OPERATORS CONSISTENT IN REGULARITY

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Abstract. If $\mathcal{S}(X)$ is an arbitrary subset of $\mathcal{L}(X)$ (where $\mathcal{L}(X)$ is the set of all bounded operators on a Banach space X), then we say that $B \in \mathcal{L}(X)$ is $\mathcal{S}$-consistent, or consistent in $\mathcal{S}(X)$, provided that for all $A \in \mathcal{L}(X)$ the following holds:

$$AB \in \mathcal{S}(X) \quad \text{if and only if} \quad BA \in \mathcal{S}(X).$$

It is convenient to take that $\mathcal{S}(X)$ is close to the set of all invertible operators on X, or that $\mathcal{S}(X)$ contains regular operators. Here "regular" means that $\mathcal{S}(X)$ is equal to the set of invertible, left (right) invertible, Fredholm, left (right) Fredholm, Weyl, or Browder operators on X. In this article we completely describe operators consistent in the previous regularities.

1. Introduction

Let X denote an arbitrary infinite dimensional complex Banach space and $\mathcal{L}(X)$ denote the set of all bounded operators on X. For $T \in \mathcal{L}(X)$ we use $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, to denote the kernel and the range of T.

Let $\mathcal{S}(X)$ denote an arbitrary subset of $\mathcal{L}(X)$. We say that $B \in \mathcal{L}(X)$ is $\mathcal{S}$-consistent, or consistent in $\mathcal{S}(X)$, provided that for all $A \in \mathcal{L}(X)$ the following holds [6]:

$$AB \in \mathcal{S}(X) \quad \text{if and only if} \quad BA \in \mathcal{S}(X).$$

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In this article we consider the cases when $\mathcal{S}(X)$ is the invertible, left or right invertible operators, giving in particular the main results of [6], as well as several kinds of Fredholm operators.

Section 2 is devoted to operators consistent in invertibility. In this section the main result for bounded operators on Banach spaces is proved. Section 2 contains generalizations of corresponding results from [6]. Also, some new aspects of operators consistent in left and right invertibility are considered. We also introduce a concept of strictly left singular and strictly right singular operators on Banach spaces. It seems to be a natural generalization of the known classes of α-strictly singular and α-strictly cosingular operators on Hilbert spaces.

Section 3 seems to be essentially new, where we consider Fredholm, left and right Fredholm, Weyl and Browder consistent operators. Also, the classes of essentially strictly left (right) singular operators are introduced. As a corollary, we obtain the main result from [6]: our Theorem 3.5 is the same as [6, Theorem 3.7].

2. Operators consistent in invertibility

Let $\mathcal{G}(X)$, $\mathcal{G}_l(X)$ and $\mathcal{G}_r(X)$, respectively, denote the set of all invertible, left invertible and right invertible operators on X. Recall that $T \in \mathcal{G}_l(X)$ if and only if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \overline{\mathcal{R}(T)}$ is a complemented subspace of X. Analogously, $T \in \mathcal{G}_r(X)$ if and only if $\mathcal{R}(T) = X$ and $\mathcal{N}(T)$ is a complemented subspace of X.

We say that $T \in \mathcal{L}(X)$ is relatively regular provided that there exists some $S \in \mathcal{L}(X)$, such that $TST = T$. In this case S is called an inner generalized inverse of T. It is well-known that T is relatively regular if and only if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are closed and complemented subspaces of X [3], [11]. If S is an inner generalized inverse of T, it is well-known that TS is a projection of X onto $\mathcal{R}(T)$, and $I - ST$ is a projection of X onto $\mathcal{N}(T)$.

We say that $U \in \mathcal{L}(X)$ is a reflexive generalized inverse of T, provided that $TUT = T$ and $UTU = U$. If S is an inner generalized inverse of T, then STS is a reflexive generalized inverse of T.

Firstly we assume that $\mathcal{S}(X) = \mathcal{G}(X)$ is the set of all invertible elements of $\mathcal{L}(X)$. The following theorem gives a complete answer to the question whether an operator is consistent or not consistent in invertibility.

Theorem 2.1. *Let $B \in \mathcal{L}(X)$. Then B is $\mathcal{G}$-consistent, if and only if one of the following five mutually disjoint cases occurs:*

(1) *B is invertible;*
(2) *$\mathcal{R}(B)$ is not closed;*
(3) *$\mathcal{N}(B) \neq \{0\}$ and $\mathcal{R}(B) = \overline{\mathcal{R}(B)} \neq X$;*
(4) *$\mathcal{N}(B) = \{0\}$, $\mathcal{R}(B) = \overline{\mathcal{R}(B)}$ and $\mathcal{R}(B)$ is not complemented in X;*
(5) *$\mathcal{N}(B) \neq \{0\}$, $\mathcal{R}(B) = X$ and $\mathcal{N}(B)$ is not complemented in X.*

Also, B is not $\mathcal{G}$-consistent if and only if one of the following two mutually disjoint cases occurs:

(6) *$\mathcal{N}(B) = \{0\}$, $\mathcal{R}(B) = \overline{\mathcal{R}(B)}$ and $\mathcal{R}(B)$ is a proper complemented subspace of X;*
(7) *$\mathcal{N}(B) \neq \{0\}$, $\mathcal{R}(B) = X$ and $\mathcal{N}(B)$ is a complemented subspace of X.*

Proof. If B is invertible, then $AB = B^{-1}(BA)B$, so (1) follows. To prove (2), suppose that $\mathcal{R}(B)$ is not closed. Then $\mathcal{R}(BA) \neq X$ so BA is not invertible for all $A \in \mathcal{L}(X)$. Suppose that there exists some $A \in \mathcal{L}(X)$ such that AB is invertible. Then B is left invertible and B is relatively regular. It follows that $\mathcal{R}(B)$ is closed, which contradicts our previous assumptions. Now, AB is not invertible and (2) follows. To prove (3), suppose that $\mathcal{N}(B) \neq \{0\}$ and $\mathcal{R}(B) \neq X$. Obviously, $\mathcal{N}(B) \subset \mathcal{N}(AB)$, so AB is not invertible for all $A \in \mathcal{L}(X)$. Also, $\mathcal{R}(BA) \subset \mathcal{R}(B) \neq X$, so BA is not invertible for all $A \in \mathcal{L}(X)$ and (3) follows.

To prove (4), suppose that $\mathcal{N}(B) = \{0\}$, $\mathcal{R}(B)$ is closed and $\mathcal{R}(B)$ is not complemented in X. It follows that BA is not invertible for all $A \in \mathcal{L}(X)$. Suppose that there exists $A \in \mathcal{L}(X)$ such that AB is invertible. Then B is left invertible, so B is relatively regular and $\mathcal{R}(B)$ is complemented. The obtained contradiction finishes the proof of (4).

We prove (5). Let $\mathcal{N}(B) \neq \{0\}$, $\mathcal{R}(B) = X$ and $\mathcal{N}(B)$ is not complemented in X. Obviously, AB is not invertible for all $A \in \mathcal{L}(X)$. Suppose that there exists $A \in \mathcal{L}(X)$ such that BA is invertible. It follows that B is right invertible, so B is relatively regular and $\mathcal{N}(B)$ is complemented in X.

To prove (6), suppose that $\mathcal{N}(B) = \{0\}$, $\mathcal{R}(B)$ is closed and a proper complemented subspace of X. It follows that BA is not invertible for all $A \in \mathcal{L}(X)$. However, B is left invertible, so there exists an operator S, such that $SB = I$. It follows that B is not $\mathcal{G}$-consistent.

The proof of (7) is similar to the proof of (6), since in that case B is right invertible and AB is not invertible for all $A \in \mathcal{L}(X)$. $\square$

Remark 2.2. *If X is a Hilbert space, the cases (4) and (5) of Theorem 2.1 are not possible. In the case when X is a Hilbert space, our Theorem 2.1 reduces to* [6, Theorem 1.1].

In [6] these kind of results are used in determining the closure of invertible operators on Hilbert spaces. The following notations and results are taken from [9], [10] and [11].

An operator $T \in \mathcal{L}(X)$ is called decomposably regular, if there exists an invertible operator $S \in \mathcal{L}(X)$, such that $TST = T$. It is well-known that T is decomposably regular if and only if T is relatively regular and $\mathcal{N}(T)$ is isomorphic to $X/\mathcal{R}(T)$ [10]. We shall use the following result [9].

Lemma 2.2. *If T is relatively regular, then $T \in cl\mathcal{G}(X)$ if and only if T is decomposably regular.*

Notice that Lemma 2.2 holds more generally, in arbitrary unital Banach algebras. This result is enlarged to the closure of Fredholm operators [12], and in more general settings [5]. The closure of Fredholm operators will be considered in Section 3.

If Z is a finite dimensional space, $\dim Z$ denotes its dimension. If Z is an infinite dimensional Banach space, we simply write $\dim Z = \infty$. On the other hand, if Z is an arbitrary Hilbert space, then $\dim_H Z$ denotes the orthogonal dimension of Z.

Let $\alpha(T) = \dim \mathcal{N}(T)$, $\beta(T) = \dim X/\mathcal{R}(T)$. The following sets of semi–Fredholm operators are well-known:

$$\Phi_+(X) = \{T \in \mathcal{L}(X) : \mathcal{R}(T) \text{ is closed and } \alpha(T) < \infty\}, \quad \text{and}$$
$$\Phi_-(X) = \{T \in \mathcal{L}(X) : \mathcal{R}(T) \text{ is closed and } \beta(T) < \infty\}.$$

The set of Fredholm operators is $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. It is well-known that the sets $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ form open multiplicative semigroups of $\mathcal{L}(X)$. For a semi-Fredholm operator T the index is defined by $i(T) = \alpha(T) - \beta(T)$. We also consider the set of Weyl operators, which is defined as $\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}$.

Recall that $\mathrm{asc}(T)$ (respectively $\mathrm{des}(T)$), the ascent (respectively descent) of T, is the smallest non-negative integer n, such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (respectively $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). If no such n exists, then $\mathrm{asc}(T) = \infty$ (respectively $\mathrm{des}(T) = \infty$) [4]. An operator $T \in \mathcal{L}(X)$ is called Browder (Riesz–Schauder), provided that $T \in \Phi(X)$ and $\mathrm{asc}(T) = \mathrm{des}(T) < \infty$ [4]. The set of all Browder operators is denoted by $\mathcal{B}(X)$. It is well-known that $T \in \mathcal{B}(X)$ is and only if $T \in \Phi(X)$ and $0 \notin \mathrm{acc}\,\sigma(T)$ [8].

We use $\mathcal{F}(X)$ to denote the set of all finite rank operators on X. Let $\mathcal{K}(X)$ denote the set (closed two-sided ideal) of all compact operators in $\mathcal{L}(X)$, and let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X) = C(X)$ denote the natural homomorphism. $C(X)$ is the Calkin algebra on X. It is well-known that $T \in \Phi(X)$ if and only if $\pi(T)$ is invertible in $C(X)$. The next classes of left and right Fredholm operators can be defined using the homomorphism π:

$$\Phi_l(X) = \{T \in \mathcal{L}(X) : \pi(T) \text{ is left invertible in } C(X)\} \quad \text{and}$$
$$\Phi_r(X) = \{T \in \mathcal{L}(X) : \pi(T) \text{ is right invertible in } C(X)\}.$$

It is well-known that $\Phi_l(X) \subset \Phi_+(X)$ and $\Phi_r(X) \subset \Phi_-(X)$. Also, if $T \in \Phi_l(X) \cup \Phi_r(X)$ then T is relatively regular. Recall that for a Hilbert space H the following holds: $\Phi_+(H) = \Phi_l(H)$ and $\Phi_-(H) = \Phi_r(H)$.

Let $\mathcal{K} \subset \mathcal{L}(X)$. Recall that the perturbation class of $\mathcal{K}$ is defined as

$$\mathcal{P}(\mathcal{K}) = \{T \in \mathcal{L}(X) : \mathcal{K} + T \subset \mathcal{K}\}.$$

If

$$(*) \qquad \mathcal{G}(X) \cdot \mathcal{K} \subset \mathcal{K} \quad \text{and} \quad \mathcal{K} \cdot \mathcal{G}(X) \subset \mathcal{K}$$

is satisfied, then $\mathcal{P}(\mathcal{K})$ is a two-sided ideal of $\mathcal{L}(X)$. Thus, if $\mathcal{K}$ satisfies (*), then the following implication holds:

$$T \text{ is } \mathcal{K}\text{-consistent and } U \in \mathcal{P}(\mathcal{K}), \text{ then } T + U \text{ is } \mathcal{K}\text{-consistent.}$$

Let $\mathcal{P}(\Phi_+(X))$ (respectively $\mathcal{P}(\Phi_+(X))$) denote the perturbation class of the set $\Phi_+(X)$ ($\Phi_-(X)$)) (see [4] for similar considerations). It is well-known that

$$\mathcal{K}(X) \subset \mathcal{P}(\Phi_+(X)) \cap \mathcal{P}(\Phi_-(X)).$$

The following result is a simple generalization of [6, Theorem 3.2].

Theorem 2.3. *Let $B \in \mathcal{L}(X)$. The following statements are equivalent:*

1. *$\alpha(B) = \beta(B)$, or $\mathcal{R}(B)$ is not closed or not complemented in X, or $\mathcal{N}(B)$ is not complemented in X;*
2. *$B + F$ is $\mathcal{G}$-consistent for all $F \in \mathcal{F}(X)$;*
3. *$B + K$ is $\mathcal{G}$-consistent for all $K \in \mathcal{P}(\Phi_+(X)) \cap \mathcal{P}(\Phi_-(X))$.*

Here $\alpha(B) = \beta(B)$ means that either $\mathcal{N}(B)$ and $X/\mathcal{R}(B)$ are finite dimensional spaces of the same dimension, or $\mathcal{N}(B)$ and $X/\mathcal{R}(B)$ are both infinite dimensional.

Proof. (2) $\Longrightarrow$ (1). Let $\mathcal{R}(B)$ be closed and complemented. Suppose that $\alpha(B) < \beta(B)$. Then $\mathcal{N}(B)$ is finite dimensional, $n = \dim \mathcal{N}(B) < \infty$ and let $y_1, \dots, y_n$ be vectors in X which are linearly independent modulo $\mathcal{R}(B)$. Denote by $F_1 : \mathcal{N}(B) \to span\{y_1, \dots, y_n\} = Y$ an arbitrary isomorphism. There exists a closed subspace M of X, such that $X = \mathcal{N}(B) \oplus M$. Define $F \in \mathcal{L}(X)$ in the following way:

$$Fx = \begin{cases} F_1 x, & x \in \mathcal{N}(B), \\ 0, & x \in M. \end{cases}$$

It is easy to verify $\alpha(B + F) = 0$. Since $\mathcal{R}(B + F) = \mathcal{R}(B) \oplus Y$, it follows that $\mathcal{R}(B + F)$ is a proper closed and complemented subspace of X. By Theorem 2.1 (6) it follows that $B + F$ is not $\mathcal{G}$-consistent.

Let $\mathcal{N}(B)$ be complemented in X and $\beta(B) < \alpha(B)$. Then $X/\mathcal{R}(B)$ is finite dimentional, $\dim X/\mathcal{R}(B) = n < \infty$ and there exists a subspace M such that $\mathcal{R}(B) \oplus M = X$, $\dim M = n$. Let $x_1, \dots, x_n \in \mathcal{N}(B)$ be linearly independent and $Z = span\{x_1, \dots, x_n\}$. There exists a closed subspace Z_1 such that $Z \oplus Z_1 = \mathcal{N}(B)$. Since $\mathcal{N}(B)$ is complemented, there exists a subspace Z_2 such that $X = \mathcal{N}(B) \oplus Z_2$. Moreover, $X = Z \oplus Z_1 \oplus Z_2$. Let $E_1 : Z \to M$ be an arbitrary isomorphism. Define $E \in \mathcal{L}(X)$ as follows:

$$Ex = \begin{cases} E_1 x, & x \in Z \\ 0, & x \in Z_1 \oplus Z_2. \end{cases}$$

It is easy to verify $\mathcal{R}(B + E) = X$ and $\mathcal{N}(B + E) = Z_1$. Since Z_1 is complemented in X, from Theorem 2.1 (7) it follows that $B + F$ is not $\mathcal{G}$-consistent.

(1) $\Longrightarrow$ (3) Let $K \in \mathcal{P}(\Phi_+(X)) \cap \mathcal{P}(\Phi_-(X))$ be arbitrary. If $\mathcal{R}(B + K)$ is not closed, then $B + K$ is $\mathcal{G}$-consistent (Theorem 2.1 (2)). Suppose that

$\mathcal{R}(B+K)$ is closed. If $\alpha(B+K)=\infty$ and $\beta(B+K)=\infty$, then $B+K$ is $\mathcal{G}$-consistent (Theorem 2.1 (3)). Suppose that $\alpha(B+K)<\infty$. Then $B+K\in\Phi_+(X)$, so $B\in\Phi_+(X)$ and $i(B+K)=i(B)=0$. If $\alpha(B+K)=0$, then $B+K$ is invertible and $\mathcal{G}$-consistent (Theorem 2.1 (1)). If $\alpha(B+K)>0$, by Theorem 2.1 (3) it follows that $B+K$ is $\mathcal{G}$-consistent.

Let $\beta(B+K)<\infty$. Then $B+K\in\Phi_-(X)$ and $B\in\Phi_-(X)$, implying $i(B+K)=i(B)=0$. If $\beta(B+K)=0$, then $B+K$ is invertible and from Theorem 2.1 (1) it follows that $B+K$ is $\mathcal{G}$-consistent. If $\beta(B+K)>0$, from Theorem 2.1 (3) it follows that $B+K$ is $\mathcal{G}$-consistent. $\square$

Now, using Lemma 2.2 and Theorem 2.3 we get the following simple corollary.

Corollary 2.4. *Let $B\in\mathcal{L}(X)$ be relatively regular. If $B\in cl\mathcal{G}(X)$, then $B+K$ is $\mathcal{G}$-consistent for all $K\in\mathcal{P}(\Phi_+(X))\cap\mathcal{P}(\Phi_-(X))$.*

We shall consider $\mathcal{G}_l$-consistent operators, where $\mathcal{G}_l(X)$ denotes the set of all left invertible operators on X. Recall that T is strictly singular if and only if T is not bounded below on every closed infinite dimensional subspace of X. In the case when H is a Hilbert space, it is convenient to use the following generalized definition. Let $\dim_H H=\alpha$, where α is an infinite cardinal. An operator $T\in\mathcal{L}(H)$ is said to be α-strictly singular, provided that the following holds: if M is a closed subspace of H and the restriction $T|_M : M\to T(M)$ is invertible, then $\dim_H M<\alpha$. We need to introduce the following property for Banach space oparators.

An operator $T\in\mathcal{L}(X)$ is called strictly left singular, if and only if for all $S\in\mathcal{G}_l(X)$ it follows that $TS\notin\mathcal{G}_l(X)$.

Obviously, if T is strictly left singular, then $T\notin\mathcal{G}_l(X)$.

Remark 2.5. (1) *If $T\in\mathcal{L}(X)$ is strictly singular, then T is strictly left singular.*

(2) *If X is a Hilbert space and* $\dim_H X = \alpha$*, then T is strictly left singular if and only if T is α-strictly singular (see also* [7, Problem 42]*).*

(3) *If X is a separable infinite dimensional Hilbert space, then T is strictly left singular if and only if T is compact (since there exists the unique closed ideal of $\mathcal{L}(X)$).*

The following theorem is our main result concerning the $\mathcal{G}_l$-consistent operators. Namely, it completely characterizes the set of $\mathcal{G}_l$-consistent operators.

Theorem 2.6. *Let $B \in \mathcal{L}(X)$. Then B is $\mathcal{G}_l$-consistent, if and only if one of the following two mutually disjoint cases occurs:*

(1) *$B \in \mathcal{G}(X)$;*
(2) *B is strictly left singular.*

Also, B is not $\mathcal{G}_l$-consistent if and only if one of the following two mutually disjoint cases occurs:

(3) *$B \in \mathcal{G}_l(X) \setminus \mathcal{G}(X)$;*
(4) *$B \notin \mathcal{G}_l(X)$ and B is not strictly left singular.*

In the case when X is an infinite dimensional Hilbert space and $\dim_H X = \alpha$*, then "strictly left singular" should be replaced by "α-strictly singular". If X is a separable infinite dimensional Hilbert space then "strictly left singular" should be replaced by "compact".*

Proof. To prove (1), suppose that $B \in \mathcal{G}(X)$. If $BA \in \mathcal{G}_l(X)$, then $A \in \mathcal{G}_l(X)$ and $AB \in \mathcal{G}_l(X)$. From the other hand, if $AB = S \in \mathcal{G}_l(X)$, then $A = SB^{-1} \in \mathcal{G}_l(X)$ and $BA \in \mathcal{G}_l(X)$, so B is $\mathcal{G}_l$-consistent.

(2) Let B be strictly left singular (hence, $B \notin \mathcal{G}_l(X)$). Then $AB \notin \mathcal{G}_l(X)$ for all $A \in \mathcal{L}(X)$. Suppose that there exists an operator $A_0 \in \mathcal{L}(X)$, such that $BA_0 \in \mathcal{G}_l(X)$. It follows that $A_0 \in \mathcal{G}_l(X)$ which contradicts the assumption that B is strictly left singular. We get that B is $\mathcal{G}_l$-consistent.

(3) Let $B \in \mathcal{G}_l(X) \setminus \mathcal{G}(X)$ and let B_1 be an arbitrary left inverse of B. Obviously, $B_1 B = I \in \mathcal{G}_l(X)$. On the other hand, B_1 is a reflexive generalized inverse of B, so BB_1 is a projection of X onto $\mathcal{R}(B)$ with a non-trivial kernel, so $BB_1 \notin \mathcal{G}_l(X)$. It follows that B is not $\mathcal{G}_l$-consistent.

(4) Finally, let $B \notin \mathcal{G}_l(X)$ and B is not strictly left singular. There exists an operator $A_0 \in \mathcal{G}_l(X)$ such that $BA_0 \in \mathcal{G}_l(X)$. Also, $AB \notin \mathcal{G}_l(X)$ for all $A \in \mathcal{L}(X)$, so B is not $\mathcal{G}_l$-consistent.

The rest of the proof follows from Remark 2.5 $\square$

Recall that an operator $T \in \mathcal{L}(X)$ is strictly cosingular provided that for every closed infinite codimensional subspace V of X, the operator $Q_V T$ is not surjective. Here $Q_V : X \to X/V$ denotes the natural homomorphism. More generally, let H be a Hilbert space and $\dim_H H = \alpha$ be an infinite cardinal. $T \in \mathcal{L}(H)$ is called α-strictly cosingular, provided that for an arbitrary closed subspace V of H the following holds: if $Q_V T$ is a surjection of H onto H/V, then $\operatorname{codim} V < \alpha$. We also introduce the following notion for Banach space operators.

An operator $T \in \mathcal{L}(X)$ is called strictly right singular, if and only if for all $S \in \mathcal{G}_r(X)$ it follows that $ST \notin \mathcal{G}_r(X)$.

If T is strictly right singular, then $T \notin \mathcal{G}_r(X)$.

We connect various aspects of singularity.

Theorem 2.7. (1) *If T is strictly cosingular, then T is strictly right singular.*

(2) *If X is a complex infinite dimensional Hilbert space and $\dim_H X = \alpha$, then T is α-strictly cosingular if and only if T is strictly right singular.*

(3) *If X is a separable infinite dimensional Hilbert space, then T is strictly right singular if and only if T is compact.*

Proof. (1) Suppose that T is strictly cosingular and $S \in \mathcal{G}_r(X)$. It follows that there exists a closed subspace M of X, such that $\mathcal{N}(S) \oplus M = X$ and the restriction $S|_M : M \to X$ is invertible, so $\dim M = \operatorname{codim} \mathcal{N}(S) = \infty$. Consider the natural homomorphism $Q_{\mathcal{N}(S)} : X \to X/\mathcal{N}(S)$. It follows that

$$X/\mathcal{N}(S) \neq \mathcal{R}(Q_{\mathcal{N}(S)}T) = \{Tx + \mathcal{N}(S) : x \in X\}.$$

There exists $y \in X$, such that $y + \mathcal{N}(S) \neq Tx + \mathcal{N}(S)$ for all $x \in X$. It follows that $y_1 = Sy \neq STx$ for all $x \in X$. We get that $\mathcal{R}(ST) \neq X$ and $ST \notin \mathcal{G}_r(X)$, so T is strictly right singular.

(2) The implication $\Longrightarrow$ follows in the same way as in (1). We only need to consider the orthogonal dimensions of closed subspaces.

To prove the opposite implication suppose that $T \in \mathcal{L}(X)$ is strictly right singular. Let V be an arbitrary closed subspace of X such that $\dim_H V^\perp = \alpha$. Let $S \in \mathcal{L}(X)$ be defined such that $S|_V = 0$ and $S|_{V^\perp} : V^\perp \to X$ is the Hilbert space isomorphism. It follows that $S \in \mathcal{G}_r(X)$ and $ST \notin \mathcal{G}_r(X)$. Since $\mathcal{N}(ST)$ is always complemented in X, we get that $\mathcal{R}(ST) \neq X$. There exists $y_0 \in X$ such that $y_0 \neq STx$ for all $x \in X$. Suppose that for all $y \in X$ there exists $x \in X$ such that $y + V = Tx + V$. We conclude $y - Tx \in V = \mathcal{N}(S)$ and $Sy = STx$. Now,

$$X = \{Sy : y \in X\} = \{STx : x \in X\} = \mathcal{R}(ST) \neq X.$$

It follows that $\mathcal{R}(Q_V T) \neq X/V$ and T is α-strictly singular. $\square$

In the following theorem we give a complete description of $\mathcal{G}_r$-consistent operators on Banach spaces.

Theorem 2.8. *Let $B \in \mathcal{L}(X)$. Then B is $\mathcal{G}_r$-consistent, if and only if one of the following two mutually disjoint cases occurs:*

(1) *$B \in \mathcal{G}(X)$;*

(2) *B is strictly right singular.*

Also, B is not $\mathcal{G}_r$-consistent if and only if one of the following two mutually disjoint cases occurs:

(3) *$B \in \mathcal{G}_r(X) \setminus \mathcal{G}(X)$;*

(4) *$B \notin \mathcal{G}_r(X)$ and B is not strictly right singular.*

In the case when X is an infinite dimensional Hilbert space and $\dim_H X = \alpha$, then "strictly right singular" should be replaced by "α-strictly cosingular". If X is a separable infinite dimensional Hilbert space then "strictly right singular" should be replaced by "compact".

The proof of Theorem 2.8 is similar to the proof of Theorem 2.6.

3. Fredholm consistent operators

As we mentioned before, in this section we shall consider Φ-consistent operators. We give a complete answer to the question whether or not an operator is Φ-consistent.

Theorem 3.1. *Let $B \in \mathcal{L}(X)$. Then B is Φ-consistent, if and only if one of the following four mutually disjoint cases occurs:*

(1) *$B \in \Phi(X)$;*
(2) *$\alpha(B) = \infty$ and $\beta(B) = \infty$;*
(3) *$\alpha(B) < \infty$, $\beta(B) = \infty$ and $\mathcal{R}(B)$ is not closed or not complemented in X;*
(4) *$\alpha(B) = \infty$, $\beta(B) < \infty$ and $\mathcal{N}(B)$ is not complemented in X;*

Also, B is not Φ-consistent if and only if one of the following two mutually disjoint cases occurs:

(5) *$\alpha(B) < \infty$, $\beta(B) = \infty$ and $\mathcal{R}(B)$ is closed and complemented in X;*
(6) *$\alpha(B) = \infty$, $\beta(B) < \infty$ and $\mathcal{N}(B)$ is complemented in X.*

Proof. (1) Suppose that $B \in \Phi(X)$. Then $\pi(B)$ is invertible in $C(X)$ and the proof follows in the same way as in Theorem 2.1 (1).

(2) Let $\alpha(B) = \infty$ and $\beta(B) = \infty$. Then for all $A \in \mathcal{L}(X)$, $\beta(BA) \geq \beta(B)$, so $BA \notin \Phi(X)$. Also, $\alpha(AB) \geq \alpha(B)$ and $AB \notin \Phi(X)$. It follows that B is Φ-consistent.

(3) Let $\alpha(B) < \infty$, $\beta(B) = \infty$ and $\mathcal{R}(B)$ is not closed or not complemented in X. Obviously, for all $A \in \mathcal{L}(X)$, $BA \notin \Phi(X)$. Suppose that there exists some $A \in \mathcal{L}(X)$, such that $AB \in \Phi(X)$. Then $\pi(B)$ is left invertible in $C(X)$ and $B \in \Phi_l(X)$. It follows that B is relatively regular and $\mathcal{R}(B)$ must be closed and complemented in X.

(4) Let $\alpha(B) = \infty$, $\beta(B) < \infty$ and $\mathcal{N}(B)$ is not complemented in X. For all $A \in \mathcal{L}(X)$ we get $AB \notin \Phi(X)$. Suppose that there exists some $A \in \mathcal{L}(X)$, such that $BA \in \Phi(X)$. It follows that $\pi(B)$ is right invertible in $C(X)$, $B \in \Phi_r(X)$ and B is relatively regular, so $\mathcal{N}(B)$ must be complemented in X.

(5) Let $\alpha(B) < \infty$, $\beta(B) = \infty$ and $\mathcal{R}(B)$ is closed and complemented in X. It follows that $BA \notin \Phi(X)$ for all $A \in \mathcal{L}(X)$. Since B is relatively regular, there exists a reflexive generalized inverse S of B. Now, SB is the projection onto $\mathcal{R}(S)$ parallel to $\mathcal{N}(B)$, so $\beta(S) < \infty$. Since $X = \mathcal{N}(S) \oplus \mathcal{R}(B)$, it follows that $\mathcal{R}(SB) = \mathcal{R}(S)$ is closed, $\alpha(SB) = \alpha(B) < \infty$ and $\beta(SB) = \beta(S) < \infty$, so $SB \in \Phi(X)$. We get that B is not Φ-consistent.

(6) Let $\alpha(B) = \infty$, $\beta(B) < \infty$ and $\mathcal{N}(B)$ is complemented in X. Obviously, $AB \notin \Phi(X)$ for all $A \in \mathcal{L}(X)$. Since B is relatively regular, there exists a reflexive generalized inverse S of B. Again, we get that $\mathcal{R}(BS) = \mathcal{R}(B)$, $\beta(BS) = \beta(B) < \infty$, $\alpha(BS) = \alpha(S) = \beta(B) < \infty$. It follows that B is not Φ-consistent. $\square$

Now, as corollaries, we consider Φ_0- and $\mathcal{B}$-consistent operators, i.e. $\mathcal{S}(X) = \Phi_0(X)$ (the set of Weyl operators), or $\mathcal{S}(X) = \mathcal{B}(X)$ (the set of Browder operators) on X.

Corollary 3.2. *An operator $B \in \mathcal{L}(X)$ is Φ_0-consistent if and only if it is Φ-consistent.*

Proof. Let B be Φ-consistent and let $B \in \Phi(X)$. Suppose that there exists $A \in \mathcal{L}(X)$ such that $AB \in \Phi_0(X)$. It follows that $BA \in \Phi(X)$ and $A \in \Phi(X)$. Now, $i(BA) = i(B) + i(A) = i(AB) = 0$, so $BA \in \Phi_0(X)$ and B

is Φ_0-consistent. Cases (2), (3) and (4) of Theorem 3.1 are analogous. If S is described in Theorem 3.1 (5) (or (6)), it follows that $i(SB) = 0$ (or $i(BS) = 0$), so B is not Φ_0-consistent. $\square$

Corollary 3.3. *An operator $B \in \mathcal{L}(X)$ is $\mathcal{B}$-consistent if and only if it is Φ-consistent.*

Proof. Suppose that B is Φ-consistent and let $AB \in \mathcal{B}(X)$ for some $A \in \mathcal{L}(X)$. It follows that $AB \in \Phi(X)$ and $0 \notin \operatorname{acc}\sigma(AB)$. Since $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ and B is Φ-consistent, we get $0 \notin \operatorname{acc}\sigma(BA)$ and $BA \in \Phi(X)$. It follows that $BA \in \mathcal{B}(X)$, so B is $\mathcal{B}$-consistent.

Suppose that B is not Φ-consistent. It follows that (5) or (6) from Theorem 3.1 holds. Let (5) hold, i.e. $\alpha(B) < \infty$, $\beta(B) = \infty$ and $\mathcal{R}(B)$ is closed and complemented in X. Since $BA \notin \Phi(X)$ for all $A \in \mathcal{L}(X)$, it follows also that $BA \notin \mathcal{B}(X)$ for all $A \in \mathcal{L}(X)$. On the other hand, if S is an arbitrary reflexive generalized inverse of B, then we know that BS is Fredholm. Also, BS is a projection, so $\operatorname{asc}(BS) = \operatorname{des}(BS) = 1$ and $BS \in \mathcal{B}(X)$. It follows that B is not $\mathcal{B}$-consistent. The proof is similar if we suppose that (6) holds from Theorem 3.1. $\square$

It is nice to formulate the corresponding result for Hilbert space operators. Let H be an arbitrary complex infinite dimensional Hilbert space. According to Theorem 3.1 we have the following

Corollary 3.4. *If $B \in \mathcal{L}(H)$, then B is Φ-consistent if and only if one of the following three mutually disjoint cases occurs:*

(1) $B \in \Phi(H)$;

(2) $\alpha(B) = \infty$ *and* $\beta(B) = \infty$;

(3) $\alpha(B) < \infty$, $\beta(B) = \infty$ *and* $\mathcal{R}(B) \neq \overline{\mathcal{R}(B)}$;

Also, B is not Φ-consistent if and only if one of the following two mutually disjoint cases occurs:

(4) $\alpha(B) < \infty$, $\beta(B) = \infty$ *and* $\mathcal{R}(B) = \overline{\mathcal{R}(B)}$;

(5) $\alpha(B) = \infty$ *and* $\beta(B) < \infty$.

If H is a separable Hilbert space, using [1, Theorem 4 and Remark 5], [6, Theorem 3.1], or [11, Theorem 5], [2, Proposition 4], we know that the following holds:

$$\begin{aligned} cl\Phi(H) &= \Phi(H) \cup cl\mathcal{G}(H) \\ &= \Phi(H) \cup \{B \in \mathcal{L}(H) : \alpha(B) = \alpha(B^*), \text{ or } \mathcal{R}(B) \neq \overline{\mathcal{R}(B)}\}. \end{aligned} \tag{3.1}$$

Here, B^* denotes the Hilbert adjoint operator of B. Using Corollary 3.4 and (3.1), we can easily prove the last main result from [6, Theorem 3.7].

Theorem 3.5. *If H is a separable complex infinite dimensional Hilbert space and $B \in \mathcal{L}(H)$, then:*

$$B \in cl\Phi(H) \quad \textit{if and only if} \quad B \textit{ is } \Phi\textit{-consistent.}$$

Proof. Suppose that $B \in cl\Phi(H)$. Using (3.1) we conclude that the following may occur:

(i) $B \in \Phi(H)$ implies B is Φ-consistent (Corollary 3.4 (1)).

(ii) $\alpha(B) = \alpha(B^*) = \infty$ implies $\beta(B) = \infty$, so B is Φ-consistent (Corollary 3.4 (2)).

(iii) $\alpha(B) = \alpha(B^*) < \infty$ and $\mathcal{R}(B) = \overline{\mathcal{R}(B)}$ imply $B \in \Phi(H)$ and B is Φ-consistent as in (i).

(iv) $\alpha(B) = \alpha(B^*) < \infty$ and $\mathcal{R}(B) \neq \overline{\mathcal{R}(B)}$ imply $\beta(B) = \infty$, so B is Φ-consistent (Corollary 3.4 (3)).

(v) $\mathcal{R}(B) \neq \overline{\mathcal{R}(B)}$ implies that the cases (4) and (5) of Corollary 3.4 can not hold, so B is Φ-consistent.

Now, suppose that B is Φ-consistent. Then the following may occur:

(i) $\mathcal{R}(B) \neq \overline{\mathcal{R}(B)}$ implies $B \in cl\Phi(H)$;

(ii) If $\mathcal{R}(B) = \overline{\mathcal{R}(B)}$, since B is Φ-consistent, by Corollary 3.4 it follows that either $B \in \Phi(H)$, or $\alpha(B) = \beta(B) = \infty$. Anyway, by (3.1) it follows that $B \in cl\Phi(H)$. $\square$

Remark 3.6. *Theorem 3.5 is proved in* [6] *using the Gelfand-Naimark-Segal Theorem for C^*-algebras.*

Now, we shall consider Φ_l-consistent operators. As in Section 2, we introduce the following notions for Banach space operators.

An operator $T \in \mathcal{L}(X)$ is said to be essentially strictly left singular, provided that $TS \notin \Phi_l(X)$ for all $S \in \Phi_l(X)$.

$T \in \mathcal{L}(X)$ is called essentially strictly right singular, provided that $ST \notin \Phi_r(X)$ for all $S \in \Phi_r(X)$.

The relationships between the introduced notions and known strictly singular and cosingular operators are given in the following theorem.

Theorem 3.7. (1) *If T is strictly singular, then T is essentially strictly left singular. If T is strictly cosingular, then T is essentially strictly right singular.*

(2) *If X is a Hilbert space and $\dim_H X = \alpha$, then T is essentially strictly left singular if and only if T is α-strictly singular. Also, T is essentially strictly right singular if and only if T is α-strictly cosingular.*

(3) *If X is a separable infinite dimensional Hilbert space, then T is essentially strictly left (or right) singular if and only if T is compact.*

Proof. (1) Let T be strictly singular and $S \in \Phi_l(X)$. Suppose that $TS \in \Phi_l(X)$. Now, $\mathcal{R}(S)$ is an infinite dimensional closed subspace of X and $\dim \mathcal{N}(T|_{\mathcal{R}(S)}) \leq \mathcal{N}(TS) < \infty$. Hence, there exists a closed infinite dimensional subspace M of $\mathcal{R}(S)$ such that $\mathcal{R}(S) = \mathcal{N}(T|_{\mathcal{R}(S)}) \oplus M$. We conclude that $T|_M : M \to T(M) = \mathcal{R}(TS)$ is an isomorphism, so T can not be strictly singular.

Now, suppose that T strictly cosingular and there exists $S \in \Phi_r(X)$ such that $ST \in \Phi_r(X)$. There exists a finite dimensional subspace M such that $\mathcal{R}(S) = \mathcal{R}(ST) \oplus M$. Also, there exists a closed subspace N such that $X = \mathcal{N}(S) \oplus N$. Notice that the truncation $S|_N : N \to \mathcal{R}(ST) \oplus N$ is invertible.

Let $K = (S|_N)^{-1}(M)$ and $L = (S|_N)^{-1}(\mathcal{R}(ST))$. Then $N = K \oplus L$ and L is infinite dimensional. Let $V = \mathcal{N}(S) \oplus K$. From $X = V \oplus L$ we see that $\operatorname{codim} V = \infty$. Let $Q_V : X \to X/V$ be the natural homomorphism. We shall prove that $Q_V T$ is epimorphism. Let $z + V \in X/V$ be arbitrary. There exists some $x \in L$ such that $z + V = x + V$. Since $Sx \in \mathcal{R}(ST)$ it follows that there exists some y such that $Sx = STy$. Consequently, $x - Ty \in \mathcal{N}(S) \subset V$ and

$$z + V = x + V = Ty + v,$$

implying $Q_V Ty = z + V$.

(2) Let X be a Hilbert space and $\dim_H X = \alpha$ be an infinite cardinal. If T is α-strictly singular, in the same way as in the proof of (1) we verify that T is essentially strictly left singular. On the other hand, if T is essentially strictly left singular, from $\mathcal{G}_l(X) \subset \Phi_l(X)$ it follows that T must be strictly left singular. By Remark 2.5 it follows that T is α-strictly singular.

If T is α-strictly cosingular, in the same way as in (1) we can prove that T must be essentially strictly right singular. We only have to consider the Hilbert dimensions of closed subspaces.

On the other hand, if T is essentially strictly right singular then T is strictly right singular. By Theorem 2.7 it follows that T is α-strictly cosingular. $\square$

In the following theorem we describe the set of all Φ_l-consistent operators.

Theorem 3.8. *Let $B \in \mathcal{L}(X)$. Then B is Φ_l-consistent if and only if one of the following two mutually disjoint cases occurs:*

(1) *$B \in \Phi(X)$;*

(2) *B is essentially strictly left singular.*

Also, B is not Φ_l-consistent if and only if one of the following two mutually disjoint cases occurs:

(3) *$B \in \Phi_l(X) \setminus \Phi(X)$;*

(4) *$B \notin \Phi_l(X)$ and B is not essentially strictly left singular.*

In the case when X is an infinite dimensional Hilbert space and $\dim_H X = \alpha$, then "essentially strictly left singular" should be replaced by α-strictly singular. If X is a separable infinite dimensional Hilbert space, then "essentially strictly left singular" should be replaced by "compact".

Proof. (1) Let $B \in \Phi(X)$ and $BA \in \Phi_l(X)$. Since $\pi(B)\pi(A)$ is left invertible in $C(X)$, it follows that $A \in \Phi_l(X)$ and $AB \in \Phi_l(X)$. On the other hand, if $S = AB \in \Phi_l(X)$, then $\pi(A) = \pi(S)\pi(B)^{-1}$ is left invertible in $C(X)$, so $BA \in \Phi_l(X)$. We conclude that B is Φ_l-consistent.

(3) Let $B \in \Phi_l(X) \setminus \Phi(X)$ and let B_1 denote an arbitrary reflexive generalized inverse of B. Denote $\mathcal{N}(BB_1) = M$ and $\mathcal{R}(B_1B) = N$. Then $\dim M = \infty$ and $\operatorname{codim} N < \infty$. Since BB_1 is the projection from X onto $\mathcal{R}(B)$ parallel to M, it follows that $BB_1 \notin \Phi_l(X)$. On the other hand, B_1B is the projection from X onto N parallel to $\mathcal{N}(B)$, implying that $B_1B \in \Phi_l(X)$. Hence, B is not Φ_l-consistent.

The cases (2) and (4) are analogous to Theorem 2.6 (2) and (4). □

Dually, we can prove the following result concerning the Φ_r-consistent operators.

Theorem 3.9. *Let $B \in \mathcal{L}(X)$. Then B is Φ_r-consistent if and only if one of the following two mutually disjoint cases occurs:*

(1) *$B \in \Phi(X)$;*

(2) *B is essentially strictly right singular.*

Also, B is not Φ_r-consistent if and only if one of the following two mutually disjoint cases occurs:

(3) *$B \in \Phi_r(X) \setminus \Phi(X)$;*

(4) *$B \notin \Phi_r(X)$ and B is not essentially strictly right singular.*

In the case when X is an infinite dimensional Hilbert space and $\dim_H X = \alpha$, then "essentially strictly right singular" should be replaced by α-strictly cosin-

gular. If X is a separable infinite dimensional Hilbert space, then "essentially strictly right singular" should be replaced by "compact".

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