–52) for the ordinary Moore-Penrose inverse, and Sun and Wei (SIAM J. Matrix Anal. Appl. **19** (1998), 772–775) for the weighted Moore-Penrose inverse. Results of Bouldin (the second paper mentioned above) for the reverse order rule for the Drazin inverse are improved. Finally, necessary and suficient conditions such that the reverse order rule holds for the group inverse are introduced.

1. Introduction

The rule $(ab)^{-1} = b^{-1}a^{-1}$, where a, b are invertible elements of a semigroup with a unit, is called the reverse order rule for the ordinary inverse. In this paper we will consider the rule $(AB)^{-} = B^{-}A^{-}$, where A^{-} denotes a generalized (outer, Moore-Penrose, Drazin) inverse of a Banach or Hilbert space operator A. As corollaries, we will get generalizations of the

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well-known results of Bouldin [2],[3] and Izumino [9] for the ordinary Moore-Penrose inverse, and Sun and Wei [13] for the weighted Moore-Penrose inverse. Results of Bouldin [3] for the reverse order rule for the Drazin inverse will be improved. Finally, necessary and suficient conditions such that the reverse order rule holds for the group inverse will be introduced.

Let X, Y, Z be Banach spaces and $\mathcal{L}(X,Y)$ be the set of all bounded linear operators from X into Y. For $B \in \mathcal{L}(X,Y)$ we use $\mathcal{N}(B)$ and $\mathcal{R}(B)$ to denote the range and the kernel of B. We say that $C \in \mathcal{L}(Y,X)$ is an *outer generalized inverse* of B, if $CBC = C$.

Recall the basic properties of outer generalized inverses with prescribed range and kernel (see [11] and [12]). Let T and S be subspaces of X and Y, respectively, such that there exists an outer generalized inverse $B^{(2)}_{T,S} \in \mathcal{L}(Y,X)$ of B with range equal to T and kernel equal to S, i.e. $B^{(2)}_{T,S}$ satisfies

$$B^{(2)}_{T,S} B B^{(2)}_{T,S} = B^{(2)}_{T,S}, \quad \mathcal{R}(B^{(2)}_{T,S}) = T, \quad \mathcal{N}(B^{(2)}_{T,S}) = S.$$

If B, T and S given as above, then $B^{(2)}_{T,S}$ exists if and only if T, S are closed complemented subspaces of X and Y respectively, the restriction of $B|_T : T \to B(T)$ is invertible and $B(T) \oplus S = Y$. In this case $B^{(2)}_{T,S}$ is unique. Hence, the notation is justified. For example, the ordinary Moore-Penrose inverse, the weighted Moore-Penrose inverse and the Drazin inverse of B can be obtain as outer generalized inverses with prescribed range and kernel, for suitable choices of T and S.

We consider a similar situation for $A \in \mathcal{L}(Y,Z)$ and $AB \in \mathcal{L}(X,Z)$. Let subspaces M of Y and N of Z be given such that there exists $A^{(2)}_{M,N} \in \mathcal{L}(Z,Y)$. Also, let subspaces K of X and L of Z be given such that there exists $(AB)^{(2)}_{K,L} \in \mathcal{L}(Z,X)$.

In this paper we will find sufficient conditions such that the reverse order rule for outer generalized inverses with prescribed range and kernel holds. Precisely, these conditions will be sufficient such that the next equality holds:

$$(AB)^{(2)}_{K,L} = B^{(2)}_{T,S} A^{(2)}_{M,N}.$$

As corollaries, we will get sufficient conditions for the reverse rule for some classes of generalized inverses.

Our aim was to investigate outer inverses with prescibed range and kernel becouse of their usefulness. Starting point for investigating outer generalized inverses are papers of Nashed and Votruba [11] and Nashed [12]. Also, related results in Banach algebras are presented in [5]. Recently, several papers concerning outer generalized inverses of matrices with prescribed range and kernel appeared, for example [15], [17] and [18]. Most of their applications are related to solving overdetermined linear systems.

On the other hand, in many papers the reverse order rule for generalized inverses is investigated. Fisrt of all, there is a classical paper of Greville [7], where the reverse order rule for the Moore-Penrose inverse of compelx matrices is proved. Usually, the Moore-Penrose inverse, or inner generalized inverses are investigated. Sometimes the reverse order rule for reflexive generalized inverses is considered. Some related results are presented in papers [6], [14], [16] and [19]. It is important to mention the reverse order rule of the form $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ is investigated by Hartwig in [8]. Results which are closely related to those presented in this paper are the following. The reverse order rule for the Moore-Penrose inverse of bounded operators on Hilbert spaces is obtained in [2], [3] and [9]. The reverse order rule for the weighted Moore-Penrose inverse is obtained in [13] and the reverse order rule for the Drazin inverse is considered in [3].

The paper is organized as follows. In Section 2 known results concerning the ordinary and the weighted Moore-Penrose inverses, the ordinary and the generalized Drazin inverses and the Drazin index of an operator are presented. The main result concerning the sufficient conditions for the reverse order rule for outer generalized inverses with prescribed range and kernel is proved in Section 3. In Section 4 we introduce generalizations of known results for the reverse order rule for the weighted Moore-Penrose inverse and the ordinary and the generalized Drazin inverses of an operator. Finally, we also find necessary and sufficient conditions such that the reverse order rule for the group inverse holds.

2. Auxiliary results

If X and Y are Hilbert spaces and $B \in \mathcal{L}(X,Y)$ has a closed range, we can take $T = \mathcal{R}(A^*)$ and $S = \mathcal{N}(A^*)$, to obtain $B^{(2)}_{T,S} = B^\dagger$. Here B^* denotes the conjugate operator of B and $B^\dagger$ denotes the Moore-Penrose inverse of B. If $\mathcal{R}(B)$ is closed, $B^\dagger$ can also be defined as the unique operator which satisfies: $BB^\dagger B = B$, $B^\dagger BB^\dagger = B^\dagger$, $(BB^\dagger)^* = BB^\dagger$ and $(B^\dagger B)^* = B^\dagger B$.

Let X, Y be Hilbert spaces and let positive and invertible operators $F \in \mathcal{L}(X)$ and $E \in \mathcal{L}(Y)$ be given. If $B \in \mathcal{L}(X,Y)$ has a closed range, then there exists the unique operator $B^\dagger_{E,F} \in \mathcal{L}(Y,X)$, such that the following hold:

$$BB^\dagger_{E,F}B = B,\ B^\dagger_{E,F}BB^\dagger_{E,F} = B^\dagger_{E,F},$$
$$(EBB^\dagger_{E,F})^* = EBB^\dagger_{E,F},\ (FB^\dagger_{E,F}B)^* = FB^\dagger_{E,F}B.$$

The operator $B^\dagger_{E,F}$ is known as the weighted Moore-Penrose inverse of B.

Notice the weighted conjugate operator $B^\# = F^{-1}B^*E$ considered in [13]. Recall that

$$BB^\dagger_{E,F} = P_{\mathcal{R}(B),E^{-1}\mathcal{N}(B^*)} = P_{\mathcal{R}(B),\mathcal{N}(B^\#)} \quad \text{and}$$
$$B^\dagger_{E,F}B = P_{F^{-1}\mathcal{R}(B^*),\mathcal{N}(B)} = P_{\mathcal{R}(B^\#),\mathcal{N}(B)}.$$

Here P_{W_1,W_2} denotes the projection onto W_1 parallel to W_2. Hence, $B^\dagger_{E,F}$ is the unique outer generalized inverse of B with the range equal to $F^{-1}\mathcal{R}(B^*)$ and the kernel equal to $E^{-1}\mathcal{N}(B^*)$.

If X is a Banach space and $V \in \mathcal{L}(X)$, then the ascent of V is the smallest nonnegative integer k (if it exists) such that $\mathcal{N}(V^k) = \mathcal{N}(V^{k+1})$ holds. The descent of V is the smallest nonnegative integer k (if it exists) such that

$\mathcal{R}(V^k) = \mathcal{R}(V^{k+1})$ holds. If the ascent and the descent of V are both finite, then they are equal and this common value is known as the Drazin index of V, denoted by $\operatorname{ind}(V)$. It is well-known that $\operatorname{ind}(V) = k$ if and only if there exists the unique operator $V^D \in \mathcal{L}(X)$, which satisfies

$$V^{k+1}V^D = V^k, \quad V^D V V^D = V^D, \quad V V^D = V^D V.$$

V^D is known as the Drazin inverse of V. If $\operatorname{ind}(V) = 0$, then V is invertible. If $\operatorname{ind}(V) \leq 1$, then V^D is known as the *group inverse* of V, denoted by V^g. For more informations about the Drazin inverse see [5, p. 186-190]. For the Drazin inverses of matrices see also [1] and [4].

If $\operatorname{ind}(V) = k$, then $X = \mathcal{R}(V^k) \oplus \mathcal{N}(V^k)$, the restriction of V to $\mathcal{R}(V^k)$ is invertible and the restriction of V to $\mathcal{N}(V^k)$ is nilpotent. Hence, the matrix representation of V has the following form:

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(V^k) \\ \mathcal{N}(V^k) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(V^k) \\ \mathcal{N}(V^k) \end{bmatrix},$$

where V_1 is invertible and $V_2^k = 0$. Also

$$V^D = \begin{bmatrix} V_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(V^k) \\ \mathcal{N}(V^k) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(V^k) \\ \mathcal{N}(V^k) \end{bmatrix}.$$

Hence, the Drazin inverse V^D is the unique outer generalized inverse of V, with the range equal to $\mathcal{R}(V^k)$ and the kernel equal to $\mathcal{N}(V^k)$.

An operator $V \in \mathcal{L}(X)$ has the group inverse (i.e. $\operatorname{ind}(V) \leq 1$) if and only if $X = \mathcal{R}(V) \oplus \mathcal{N}(V)$. In particular, it follows that $\mathcal{R}(V)$ is closed. In this case V has the following matrix decomposition

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(V) \\ \mathcal{N}(V) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(V) \\ \mathcal{N}(V) \end{bmatrix},$$

where V_1 is invertible. In this case the group inverse of V has the form

$$V^g = \begin{bmatrix} V_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

It is also well-known that $V \in \mathcal{L}(X)$ has the Drazin inverse, if and only if the point $\lambda = 0$ is pole of the resolvent $\lambda \mapsto (\lambda - V)^{-1}$. The order of this pole is equal to $\mathrm{ind}(V)$ (see [5, p.188]).

In [10] Koliha introduced the concept of a generalized Drazin inverse V^d of a bounded operator $V \in \mathcal{L}(X)$. If 0 is not the point of accumulation of the spectrum $\sigma(V)$ of V, denote by P the spectral idempotent of V corresponding to $\{0\}$. Let $X_1 = \mathcal{N}(P)$ and $X_2 = \mathcal{R}(P)$. Then the decomposition $X = X_1 \oplus X_2$ completely reduces V, the restriction $V_1 = V|_{X_1}$ is invertible and the restriction $V_2 = V|_{X_2}$ is quasinilpotent. In this case V has the following matrix form

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

The generalized Drazin inverse of V can be defined as

$$V^d = \begin{bmatrix} V_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Other alternative definitions of V^d can be found in [10]. Obviously, if V has the generalized Drazin inverse, then V^d is the unique outer generalized inverse of V, whose range is equal to $\mathcal{N}(P)$ and kernel is equal to $\mathcal{R}(P)$. If V has the Drazin inverse, then $V^D = V^d$.

3. Main results

Let $B \in \mathcal{L}(X, Y)$ and assume that $B^{(2)}_{T,S} \in \mathcal{L}(Y, X)$ exists for given closed subspaces T of X and S of Y. Let $B'_{T,S} \in \mathcal{L}(Y, X)$ be any operator which satisfies $\mathcal{R}(B'_{T,S}) = T$ and $\mathcal{N}(B'_{T,S}) = S$ (this operator obviously exists). For example, if X and Y are Hilbert spaces, $T = \mathcal{N}(B)^{\perp}$, $S = \mathcal{R}(B)^{\perp}$, then we can take $B'_{T,S} = B^*$. We prove the following useful result.

Lemma 3.1. *Suppose that for $B \in \mathcal{L}(X,Y)$ and given closed subspaces T and S of X and Y, respectively, there exists $B^{(2)}_{T,S} \in \mathcal{L}(Y,X)$. Then*

$$B^{(2)}_{T,S} = B'_{T,S}(BB'_{T,S})^g,$$

where $B'_{T,S} \in \mathcal{L}(Y,X)$ is an arbitrary operator satisfying $\mathcal{R}(B'_{T,S}) = T$ and $\mathcal{N}(B'_{T,S}) = S$.

Proof. There exists a closed complementary subspace T_1 of T in X. Consider the following matrix decomposition of B:

$$B = \begin{bmatrix} U_1 & U_2 \\ 0 & U_3 \end{bmatrix} : X = \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} B(T) \\ S \end{bmatrix} = Y.$$

The restriction $U_1 = B|_T : T \to B(T)$ is invertible and obviously

$$B^{(2)}_{T,S} = \begin{bmatrix} U_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} B(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

From the properties of $B'_{T,S}$ we conclude that it has the following form

$$B'_{T,S} = \begin{bmatrix} U'_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} B(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix},$$

where $U'_1 = B'_{T,S}|_{B(T)} : B(T) \to T$ is invertible. Hence

$$BB'_{T,S} = \begin{bmatrix} U_1U'_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} B(T) \\ S \end{bmatrix} \to \begin{bmatrix} B(T) \\ S \end{bmatrix}.$$

Since $U_1U'_1$ is invertible, we conclude that $\mathrm{ind}(BB'_{T,S}) \leq 1$ and

$$(BB'_{T,S})^g = \begin{bmatrix} (U'_1)^{-1}U_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} B(T) \\ S \end{bmatrix} \to \begin{bmatrix} B(T) \\ S \end{bmatrix}.$$

Now we get

$$B'_{T,S}(BB'_{T,S})^g = B^{(2)}_{T,S}. \quad \square$$

Remark 3.2. If X, Y are Hilbert spaces and $B \in \mathcal{L}(X,Y)$ has a closed range, then Lemma 3.1 implies $B^\dagger = B^*(BB^*)^g = B^*(BB^*)^\dagger$ (see [3, Theorem 3.4]). Also, Wei proved the result of Lemma 3.1 for finite dimensional spaces in [18, Theorem 2.1].

Now, we will prove the main result of this paper.

Theorem 3.3. *Let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$, and let $A^{(2)}_{M,N} \in \mathcal{L}(Z,Y)$, $B^{(2)}_{T,S} \in \mathcal{L}(Y,X)$ and $(AB)^{(2)}_{K,L} \in \mathcal{L}(Z,X)$ be outer inverses of A, B and AB with subspaces $K,T \subset X$, $M,S \subset Y$ and $N,L \subset Z$. Let operators $A'_{M,N} \in \mathcal{L}(Z,Y)$, $B'_{T,S} \in \mathcal{L}(Y,X)$ and $(AB)'_{K,L} \in \mathcal{L}(Z,X)$ satisfy*

$$\mathcal{R}(A'_{M,N}) = M,\ \mathcal{N}(A'_{M,N}) = N,\ \mathcal{R}(B'_{T,S}) = T,\ \mathcal{N}(B'_{T,S}) = S,$$
$$\mathcal{R}(B'_{T,S}A'_{M,N}) = \mathcal{R}((AB)'_{K,L}) = K,\ \mathcal{N}(B'_{T,S}A'_{M,N}) = \mathcal{N}((AB)'_{K,L}) = L.$$

If $A^{(2)}_{M,N}A$ commutes with $BB'_{T,S}$ and $BB^{(2)}_{T,S}$ commutes with $A'_{M,N}A$, then

$$(AB)^{(2)}_{K,L} = B^{(2)}_{T,S}A^{(2)}_{M,N}.$$

Proof. Taking $M_1 = \mathcal{N}(A^{(2)}_{M,N}A)$ and $T_1 = \mathcal{N}(B^{(2)}_{T,S}B)$ we get $Y = M \oplus M_1$ and $X = T \oplus T_1$. Consider the following decompositions

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} A(M) \\ N \end{bmatrix},$$

where $A_1 = A|_M : M \to A(M)$ is invertible,

$$A^{(2)}_{M,N} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(M) \\ N \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix},$$

$$A'_{M,N} = \begin{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(M) \\ N \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix},$$

where $A'_1 = A'_{M,N}|_{A(M)} : A(M) \to M$ is invertible. Operators related to B have the following decompositions:

$$B = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix},$$

$$B'_{T,S} = \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

We know that $A^{(2)}_{M,N}A$ is the projection from Y onto M parallel to M_1. From

$$A^{(2)}_{M,N}A = \begin{bmatrix} I & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix}$$

we get $A_2 = 0$, $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}$ and $A^{(2)}_{M,N}A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix}$. Since

$$BB'_{T,S} = \begin{bmatrix} B_1B'_1 & B_1B'_2 \\ B_2B'_1 & B_2B'_2 \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix},$$

it follows that

$$BB'_{T,S}A^{(2)}_{M,N}A = \begin{bmatrix} B_1B'_1 & 0 \\ B_2B'_1 & 0 \end{bmatrix}$$

and

$$A^{(2)}_{M,N}ABB'_{T,S} = \begin{bmatrix} B_1B'_1 & B_1B'_2 \\ 0 & 0 \end{bmatrix}.$$

Since $BB'_{T,S}$ commutes with $A^{(2)}_{M,N}A$, we conclude that

$$B_2B'_1 = 0, \quad B_1B'_2 = 0$$

and

$$BB'_{T,S} = \begin{bmatrix} B_1B'_1 & 0 \\ 0 & B_2B'_2 \end{bmatrix}.$$

Using Lemma 3.1, from $\operatorname{ind}(BB'_{T,S}) \le 1$ it follows that $\operatorname{ind}(B_1B'_1) \le 1$, $\operatorname{ind}(B_2B'_2) \le 1$ and

$$(BB'_{T,S})^g = \begin{bmatrix} (B_1B'_1)^g & 0 \\ 0 & (B_2B'_2)^g \end{bmatrix}.$$

From Lemma 3.1 we also get

$$B^{(2)}_{T,S} = B'_{T,S}(BB'_{T,S})^g = \begin{bmatrix} B'_1(B_1B'_1)^g & B'_2(B_2B'_2)^g \\ 0 & 0 \end{bmatrix}.$$

Since

$$AB = \begin{bmatrix} A_1B_1 & A_1B_3 \\ A_3B_2 & A_3B_4 \end{bmatrix}$$

and

$$B^{(2)}_{T,S}A^{(2)}_{M,N} = \begin{bmatrix} B'_1(B_1B'_1)^gA_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

we easily get

$$B^{(2)}_{T,S}A^{(2)}_{M,N}ABB^{(2)}_{T,S}A^{(2)}_{M,N} = B^{(2)}_{T,S}A^{(2)}_{M,N}.$$

Using $B_2B_1' = 0$ and $B_1B_2' = 0$ we compute

$$BB^{(2)}_{T,S} = \begin{bmatrix} B_1B_1'(B_1B_1')^g & 0 \\ 0 & B_2B_2'(B_2B_2)' \end{bmatrix}.$$

Obviously, $BB^{(2)}_{T,S}$ is the projection from Y onto $\mathcal{R}(B_1B_1') \oplus \mathcal{R}(B_2B_2')$ with respect to the following decomposition of the space

$$Y = [\mathcal{R}(B_1B_1') \oplus \mathcal{N}(B_1B_1')] \oplus [\mathcal{R}(B_2B_2') \oplus \mathcal{N}(B_2B_2')].$$

On the other hand $BB^{(2)}_{T,S}$ must be the projection from Y onto $B(T)$ parallel to S.

Since the restriction $B|_T : T \to Y$ is one-to-one, we conclude that $\mathcal{N}(B_1) \cap \mathcal{N}(B_2) = \{0\}$. From $B_1B_2' = 0$ and $B_2B_1' = 0$ we get $\mathcal{R}(B_1') \cap \mathcal{R}(B_2') \subset \mathcal{N}(B_2) \cap \mathcal{N}(B_1) = \{0\}$. Hence

$$T = \mathcal{R}(B'_{T,S}) = \mathcal{R}(B_1') + \mathcal{R}(B_2') = \mathcal{R}(B_1') \oplus \mathcal{R}(B_2'),$$

$\mathcal{N}(B_1) = \mathcal{R}(B_2')$ and $\mathcal{N}(B_2) = \mathcal{R}(B_1')$. Now it follows that $\mathcal{R}(B_1) = \mathcal{R}(B_1B_1')$ and $\mathcal{R}(B_2) = \mathcal{R}(B_2B_2')$. In particular, we get that $\mathcal{R}(B_1)$ and $\mathcal{R}(B_2)$ are closed subspaces of Y and $B(T) = \mathcal{R}(B_1) \oplus \mathcal{R}(B_2)$.

The inclusion $\mathcal{N}(B_1') \subset \mathcal{N}(B_1B_1')$ is obvious. Suppose that there exists some $x \in \mathcal{N}(B_1B_1') \setminus \mathcal{N}(B_1')$. Then $B_1B_1'x = 0$, $B_1'x \in \mathcal{N}(B_1)$ and hence $B_1'x = 0$. We get $\mathcal{N}(B_1B_1') = \mathcal{N}(B_1')$ and similarly $\mathcal{N}(B_2B_2') = \mathcal{N}(B_2')$. It follows that

$$M = \mathcal{R}(B_1) \oplus \mathcal{N}(B_1'), \quad M_1 = \mathcal{R}(B_2) \oplus \mathcal{N}(B_2')$$

and these decompositions completely reduce B_1B_1' and B_2B_2' respectively.

Since $BB^{(2)}_{T,S}$ commutes with $A'_{M,N}A$, we obtain

$$A_1'A_1B_1B_1'(B_1B_1')^g = B_1B_1'(B_1B_1')^gA_1'A_1.$$

Hence the decomposition $M = \mathcal{R}(B_1) \oplus \mathcal{N}(B_1')$ completely reduces $A_1'A_1$. Consider the following decomposition:

$$A(M) = A_1(\mathcal{R}(B_1)) \oplus A_1(\mathcal{N}(B_1')).$$

There exist invertible operators C_1, C_2, C_1', C_2' such that

$$A_1 = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B_1) \\ \mathcal{N}(B_1') \end{bmatrix} \to \begin{bmatrix} A_1(\mathcal{R}(B_1)) \\ A_1(\mathcal{N}(B_1')) \end{bmatrix}, \quad A_1^{-1} = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{bmatrix}$$

and

$$A_1' = \begin{bmatrix} C_1' & 0 \\ 0 & C_2' \end{bmatrix} : \begin{bmatrix} A_1(\mathcal{R}(B_1)) \\ \mathcal{A}_1(N(B_1')) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B_1) \\ \mathcal{N}(B_1') \end{bmatrix}.$$

There exists an invertible operator $D \in \mathcal{L}(\mathcal{R}(B_1))$ such that

$$B_1B_1' = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B_1) \\ \mathcal{N}(B_1') \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B_1) \\ \mathcal{N}(B_1') \end{bmatrix} \quad \text{and} \quad (B_1B_1')^g = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since

$$B_{T,S}'A_{M,N}' = \begin{bmatrix} B_1'A_1' & 0 \\ 0 & 0 \end{bmatrix},$$

we find

$$\mathcal{N}(B_{T,S}'A_{M,N}') = N \oplus \mathcal{N}(B_1'A_1') = N \oplus A_1(\mathcal{N}(B_1')).$$

It is easy to verify the following

$$\mathcal{N}(B_{T,S}^{(2)}A_{M,N}^{(2)}) = N \oplus A_1(\mathcal{N}(B_1')) = \mathcal{N}(B_{T,S}'A_{M,N}') = \mathcal{N}((AB)_{K,L}') = L.$$

We also find that

$$\mathcal{R}(B_{T,S}'A_{M,N}') = \mathcal{R}(B_1')$$

and

$$\begin{aligned}\mathcal{R}(B_{T,S}^{(2)}A_{M,N}^{(2)}) &= \mathcal{R}(B_1'(B_1B_1')^gA_1^{-1}) = \mathcal{R}(B_1') \\ &= \mathcal{R}(B_{T,S}'A_{M,N}') = \mathcal{R}((AB)_{K,L}') = K.\end{aligned}$$

We have just proved

$$(AB)_{K,L}^{(2)} = B_{T,S}^{(2)}A_{M,N}^{(2)}. \quad \square$$

Lots of results concerning the reverse order rule for generalized inverses are already known (see, for example, [2], [3], [6], [7], [8], [9], [13], [14], [16] [19]). Most of them deal with the Moore-Penrose inverse, inner of reflexive (both inner and outer) generalized inverse. Also, most of these results are specialized to complex matrices. As far as we know, results contained in our Theorem 3.3 are not published before.

4. Applications

Bouldin [2, Theorem 3.1, Remark 3.2], [3, Theorem 3.3] and Izumino [9, Corollary 3.11] found the necessary and sufficient conditions such that the reverse order rule for the ordinary Moore-Penrose inverse of operators on Hilbert spaces holds:

Proposition 4.1. *If X, Y, Z are Hilbert spaces and $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ and $AB \in \mathcal{L}(X,Z)$ have closed ranges, then the following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $A^\dagger A$ *commutes with* BB^* *and* $BB^\dagger$ *commutes with* A^*A;

(3) $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$.

In [13, Theorem 2.4] Sun and Wei proved the following result:

Proposition 4.2. *Let X, Y, Z be Hilbert spaces and let positive and invertible operators $E \in \mathcal{L}(X)$, $F \in \mathcal{L}(Y)$ and $G \in \mathcal{L}(Z)$ be positive and invertible operators. Then the following statements are equivalent:*

(1) $(AB)^\dagger_{G,E} = B^\dagger_{F,E} A^\dagger_{G,F}$;

(2) $\mathcal{R}(A^\# AB) \subset \mathcal{R}(B)$ *and* $\mathcal{R}(BB^\# A^\#) \subset \mathcal{R}(A^\#)$.

Here $A^\# = F^{-1}A^*G$ *and* $B^\# = E^{-1}B^*F$.

Although Sun and Wei considered complex matrices, a careful reading shows that their method is valid for bounded Hilbert space operators with closed ranges. Notice that in Proposition 4.2 the same weight for A and B on the space Y is considered. We can prove a more general result, considering different weights for A and B on the space Y.

The next Corollary 4.3 is a generalization of known results considering the reverse order rule for the weighted Moore-Penrose inverse.

Corollary 4.3. *Let X, Y, Z be Hilbert spaces and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ and $AB \in \mathcal{L}(X, Z)$ have closed ranges. Let $E \in \mathcal{L}(X)$, $F, H \in \mathcal{L}(Y)$ and $G \in \mathcal{L}(Z)$ be positive and invertible and let $A^{\#} = H^{-1}A^*G$, $B^{\#} = E^{-1}B^*F$ and $(AB)^{\#} = E^{-1}(AB)^*G$. Consider the following statements:*

(1) *$A^{\dagger}_{G,H}A$ commutes with $BB^{\#}$ and $BB^{\dagger}_{F,E}$ commutes with $A^{\#}A$;*
(2) *$(AB)^{\dagger}_{G,E} = B^{\dagger}_{F,E}A^{\dagger}_{G,H}$.*

If $\mathcal{R}((AB)^{\#}) = \mathcal{R}(B^{\#}A^{\#})$ and $\mathcal{N}((AB)^{\#}) = \mathcal{N}(B^{\#}A^{\#})$, then (1) $\Longrightarrow$ (2). If $F = H$, then (1) $\Longleftrightarrow$ (2).

Proof. The implication (1) $\Longrightarrow$ (2) follows from Theorem 3.3 taking $T = E^{-1}\mathcal{R}(B^*) = \mathcal{R}(B^{\#})$, $S = F^{-1}\mathcal{N}(B^*) = \mathcal{N}(B^{\#})$, $M = H^{-1}\mathcal{R}(A^*) = \mathcal{R}(A^{\#})$, $N = G^{-1}\mathcal{N}(A^*) = \mathcal{N}(A^{\#})$, $A'_{M,N} = A^{\#}$, $B'_{T,S} = B^{\#}$ and $(AB)'_{K,L} = (AB)^{\#} = E^{-1}(AB)^*G$. If $F = H$, then the implication (2) $\Longrightarrow$ (1) is a result from Proposition 4.2. $\square$

Let X be a Banach space and let $A, B \in \mathcal{L}(X)$ have Drazin inverses. Bouldin [3, Theorem 4.3] proved the following result:

Proposition 4.4. *If B^DB commutes with A, A^DA commutes with B and*

$$\mathcal{N}((AB)^j) \supset \mathcal{N}(A^D) \cup \mathcal{N}(B^D) \tag{1}$$

holds for some nonnegative integer j, then

$$(AB)^D = B^DA^D,$$

$$\mathcal{N}(AB)^D = \operatorname{span}\{\mathcal{N}(A^D), \mathcal{N}(B^D)\}, \ \mathcal{R}((AB)^D) = \mathcal{R}(A^D) \cap \mathcal{R}(B^D)$$

and the least j for which (1) *holds is the Drazin index of AB.*

We will prove a result concerning the generalized Drazin inverse.

Corollary 4.5. *If $A, B, AB \in \mathcal{L}(X)$ have generalized Drazin inverses, AA^d commutes with BB^d, $\mathcal{R}(B^dA^d) = \mathcal{R}((AB)^d)$ and $\mathcal{N}(B^dA^d) = \mathcal{N}((AB)^d)$, then $(AB)^d = B^dA^d$.*

Proof. Let P, Q, R denote the spectral idempotents of A, B, AB, respectively, corresponding to $\{0\}$. In Theorem 3.3 we have to take $T = \mathcal{N}(Q)$, $S = \mathcal{R}(Q)$, $M = \mathcal{N}(P)$, $N = \mathcal{R}(P)$, $K = \mathcal{N}(R)$, $L = \mathcal{R}(R)$, $A'_{M,N} = A^d$, $B'_{T,S} = B^d$, $(AB)'_{K,L} = (AB)^d$. □

In the case of the ordinary Drazin inverse, we compare Corollary 4.5 with Proposition 4.4.

Proposition 4.6. *If the assumptions of Proposition 4.4 hold, then so do the assumptions of Corollary 4.5.*

Proof. If the assumptions of Proposition 4.4 hold, i.e. A^D and B^D exists, B^DB commutes with A, A^DA commutes with B and $\mathcal{N}((AB)^j) \supset \mathcal{N}(A^D) \cup \mathcal{N}(B^D)$, then AA^D commutes with BB^D, the reverse order rule for the Drazin inverse holds, and hence the assumptions of Corollary 4.5 are valid. □

However, the converse of Proposition 4.6 is not true in general. For the convenience of the reader, we use one Bouldin's example from [3, p. 243-244].

Example 4.7. Let A be a nonzero nilpotent operator on a Hilbert space X, and let B be the orthogonal projection onto the orthogonal complement of $\mathcal{N}(A)$. Then $AB = A$, $A^D = 0$, $(AB)^D = 0$, $B^D = B$, $AA^DBB^D = BB^DAA^D = 0$, $(AB)^D = B^DA^D$, but $B^DB = B^2 = B$ does not commute with A. Otherwise, $\mathcal{R}(B) = \mathcal{N}(A)^\perp$ would be invariant under A, so $\mathcal{R}(A)$ would be contained in $\mathcal{N}(A)^\perp$. It follows that $\mathcal{N}(A^2) = \mathcal{N}(A)$, hence $A = 0$.

In the case of the group inverse, we can prove the reverse implication in Corollary 4.5.

Theorem 4.8. *Let $A, B, AB \in \mathcal{L}(X)$ have group inverses. Then the following statements are equivalent:*

(1) *AA^g commutes with BB^g, $\mathcal{R}((AB)^g) = \mathcal{R}(B^gA^g)$ and $\mathcal{N}((AB)^g) = \mathcal{N}(B^gA^g)$.*

(2) *$(AB)^g = B^gA^g$.*

Proof. According to Corollary 4.5 we only have to prove (2) $\Longrightarrow$ (1). The main problem is to prove that AA^g commutes with BB^g.

We improve notations from Theorem 3.3:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where A_1 is invertible,

$$A^g = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Since $\mathcal{R}(B^g) = \mathcal{R}(B)$, we conclude that B^g must have the following form

$$B^g = \begin{bmatrix} B_1' & B_2' \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

Since

$$B^gA^g = \begin{bmatrix} B_1'A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$B^gA^gABB^gA^g = \begin{bmatrix} B_1'B_1B_1'A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

we conclude that $B_1'B_1B_1' = B_1'$. From

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$ABB^gA^gAB = \begin{bmatrix} A_1B_1B_1'B_1 & 0 \\ 0 & 0 \end{bmatrix},$$

we conclude that $B_1B_1'B_1 = B_1$. Hence $B_1'B_1$ is the projection from $\mathcal{R}(B)$ onto $\mathcal{R}(B_1')$ parallel to $\mathcal{N}(B_1)$ and $\mathcal{R}(B_1) = B_1(\mathcal{R}(B_1'))$. In particular, it follows that $\mathcal{R}(B_1)$ is closed. Also,

$$B^gB = \begin{bmatrix} B_1'B_1 + B_2'B_2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

But B^gB must be a projection from X onto $\mathcal{R}(B)$ parallel to $\mathcal{N}(B)$, so $B_1'B_1 + B_2'B_2 = I|_{\mathcal{R}(B)}$. We conclude that $B_2'B_2$ is the projection from $\mathcal{R}(B)$ onto $\mathcal{N}(B_1)$ parallel to $\mathcal{R}(B_1')$.

We will prove that $B_2|_{\mathcal{R}(B_1')} = 0$.

Notice that $\mathcal{R}(AB) = \mathcal{R}((AB)^g) = \mathcal{R}(B^gA^g) = \mathcal{R}(B_1')$. Hence $\mathcal{R}(B_1') \subset \mathcal{R}(A) \cap \mathcal{R}(B) \subset \mathcal{R}(B_1)$. Since $\mathcal{R}(B_1)$ is closed and $\mathcal{R}(B_1')$ is complemented in X, we conclude that $\mathcal{R}(B_1')$ is complemented in $\mathcal{R}(B_1)$. There exists a subspace U, such that $\mathcal{R}(B_1') \oplus U = \mathcal{R}(B_1)$. Notice that we can always take $U \subset \mathcal{N}(B_1) \oplus \mathcal{N}(B)$ and U need not to be closed. Define the operator $\overline{B_1} \in \mathcal{L}(X)$ in the following way: $\overline{B_1}|_{\mathcal{R}(B)} = B_1$ and $\overline{B_1}|_{\mathcal{N}(B)} = 0$. It follows that $\mathcal{R}(\overline{B_1}) = \mathcal{R}(B_1)$ and $\mathcal{N}(\overline{B_1}) = \mathcal{N}(B) \oplus \mathcal{N}(B_1)$. Notice that

$$\mathcal{R}(\overline{B_1}^2) = B_1(\mathcal{R}(B_1')) + \overline{B_1}(U) = \mathcal{R}(B_1) = \mathcal{R}(\overline{B_1}).$$

Also

$$\mathcal{N}(\overline{B_1}^2) = [\mathcal{N}(B) \oplus \mathcal{N}(B_1)] + U = \mathcal{N}(\overline{B_1}).$$

We conclude that $\mathrm{ind}(\overline{B_1}) \leq 1$, $U = \{0\}$ and

$$X = \mathcal{R}(\overline{B_1}) \oplus \mathcal{N}(\overline{B_1}) = \mathcal{R}(\overline{B_1}) \oplus \mathcal{N}(B_1) \oplus \mathcal{N}(B) = \mathcal{R}(B_1') \oplus \mathcal{N}(B_1) \oplus \mathcal{N}(B).$$

From $\mathcal{R}(B_1') \subset \mathcal{R}(A) \cap \mathcal{R}(B) \subset \mathcal{R}(B_1)$ we conclude that $\mathcal{R}(B_1') = \mathcal{R}(B_1) = \mathcal{R}(A) \cap \mathcal{R}(B)$. Since $\mathcal{R}(B) \subset \mathcal{R}(B_1) \oplus \mathcal{R}(B_2)$ we find $B_2|_{\mathcal{R}(B_1')} = 0$ and $\mathcal{R}(B) = \mathcal{R}(B_1) \oplus \mathcal{R}(B_2)$.

We will prove $B_1|_{\mathcal{R}(B_2')} = 0$.

Since $B|_{\mathcal{R}(B)}$ is one-to-one, we get $\mathcal{N}(B_1) \cap \mathcal{N}(B_2) = \{0\}$, hence $\mathcal{N}(B_2) = \mathcal{R}(B_1')$. We know the following

$$\begin{aligned}\mathcal{R}(B_1') + \mathcal{R}(B_2') = \mathcal{R}(B) = \mathcal{R}(B^g) = B^g(\mathcal{R}(B)) &= B_1'(\mathcal{R}(B_1)) + B_2'(\mathcal{R}(B_2)) \\ &= \mathcal{R}(B_1') \oplus \mathcal{N}(B_1),\end{aligned}$$

since $B_1'B_1 + B_2'B_2 = I|_{\mathcal{R}(B)}$ and $B_1'B_1$ is the projection from $\mathcal{R}(B)$ onto $\mathcal{R}(B_1')$ parallel to $\mathcal{N}(B_1')$. Since B^g is nonzero only on the subspace $\mathcal{R}(B) = \mathcal{R}(B_1) \oplus \mathcal{R}(B_2)$, we conclude that $\mathcal{R}(B_1') = B_1'(\mathcal{R}(B_1))$ and $\mathcal{R}(B_2') = B_2'(\mathcal{R}(B_2)) = \mathcal{N}(B_1)$. Hence, $B_1|_{\mathcal{R}(B_2')} = 0$.

Now we know

$$BB^g = \begin{bmatrix} B_1B_1' & 0 \\ 0 & B_2B_2' \end{bmatrix}$$

and BB^g commutes with

$$A^gA = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the proof is completed. □

Remark 4.9. The following quaestion is interesting: is Theorem 4.8 a new result? Our Theorem 4.8 shows that if one wants to get the reverse order rule for the group inverse, then the strong commutativity condition $AA^gBB^g = BB^gAA^g$ cannot be avoided.

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