ON INTEGRAL REPRESENTATION OF THE GENERALIZED INVERSE $\mathbf{A}^{(2)}_{\mathbf{T},\mathbf{S}}$

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Abstract

We present a general integral representation for the generalized inverse $A^{(2)}_{T,S}$, which extends earlier result on the Moore-Penrose inverse, weighted Moore-Penrose inverse and Drazin inverse.

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1 Introduction

Goretsch [3] presented an integral represntation of the Moore-Penrose inverse $T^\dagger$ of a bounded linear operator $T \in \mathcal{L}(H_1, H_2)$ with closed range $\mathcal{R}(T)$ in Hilbert space

$$T^\dagger = \int_0^\infty exp(-T^*Tt)T^* dt, \tag{1}$$

where H_1, H_2 are Hilbert spaces.

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Wei and Wu [7] extended the result of Groetsch to the weighted Moore-Penrose inverse of matrix $A \in \mathbf{C}^{m\times n}$,

$$A^{\dagger}_{M,N} = \int_0^{\infty} exp(-A^{\#}At)A^{\#}dt, \tag{2}$$

where $A^{\#} = N^{-1}A^{*}M$, M and N are Hermitian positive definite matrices of order m and n, respectively.

Gonzalez, Koliha and Wei [2] gave a simple integral representation of the Drazin inverse a^D in Banach algebras: let $a \in \mathcal{A}$ be a Drazin invertible element of a finite Drazin index $k \geq 1$ such that the nonzero spectrum of a^{m+1} lies in the open right half of the complex plane for some $m \geq k$. Then

$$a^D = \int_0^{\infty} exp(-a^{m+1}t)a^m dt. \tag{3}$$

The above-mentioned results motivate us to investigate the outer inverse $A^{(2)}_{T,S}$ of a matrix $A \in \mathbf{C}^{m\times n}$, since we have observed that the traditional generalized inverses (see [1]), such as the Moore-Penrose inverse $A^{\dagger}$, the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$, the Drazin inverse A^D, the group inverse A^g, etc., are outer inverses with prescribed range and kernel. The generalized inverse $A^{(2)}_{T,S}$ of $A \in \mathbf{C}^{m\times n}$ is the matrix $X \in \mathbf{C}^{n\times m}$ satisying

$$XAX = X,\ \mathcal{R}(X) = T,\ \mathcal{N}(X) = S.$$

Recently, Wei [6] established the integral representaton for the generalized inverse $A_{T,S}{}^{(}2)$. Let $A \in \mathbf{C}^{m\times n}$, T and S be subspaces of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively. Suppose $G \in \mathbf{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. If any nonzero eigenvalue λ of GA satisfy $Re\,\lambda > 0$, then

$$A^{(2)}_{T,S} = \int_0^{\infty} exp(-GAt)Gdt. \tag{4}$$

In this paper we will give a general integral representation for the generalized inverse $A^{(2)}_{T,S}$ which drops the restriction on the spectrum of GA and extends the earlier result on Drazin inverse [2].

Funtamental lemmas are needed in what follows.

Lemma 1.1 *Let $A \in \mathbf{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbf{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbf{C}^m$ of dimension $m - s$. In addition, suppose $G \in \mathbf{C}^{n\times n}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. If A has an outer inverse $A^{(2)}_{T,S}$, then $ind(GA) = ind(AG) = 1$. Further, we have*

$$A^{(2)}_{T,S} = (GA)^g G = G(AG)^g. \tag{5}$$

Lemma 1.2 *Let $A \in \mathbf{C}^{n\times n}$ be a nonsingular matrix with $Re\,\sigma(A) > 0$. Then*

$$A^{-1} = \int_0^\infty ext(-At)dt. \tag{6}$$

In this paper for any matrix $A \in \mathbf{C}^{n\times n}$ we denote its spectrum by $\sigma(A)$. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represents the range and the null space of A, respectively. We define the index of A, written $\mathrm{ind}\,(A)$, to be the least nonnegative k for which $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$ holds.

2 Main results

In this section we will present a general integral representation of the generalized inverse $A^{(2)}_{T,S}$. Throughout this section, we let A, T and S to be the same as in Lemma 1.1. In addition, let $G \in \mathbf{C}^{n\times m}$ be such that

$$\mathcal{R}(G) = T \quad \text{and} \quad \mathcal{N}(G) = S. \tag{7}$$

First we develop the algebraic structures of A and G.

Theorem 2.1 *Let A, T and S be the same as in Lemma 1.1 and $G \in \mathbf{C}^{n\times m}$ satisfies (7). Then we have*

$$A = Q^{-1}\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix}P,\ G = P^{-1}\begin{bmatrix} G_{11} & 0 \\ 0 & 0\end{bmatrix}Q,\ A^{(2)}_{T,S} = P^{-1}\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0\end{bmatrix}Q, \tag{8}$$

where P, Q, A_{11} and G_{11} are nonsingular matrices.

Proof. It follows from Lemma 1.1 that

$$\mathrm{ind}\,(AG) = \mathrm{ind}\,(GA) = 1.$$

There is a Jordan canonical form of AG and GA as folows:

$$GA = P^{-1}\begin{bmatrix} C & 0 \\ 0 & 0\end{bmatrix}P, \quad AG = Q^{-1}\begin{bmatrix} D & 0 \\ 0 & 0\end{bmatrix}Q,$$

where C and D are invertible matrices of the same order. Partition A and G as

$$A = Q^{-1}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}P, \quad G = P^{-1}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}Q.$$

It is easy to check that

$$\begin{aligned}(GA)^g G &= P^{-1}\begin{bmatrix} C^{-1} & 0 \\ 0 & 0\end{bmatrix}PP^{-1}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}Q \\ &= P^{-1}\begin{bmatrix} C^{-1}G_{11} & C^{-1}G_{12} \\ 0 & 0\end{bmatrix}Q.\end{aligned}$$

Similarly, we have

$$G(AG)^g = P^{-1} \begin{bmatrix} G_{11}D^{-1} & 0 \\ G_{21}D^{-1} & 0 \end{bmatrix} Q.$$

Since $A^{(2)}_{T,S} = (GA)^g G = G(AG)^g$, we have

$$C^{-1}G_{12} = 0 \quad \text{and} \quad G_{21}D^{-1} = 0,$$

i.e.

$$G_{12} = 0 \quad \text{and} \quad G_{21} = 0.$$

Applying a little algebra, we obtain

$$GA = P^{-1} \begin{bmatrix} G_{11}A_{11} & G_{11}A_{12} \\ G_{22}A_{21} & G_{22}A_{22} \end{bmatrix} P = P^{-1} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} P,$$

and

$$AG = Q^{-1} \begin{bmatrix} A_{11}G_{11} & A_{12}G_{22} \\ A_{21}G_{11} & A_{22}G_{22} \end{bmatrix} Q = Q^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q.$$

We deduce that

$$G_{11}A_{11} = C \text{ (nonsingular)}, \; A_{11}G_{11} = D \text{ (nonsingular)},$$

that is to say both A_{11} and G_{11} are invertible. From $G_{11}A_{12} = 0$ and $A_{21}G_{11} = 0$ we obtain

$$A_{12} = 0 \quad \text{and} \quad A_{21} = 0.$$

Finally, from the facts $G = GAA^{(2)}_{T,S} = A^{(2)}_{T,S}AG$ and $AA^{(2)}_{T,S} = AG(AG)^g$, with $A^{(2)}_{T,S}A = (GA)^g GA$, we have

$$\begin{aligned} G &= P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} Q = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} QQ^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q \\ &= P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q, \end{aligned}$$

i.e. $G_{22} = 0$.

Thus, we get

$$A = Q^{-1} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} P, \quad G = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q$$

and

$$A^{(2)}_{T,S} = (GA)^g G = P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

We have just finished the proof.

Now we are in a position to derive the general integral representation for the generalized inverse $A^{(2)}_{T,S}$.

Theorem 2.2 *Suppose that A, T and S be the same as in Lemma 1.1 and $G \in \mathbf{C}^{n\times m}$ satisfying (7). Then we have*

$$A^{(2)}_{T,S} = \int_0^\infty exp\Big[- G(GAG)^* GAt\Big] G(GAG)^* G dt. \tag{9}$$

Proof. It follows from Theorem 2.1 that

$$A = Q^{-1} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} P,\ G = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q \text{ and } A^{(2)}_{T,S} = P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

We denote

$$QQ^* = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \text{ and } (P^{-1})^* P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

It is obviously that Q_{11} and P_{11} are Hermitian positive definite matrices, their square roots $Q_{11}^{1/2}$ and $P_{11}^{1/2}$ are also Hermitian positive definite matrices. By a direct computation we have

$$G(GAG)^* Q = P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q$$

and

$$G(GAG)^* GA = P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} A_{11} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

We notice that

$$\begin{aligned} & \sigma[G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} A_{11}] = \\ = \ & \sigma[Q_{11}^{1/2} Q_{11}^{1/2} (G_{11}A_{11}G_{11})^* P_{11}^{1/2} P_{11}^{1/2} (G_{11}A_{11}G_{11})] \\ = \ & \sigma[(P_{11}^{1/2} G_{11}A_{11}G_{11} Q_{11}^{1/2})^* (P_{11}^{1/2} G_{11}A_{11}G_{11} Q_{11}^{1/2})] > 0. \end{aligned}$$

It follows from Lemma 1.2 that

$$\begin{aligned} & \int_0^\infty exp\Big[- G(GAG)^* GAt\Big] G(GAG)^* G dt \\ = \ & P^{-1} \begin{bmatrix} \int_0^\infty [-G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} A_{11} t] dt & 0 \\ 0 & 0 \end{bmatrix} P \times \\ \times \ & P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q \\ = \ & P^{-1} \begin{bmatrix} [G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} A_{11}]^{-1} G_{11}Q_{11}(G_{11}A_{11}G_{11})^* P_{11} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q \\ = \ & P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q = A^{(2)}_{T,S}. \end{aligned}$$

The proof is complete.

3 Concluding remarks

In this paper we have developed the integral representation for the generalized inverse $A_{T,S}^{(2)}$ of a complex matrix A. In our oppinion, it is worth establishing the same result in Hilbert spaces or C^*-algebras.

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