ADDITIVE RESULTS FOR THE GENERALIZED DRAZIN INVERSE

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Abstract

Additive perturbation results for the generalized Drazin inverse of Banach space operators are presented. Precisely, if A^d denotes the generalized Drazin inverse of a bounded linear operator A on an arbitrary complex Banach space, then in some special cases $(A+B)^d$ is computed in terms of A^d and B^d. Thus, recent results of R. E. Hartwig, G. Wang and Y. Wei are extended to infinite dimensional settings with simplified proof.

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1 Introduction

Let X denote an arbitrary complex Banach space and $\mathcal{L}(X)$ denote the Banach algebra of all bounded operators on X. If $A \in \mathcal{L}(X)$, then $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\sigma(A)$, respectively, denote the range, kernel and spectrum of A.

If S is a subset of the complex plane, then $\operatorname{acc} S$ and $\operatorname{iso} S$, respectively, denote the set of all points of accumulation and the set of all isolated points of S.

If $0 \notin \operatorname{acc}\sigma(A)$, then the function $z \mapsto f(z)$ can be defined as $f(z) = 0$ in a neighbourhood of 0 and $f(z) = 1/z$ in a neighbourhood of $\sigma(A) \setminus \{0\}$. Then $z \mapsto f(z)$ is regular in a neighbourhood of $\sigma(A)$ and the generalized Drazin inverse of A is defined using the functional calculus as $A^d = f(A)$ (see the well-known Koliha's paper [12]). Notice that $A^dA = AA^d$ and $A^dAA^d = A^d$. We say that $A \in \mathcal{L}(X)$ is GD-invertible, if $0 \notin \operatorname{acc}\sigma(A)$. If A is GD-invertible, then the spectral idempotent P of A corresponding to $\{0\}$ is given by $P = I - AA^d$. The matrix

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form of A with respect to the decomposition $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$ is given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 is invertible and A_2 is quasinilpotent. We can also write

$$A = C_A + Q_A, \quad C_A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}, \tag{1}$$

and $A = C_A + Q_A$ is known as the core-quasinilpotent decomposition of A.

If A is GD-invertible, then the resolvent function $z \mapsto (zI - A)^{-1}$ is defined in a punctured neighbourhood of $\{0\}$. If $z = 0$ is the pole of the resolvent function of A, then the order of this pole is known as the Drazin index of A, denoted by $\operatorname{ind}(A)$. In this case we say that A is D-invertible. If $\operatorname{ind}(A) = k$, then A^d reduces to the ordinary Drazin inverse of A, denoted by A^D. Thus, A^D is the unique operator satisfying conditions

$$A^{k+1}A^D = A^k, \quad A^D A A^D = A^D, \quad AA^D = A^D A, \tag{2}$$

and k is the least integer such that (2) is satisfied. Now the core-quasinilpotent decomposition reduces to the core-nilpotent decomposition. Precisely, $\operatorname{ind}(A) = k$ holds if and only if k is the least integer (if it exists) such that $A_2^k = 0$ (recall notations from (1)).

If $\operatorname{ind}(A) \leq 1$, then A^D is known as the group inverse of A, denoted by $A^\#$. Also, $\operatorname{ind}(A) = 0$ if and only if A is invertible and in this case $A^D = A^{-1}$. Notice that $\operatorname{ind}(C_A) \leq 1$ always holds and (see (1))

$$C_A^\# = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The Drazin inverse in semigroups and associative rings is first introduced in [6]. The Drazin inverse of complex square matrices is investigated, among other papers and books, in [1] and [2]. A detailed treatment of the Drazin inverse in infinite dimensional spaces is given in [3] and [8]. The generalized Drazin inverse in Banach algebras is introduced in [12]. We mention that an alternative definition of a generalized Drazin inverse in a ring is also given in [7, 8, 9]. These two concepts of a generalized Drazin inverse are equivalent in the case when the ring is actually a complex Banach algebra with a unit.

In this paper we investigate the generalized Drazin inverse $(A + B)^d$ in terms of A^d and B^d. Hartwig, Wang and Wei investigated this problem in a finite dimensional case (see [11]). In this paper we generalize their results to infinite dimensional setting, using an alternative approach based on matrix representation of operaotrs relative to a space decomposition. This enables us to simplify the proofs.

2 Results

First we state the following auxilliary result.

Lemma 2.1 *If $P, Q \in \mathcal{L}(X)$ are quasinilpotent and $PQ = 0$ or $PQ = QP$, then $P + Q$ is also quasinilpotent. Hence, $(P + Q)^d = 0$.*

Proof. Case I. Let $PQ = 0$ and $\lambda \neq 0$. Then $\lambda - P$ and $\lambda - Q$ are invertible and consequently

$$\lambda - P - Q = \lambda^{-1}(\lambda - P)(\lambda - Q)$$

is invertible.

Case II. Let $PQ = QP$ and let $\sigma(P, Q)$ denote the Harte joint spectrum of (P, Q). It follows that $\sigma(P, Q) = \{(0, 0)\}$. An application of the spectral mapping theorem for this spectrum leads to $\sigma(P + Q) = \{0\}$ (see [8] for details). Thus, the proof is completed.

The following result is a generalization of Lemma 1.1 from [11].

Theorem 2.1 *Let $A, B, A + B \in \mathcal{L}(X)$ be GD-invertible and $AB = BA$. Then*

$$(A + B)^d = (C_A + C_B)^d[I + (C_A + C_B)^d(Q_A + Q_B)]^{-1}.$$

Proof. Let P be the spectral idempotent of A corresponding to $\{0\}$. Then A has the matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

with respect to the decomposition $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$, where A_1 is invertible and A_2 is quasinilpotent. Now we have

$$C_A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Q_A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Since B commutes with P, we conclude that B has the form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}.$$

From $\sigma(B) = \sigma(B_1) \cup \sigma(B_2)$ we get that B_1 and B_2 are GD-inveritble. From the definition of the generalized Drazin inverse and properties of the functional calculus we get that

$$B^d = \begin{bmatrix} B_1^d & 0 \\ 0 & B_2^d \end{bmatrix},$$

and

$$C_B = \begin{bmatrix} C_{B_1} & 0 \\ 0 & C_{B_2} \end{bmatrix}, \quad Q_B = \begin{bmatrix} Q_{B_1} & 0 \\ 0 & Q_{B_2} \end{bmatrix}.$$

From $AB = BA$ we conclude $A_iB_i = B_iA_i$ for $i = 1, 2$. Let P_1 be the spectral idempotent of B_1 in the Banach algebra $\mathcal{L}(\mathcal{N}(P))$ corresponding to $\{0\}$ and let P_2 be the spectral idempotent of B_2 in the algebra $\mathcal{L}(\mathcal{R}(P))$ corresponding to $\{0\}$. We have the matrix form of B_1

$$B_1 = \begin{bmatrix} B_3 & 0 \\ 0 & B_4 \end{bmatrix}$$

with respect to the decomposition $\mathcal{N}(P) = \mathcal{N}(P_1) \oplus \mathcal{R}(P_1)$, where B_3 is invertible and B_4 is quasinilpotent. The matrix form of B_2 is given as

$$B_2 = \begin{bmatrix} B_5 & 0 \\ 0 & B_6 \end{bmatrix},$$

with respect to the decomposition $\mathcal{R}(P) = \mathcal{N}(P_2) \oplus \mathcal{R}(P_2)$, where B_5 is invertible and B_6 is quasinilpotent. Since A_i commutes with P_i for $i = 1, 2$, we conclude that A_1 and A_2, respectively, have the following matrix forms:

$$A_1 = \begin{bmatrix} A_3 & 0 \\ 0 & A_4 \end{bmatrix},$$

where A_1 and A_3 are invertible, and

$$A_2 = \begin{bmatrix} A_5 & 0 \\ 0 & A_6 \end{bmatrix},$$

where A_5 and A_6 are quasinilpotent. The last statement follows from the fact $\sigma(A_2) = \sigma(A_5) \cup \sigma(A_6)$.

Notice that

$$A + B = \begin{bmatrix} A_3 + B_3 & 0 & 0 & 0 \\ 0 & A_4 + B_4 & 0 & 0 \\ 0 & 0 & A_5 + B_5 & 0 \\ 0 & 0 & 0 & A_6 + B_6 \end{bmatrix}.$$

Since $0 \notin \operatorname{acc} \sigma(A+B)$ and $\sigma(A+B) = \bigcup_{i=3}^{6} \sigma(A_i + B_j)$, we conclude that $A_i + B_i$ is GD-invertible for all $i = 3, 4, 5, 6$. Since A_4 is invertible and B_4 is quasinilpotent, we get

$$A_4 + B_4 = A_4(I + A_4^{-1}B_4).$$

Also, A_4 commutes with B_4. Hence, $A_4^{-1}B_4$ is quasinilpotent and $I + A_4^{-1}B_4$ is invertible. We get that $A_4 + B_4$ is invertible. Analogously, $A_5 + B_5$ is invertible. Finally, $A_6 + B_6$ is a sum of mutually commuting quasinilpotent elements. From Lemma 2.1 it follows that $A_6 + B_6$ is quasinilpotent.

We have

$$C_{B_1} = \begin{bmatrix} B_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{B_2} = \begin{bmatrix} B_5 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$C_A + C_B = \begin{bmatrix} A_3 + B_3 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 \\ 0 & 0 & B_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $A_3 + B_3$ is GD-invertible and A_4 and B_5 are invertible, we conclude that $C_A + C_B$ is GD-invertible and

$$(C_A + C_B)^d = \begin{bmatrix} (A_3 + B_3)^d & 0 & 0 & 0 \\ 0 & A_4^{-1} & 0 & 0 \\ 0 & 0 & B_5^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We also have

$$Q_A + Q_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_4 & 0 & 0 \\ 0 & 0 & A_5 & 0 \\ 0 & 0 & 0 & A_6 + B_6 \end{bmatrix}$$

and

$$(C_A + C_B)^d (Q_A + Q_B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_4^{-1} B_4 & 0 & 0 \\ 0 & 0 & B_5^{-1} A_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now it follows that

$$[I + (C_A + C_B)^d (Q_A + Q_B)]^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & (I + A_4^{-1} B_4)^{-1} & 0 & 0 \\ 0 & 0 & (I + A_5 B_5^{-1})^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and we easilly conclude that

$$\begin{aligned} &(C_A + C_B)^d [I + (C_A + C_B)^d (Q_A + Q_B)] \\ = &\begin{bmatrix} (A_3 + B_3)^d & 0 & 0 & 0 \\ 0 & (A_4 + B_4)^{-1} & 0 & 0 \\ 0 & 0 & (A_5 + B_5)^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (A + B)^d. \end{aligned}$$

Thus, the proof is completed.

Now we consider the non-commutative case. The following result is proved in [5] (see also [10] and [15] for a finite dimensional case).

Lemma 2.2 *If $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are GD-invertible, $C \in \mathcal{L}(Y, X)$ and $D \in \mathcal{L}(X, Y)$, then*

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$$

are also GD-invertible and

$$M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix}, \quad N^d = \begin{bmatrix} A^d & 0 \\ R & B^d \end{bmatrix},$$

where

$$\begin{aligned} S &= (A^d)^2 \left[\sum_{n=0}^{\infty} (A^d)^n C B^n \right] (I - BB^d) + \\ &\quad + (I - AA^d) \left[\sum_{n=0}^{\infty} A^n C (B^d)^n \right] (B^d)^2 - A^d C B^d \end{aligned}$$

and

$$\begin{aligned} R &= (B^d)^2 \left[\sum_{n=0}^{\infty} (B^d)^n D A^n \right] (I - AA^d) + \\ &\quad + (I - BB^d) \left[\sum_{n=0}^{\infty} B^n D (A^d)^n \right] (A^d)^2 - B^d D A^d. \end{aligned}$$

We need one particular case of our main result.

Theorem 2.2 *If $P, Q \in \mathcal{L}(X)$ are GD-invertible, $Q \in \mathcal{L}(X)$ is quasinilpotent and $PQ = 0$, then $P + Q$ is GD-invertible and*

$$(P+Q)^d = \sum_{n=0}^{\infty} Q^n (P^d)^{n+1}.$$

Proof. Since P is GD-invertible, we conclude that P has the matrix form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

where P_1 is invertible and P_2 is quasinilpotent. From $PQ = 0$ we conclude that Q has the matrix form

$$Q = \begin{bmatrix} 0 & 0 \\ Q_1 & Q_2 \end{bmatrix},$$

where $P_2Q_1 = 0$ and $P_2Q_2 = 0$. Since $\{0\} = \sigma(Q) = \{0\} \cup \sigma(Q_2)$, we conclude that Q_2 is quasinilpotent. Now we have

$$P + Q = \begin{bmatrix} P_1 & 0 \\ Q_1 & P_2 + Q_2 \end{bmatrix},$$

where P_1 is invertible and P_2+Q_2 is quasinilpotent (see Lemma 2.1). From Lemma 2.2 we get that

$$(P+Q)^d = \begin{bmatrix} P_1^{-1} & 0 \\ \sum\limits_{n=0}^{\infty}(P_2+Q_2)^n Q_1 P_1^{-n-2} & 0 \end{bmatrix}.$$

By the induction on n and using $P_2Q_1 = 0$ and $P_2Q_2 = 0$, we prove that $(P_2 + Q_2)^n Q_1 = Q_2^n Q_1$ holds for all $n \geq 0$. Hence,

$$\sum_{n=0}^{\infty}(P_2+Q_2)^n Q_1 P_1^{-n-2} = \sum_{n=0}^{\infty} Q_2^n Q_1 P_1^{-n-2}.$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} Q^n (P^d)^{n+1} &= P^d + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ Q_2^{n-1}Q_1 & Q_2^n \end{bmatrix} \begin{bmatrix} P_1^{-n-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} P_1^{-1} & 0 \\ \sum\limits_{n=0}^{\infty} Q_2^n Q_1 P_1^{-n-2} & 0 \end{bmatrix}. \end{aligned}$$

Thus, the proof is completed.

The following result is a generalization of [11, Theorem 2.1].

Theorem 2.3 *If $P, Q \in \mathcal{L}(X)$ are GD-invertible and $PQ = 0$, then $P + Q$ is GD-invertible and*

$$\begin{aligned} (P+Q)^d &= (I - QQ^d)\left[\sum_{n=0}^{\infty} Q^n (P^d)^n\right] P^d \\ &\quad + Q^d \left[\sum_{n=0}^{\infty} (Q^d)^n P^n\right] (I - PP^d). \end{aligned}$$

Proof. Since Q is GD-invertible, we can write

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

where Q_1 is invertible and Q_2 is quasinilpotent. From $PQ = 0$ we obtain the following matrix form of P

$$P = \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix},$$

where $P_1Q_2 = 0$ and $P_2Q_2 = 0$. From $\sigma(P) = \{0\} \cup \sigma(P_2)$ and $0 \notin \operatorname{acc}\sigma(P)$ we conclude that P_2 is GD-invertible. Now we have

$$P + Q = \begin{bmatrix} Q_1 & P_1 \\ 0 & P_2 + Q_2 \end{bmatrix}.$$

From Theorem 2.2 we know that $P_2 + Q_2$ is GD-invertible. Hence, using Lemma 2.2 we get that $P + Q$ is GD-invertible and

$$(P+Q)^d = \begin{bmatrix} Q_1^{-1} & S \\ 0 & (P_2+Q_2)^d \end{bmatrix},$$

where

$$S = \left[\sum_{n=0}^{\infty} Q_1^{-n-2} P_1 (P_2+Q_2)^n\right]\left[I - (P_2+Q_2)(P_2+Q_2)^d\right] - Q_1^{-1}P_1(P_2+Q_2)^d.$$

Using $P_1Q_2 = 0$ and $P_2Q_2 = 0$ we prove $P_1(P_2+Q_2)^n = P_1P_2^n$ for all $n \geq 0$. Now, using Lemma 2.2, Theorem 2.2, and facts $P_1Q_2 = 0$ and $P_2Q_2 = 0$, we compute

$$\begin{aligned}
&\left[\sum_{n=0}^{\infty} Q_1^{-n-2} P_1 (P_2+Q_2)^n\right]\left[I - (P_2+Q_2)(P_2+Q_2)^d\right] \\
= \ & \left[Q_1^{-2}P_1 + \sum_{n=1}^{\infty} Q_1^{-n-2}P_1P_2^n\right]\left[I - (P_2+Q_2)\left(P_2^d + \sum_{n=1}^{\infty} Q_2^n (P_2^d)^{n+1}\right)\right] \\
= \ & \left[Q_1^{-2}P_1 + \sum_{n=1}^{\infty} Q_1^{-n-2}P_1P_2^n\right]\left[I - P_2P_2^d - \sum_{n=1}^{\infty} Q_2^n (P_2^d)^{n}\right] \\
= \ & Q_1^{-2}P_1(I - P_2P_2^d) - Q_1^{-2}P_1\sum_{n=1}^{\infty} Q_2^n(P_2^d)^n + \sum_{n=1}^{\infty} Q_1^{-n-2}P_1P_2^n(I - P_2P_2^d) \\
& - \left[\sum_{n=1}^{\infty} Q_1^{-n-2}P_1P_2^n\right]\left[\sum_{n=1}^{\infty} Q_2^n(P_2^d)^n\right] \\
= \ & \sum_{n=0}^{\infty} Q_1^{-n-2}P_1P_2^n(I - P_2P_2^d).
\end{aligned}$$

Also, using Theorem 2.2 we get

$$Q_1^{-1}P_1(P_2+Q_2)^d = Q_1^{-1}P_1\sum_{n=0}^{\infty} Q_2^n(P_2^d)^{n+1} = Q_1^{-1}P_1P_2^d.$$

On the other hand, notice that from Lemma 2.2 we get

$$P^d = \begin{bmatrix} 0 & P_1(P_2^d)^2 \\ 0 & P_2^d \end{bmatrix}.$$

Hence we have

$$\begin{aligned}
& (I - QQ^d)\left[P^d + \sum_{n=1}^{\infty} Q^n (P^d)^{n+1}\right] + \left[Q^d + \sum_{n=1}^{\infty} (Q^d)^{n+1} P^n\right](I - PP^d) \\
= & \begin{bmatrix} 0 & 0 \\ 0 & P_2^d \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_1^n & 0 \\ 0 & Q_2^n \end{bmatrix} \begin{bmatrix} 0 & P_1(P_2^d)^{n+2} \\ 0 & (P_2^d)^{n+1} \end{bmatrix} \\
& + \begin{bmatrix} Q_1^{-1} & -Q_1^{-1} P_1 P_2^d \\ 0 & 0 \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & Q_1^{-n-1} P_1 P_2^{n-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -P_1 P_2^d \\ 0 & I - P_2 P_2^d \end{bmatrix} \\
= & \begin{bmatrix} Q_1^{-1} & \sum\limits_{n=1}^{\infty} Q_1^{-n-1} P_1 P_2^{n-1}(I - P_2 P_2^d) - Q_1^{-1} P_1 P_2^d \\ 0 & \sum\limits_{n=0}^{\infty} Q_2^n (P_2^d)^{n+1} \end{bmatrix}.
\end{aligned}$$

Thus, the proof is completed.

We can also prove the following result, generalizing [11, Corollary 2.2].

Theorem 2.4 *Let $A, H \in \mathcal{L}(X)$ and let A be a GD-invertible operator. Let $F \in \mathcal{L}(X)$ be an idempotent commuting with A such that $FH = H$. If $R = (A - H)F$ is GD-invertible, then $A - H$ is GD-invertible and*

$$\begin{aligned}
(A - H)^d = \; & R^d - \sum_{n=0}^{\infty} (R^d)^{n+2} H (I - F) A^n (I - AA^d) \\
& + (I + R^d H)(I - F) A^d \\
& - (I - RR^d) \sum_{n=0}^{\infty} (A - H)^n H (I - F)(A^d)^{n+2}.
\end{aligned}$$

Proof. Since $F^2 = F$, we have $X = \mathcal{R}(F) \oplus \mathcal{N}(F)$ and $F = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ with respect to this decomposition. Operators A and F mutually commute, hence $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. From $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ we know that $A_1 \in \mathcal{L}(\mathcal{R}(F))$ and $A_2 \in \mathcal{L}(\mathcal{N}(F))$ are GD-invertible and $A^d = \begin{bmatrix} A_1^d & 0 \\ 0 & A_2^d \end{bmatrix}$. Since $FH = H$, we conclude that the matrix form of H is given by $H = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix}$. Now we have

$$A - H = \begin{bmatrix} A_1 - H_1 & -H_2 \\ 0 & A_2 \end{bmatrix}.$$

Notice that

$$R=\begin{bmatrix} A_1-H_1 & 0 \\ 0 & 0 \end{bmatrix}$$

is GD-invertible, implying that $A_1-H_1\in\mathcal{L}(\mathcal{R}(F))$ is GD-invertible. From Lemma 2.2 it follows that $A-H$ is GD-invertible and

$$(A-H)^d=\begin{bmatrix} (A_1-H_1)^d & 0 \\ 0 & A_2^d \end{bmatrix}+\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} S &= (A_1-H_1)^dH_2A_2^d-\sum_{n=0}^{\infty}[(A_1-H_1)^d]^{n+2}H_2A_2^n(I-A_2A_2^d) \\ &\quad -[I-(A_1-H_1)(A_1-H_1)^d]\sum_{n=0}^{\infty}(A_1-H_1)^nH_2(A_2^d)^{n+2}. \end{aligned}$$

Notice that $R^d=\begin{bmatrix} (A_1-H_1)^d & 0 \\ 0 & 0 \end{bmatrix}$. Now a straightforward computation shows:

$$\begin{aligned} &(I-RR^d)(A-H)^nH(I-F)(A^d)^{n+2}= \\ &=\begin{bmatrix} 0 & [I-(A_1-H_1)(A_1-H_1)^d](A_1-H_1)^nH_2(A_2^d)^{n+2} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$(I+R^dH)(I-F)A^d=\begin{bmatrix} 0 & (A_1-H_1)^dH_2A_2^d \\ 0 & A_2^d \end{bmatrix}$$

and

$$(R^d)^{n+2}H(I-F)A^n(I-AA^d)=\begin{bmatrix} 0 & \left((A_1-H_1)^d\right)^{n+2}H_2A_2^n(I-A_2A_2^d) \\ 0 & 0 \end{bmatrix}.$$

Hence, the proof is completed.

Campbell and Meyer (see [2]) investigated the continuity properties of the Drazin inverse of complex square matrices, but they didnot establish the norm estimates for the perturbation of the Drazin inverse. Their results are extended to infinite dimensional settings by Rakočević (see [17]) and Koliha and Rakočević (see [13]), but norm estimates are not established there. It is interesting to mention that special cases of these preturbation results are already known. For example, see [20] for complex square matrices, [18] for complex Banach algebras and [4] for unbounded operators on Banach spaces. See also Case (4) of the following section. Hence, in this paper we partially solve the previous problem of Campbell and Meyer and extend some well-known results from previous papers.

3 Special cases

Many interesting special cases of our Theorem 2.4 are considered in [11] for matrices. Some of them are generalizations of well-known resuts.

Case (1). If $HF = 0$, then $(A-H)^n FH = A^n H$ for $n \geq 0$. Consequently,

$$\begin{aligned}(A-H)^d &= A^d F - \sum_{n=0}^{\infty} (A^d)^{n+2} HA^n (I - AA^d) \\ &\quad + (I - F + A^d H)A^d - \sum_{n=0}^{\infty} A^n (I - AA^d) H (A^d)^{n+2}.\end{aligned}$$

Case (1a). If $HF = 0$ and $F = AA^d$, then $HA^d = HAA^d = 0$ and

$$(A-H)^d = A^d - \sum_{n=0}^{\infty} (A^d)^{n+2} HA^n.$$

Case (1b). If $HF = 0$ and $F = I - AA^d$, then $A^d H = 0$ and

$$(A-H)^d = A^d - \sum_{n=0}^{\infty} A^n H (A^d)^{n+2}.$$

Case (2). If $F = AA^d$, then

$$(A-H)^d = R^d - \sum_{n=0}^{\infty} (R^d)^{n+2} HA^n (I - AA^d).$$

Case (2a). Let $F = AA^d$ and let $U = I - A^d HAA^d$ be invertible. Recall notation from Theorem 2.4, noticing that A_1 is invertible and A_2 is quasinilpotent. Then $U = \begin{bmatrix} I - A_1^{-1} H_1 & 0 \\ 0 & I \end{bmatrix}$. Since U is invertible, then $I - A_1^{-1} H_1$ is invertible and $A_1 - H_1 = A_1 (I - A_1^{-1} H_1)$ is invertible. Hence, $\operatorname{ind}(R) \leq 1$,

$$R^\# = \begin{bmatrix} (I - A_1^{-1} H_1)^{-1} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$(A-H)^d = R^\# - \sum_{n=0}^{\infty} (R^\#)^{n+2} HA^n (I - AA^d).$$

Let $V = I - AA^d HA^d = \begin{bmatrix} I - H_1 A_1^{-1} & 0 \\ 0 & I \end{bmatrix}$ and

$$W = I - A^d H = \begin{bmatrix} I - A_1^{-1} H_1 & -A_1^{-1} H_2 \\ 0 & I \end{bmatrix}.$$

Notice that U is invertible if and only if $1 \notin \sigma(A_1^{-1}H_1)$, if and only if $1 \notin \sigma(H_1A_1^{-1})$, since it is well-known that for $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,X)$

$$\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}. \tag{3}$$

Hence, U is invertible if and only if V is invertible if and only if W is invertible. We also easilly verify $R^{\#} = U^{-1}A^d = A^dV^{-1}$. Hence, this result generalizes the main results in [14, 16, 20, 19].

Case (2b). If $F = I - AA^d$, then $A^dH = A^dF = 0$,

$$R = A(I - AA^d) - (I - AA^d)H(I - AA^d)$$

and

$$(A - H)^d = R^d + (I + R^dH)A^d - (I - RR^d)\sum_{n=0}^{\infty}(A - H)^nH(A^d)^{n+2}.$$

Case (3). Let $AA^dHF = HFAA^d = HF$, let $U = I - A^dHF$ be invertible and $(AF)^{\#}$ exists. Recall notation from Theorem 2.4. Then $AF = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, implying $\operatorname{ind}(A_1) \le 1$. From $AA^dHF = HFAA^d = HF$ we conclude $A_1A_1^{\#}H_1 = H_1A_1A_1^{\#} = H_1$. We also have

$$U = \begin{bmatrix} I - A_1^{\#}H_1 & 0 \\ 0 & I \end{bmatrix}.$$

Let

$$V = I - HFA^d = V = \begin{bmatrix} I - H_1A_1^{\#} & 0 \\ 0 & I \end{bmatrix}.$$

By (3) we conclude that U is invertible if and only if V is invertible. Now a matrix computation shows $R = AFU = VFA$. From $A_1^{\#}(I - H_1A_1^{\#}) = (I - A_1^{\#}H_1)A_1^{\#}$ we get the equality $A^dFV = UA^dF$. Let $Y = U^{-1}A^dF = A^dFV^{-1}$. Then it is easy to verify $Y = R^{\#}$. Finally, we have

$$(A - H)^d = R^{\#} - \sum_{n=0}^{\infty}(R^{\#})^{n+2}H(I - AA^d)A^n.$$

Case (4). If $FH = HF = F$, then

$$(A - H)^d = R^d + (I - F)A^d.$$

Moreover, if $F = AA^d$ and $U = I - A^dH$ is invertible, then

$$(A - H)^d = R^d = U^{-1}A^d.$$

Case (4) shows that results of this paper are more general than the corresponding results in [20]. Analogous results are proved in complex Banach algebras in [18]. Results of this paper are more general then results in [18] if we consider a Banach algebra of all operators on a fixed complex Banach space. We expect that all results of this paper should be valid in an arbitrary complex Banach algebra with a unit, but this will be a matter of further investigations. It is interesting to mention results related to this Case (4), concerning the Drazin inverse for closed linear operators, as it is done in [4].

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications* (Wiley-Interscience, New York, 1974).

[2] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations* (Pitman, New York, 1979).

[3] S. R. Caradus, 'Generalized Inverses and Operator Theory', Queen's Paper in Pure and Applied Mathematics (Queen's University, Kingston, Ontario, 1978).

[4] N. Castro Gonzalez and J. J. Koliha, *Perturbation of the Drazin inverse for closed linear operators*, Integral Eq. Operator Theory **36** (2000), 92–106.

[5] D. S. Djordjević and P. S. Stanimirović, *On the generalized Drazin inverse and generalized resolvent*, Czechoslovak Math. J. (2001) (to appear).

[6] M. P. Drazin, *Pseudoinverses in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506–514.

[7] R. E. Harte, *Spectral projections*, Irish Math. Soc. Newsletter **11** (1984), 10–15.

[8] R. E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, (New York, Marcel Dekker, 1988).

[9] R. E. Harte, *On quasinilpotents in rings*, PanAm. Math. J. **1** (1991), 10–16.

[10] R. E. Hartwig and J. M. Shoaf, *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, Austral. J. Math. Ser. A **24** (1977), 10–34.

[11] R. E. Hartwig, G. Wang and Y. Wei, *Some additive results on Drazin inverse*, Linear Algebra Appl. **322** (2001), 207–217.

[12] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. **38** (1996), 367–381.

[13] J. J. Koliha and V. Rakočević, *Continuity of the Drazin inverse II*, **131** (1998), 167–177.

[14] C. D. Meyer, Jr., *The condition number of a finite Markov chains and perturbation bounds for the limiting probabilities*, SIAM J. Algebraic Discrete Methods **1** (1980), 273–283.

[15] C. D. Meyer, Jr. and N. J. Rose, *The index and the Drazin inverse of block triangular matrices*, SIAM J. Appl. Math. **33**, 1 (1977), 1–7.

[16] C. D. Meyer, Jr. and J. M. Shoaf, *Updating finite Markov chains by using techniques of group inversion*, J. Statist. Comput. Simulation **11** (1980), 163–181.

[17] V. Rakočević, *Continuity of the Drazin inverse*, J. Operator Theory **41** (1999), 55–68.

[18] V. Rakočević and Y. Wei, *The perturbation theory for the Drazin inverse and its applications II*, J. Australian Math. Soc. **70** (2001), 189-197.

[19] Y. Wei, *On the perturbation of the group inverse and the oblique projection*, Appl. Math. Comput. **98** (1999), 29–42,

[20] Y. Wei and G. Wang, *The perturbation theory for the Drazin inverse and its applications*, Linear Algebra Appl. **258** (1997), 179–186.

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