ON DAVIS-KAHAN-WEINBERGER EXTENSION THEOREM

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Abstract. If $R = \begin{bmatrix} H \\ B \end{bmatrix}$, where $H = H^*$, we find a pseudo-inverse form of all solutions $W = W^*$, such that $\|A\| = \|R\|$, where $A = \begin{bmatrix} H & B^* \\ B & W \end{bmatrix}$ and $\|H\| \leq \|R\|$. In this paper we extend well-known results in a finite dimensional setting, proved by Dao-Sheng Zheng (SIAM J. Matrix Anal. Appl. **17** (3) (1996), 621–631). Thus, a pseudo inverse form of solutions of the Davis-Kahan-Weinberger theorem is established.

1. Motivation

Let $\mathcal{Z}$ denote an arbitrary Hilbert space and let $\mathcal{H}$ and $\mathcal{K}$ denote closed mutually orthogonal subspaces of $\mathcal{Z}$, such that $\mathcal{Z} = \mathcal{H} \oplus \mathcal{K}$. We use $\mathcal{L}(\mathcal{H}, \mathcal{K})$ to denote the set of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$ and $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ let $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively, denote the range and the kernel of T.

Let $H = H^* \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be given operators, such that $\rho = \|R\|$, where

$$R = \begin{bmatrix} H \\ B \end{bmatrix} : [\, \mathcal{H} \,] \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}.$$

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Notice that $\|H\| \leq \|R\|$ always holds. We consider the following problem. Find an operator $W = W^* \in \mathcal{L}(\mathcal{K})$, such that the selfadjoint operator

$$A = \begin{bmatrix} H & B^* \\ B & W \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$$

satisfies the norm condition $\|A\| = \|R\| = \rho$.

This is a typical selfadjoint dilation problem. We mention that a non-selfadjoint form is also important.

The result which is known as the Davis-Kahan-Weinberger theorem is proved in [5, Theorem 1.2] and stated as follows:

Theorem (DKW). *Let H, B, C satisfy $\left\| \left[\begin{smallmatrix} H \\ B \end{smallmatrix}\right] \right\| \leq \mu$, $\|[\, H \ C \,]\| \leq \mu$ and $\|H\| < \mu$. Then there exists W such that $\left\| \left[\begin{smallmatrix} H & C \\ B & W \end{smallmatrix}\right] \right\| \leq \mu$. Indeed those W which have this property are exactly those of the form*

$$W = -KH^*L + \mu(I - KK^*)^{1/2} Z (I - L^*L)^{1/2},$$

where

$$K^* = (\mu^2 I - H^*H)^{-1/2} B^*, \quad L = (\mu^2 - HH^*)^{-1/2} C$$

and Z is an arbitrary contraction. If H is compact then W may be chosen compact.

The selfadjoint version of the previous theorem follows (see [5, Corollary 1.3]):

Corollary (DKW-SA). *Let H be selfadjoint and $\left\| \left[\begin{smallmatrix} H \\ B \end{smallmatrix}\right] \right\| \leq \mu$ and $\|H\| < \mu$. Then there exists selfadjoint W such that $\left\| \left[\begin{smallmatrix} H & B^* \\ B & W \end{smallmatrix}\right] \right\| \leq \mu$. Indeed those W which have this property are exactly those such that*

$$-\mu I + B(\mu I + H)^{-1} B^* \leq W \leq \mu I - B(\mu I - H)^{-1} B^*.$$

The following result is a central solution obtained from Corollary (DKW-SA) (see [5, (1.7)]). One strightforward proof of this result is given in [15, Lemma 3.1] (although the proof is given for complex matrices, a careful reading shows that it is valid for operators on arbitrary Hilbert spaces also).

Corollary (DKW-central). *Let* $R = \begin{bmatrix} H \\ B \end{bmatrix} : [\mathcal{H}] \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$, *where* $H = H^*$, $\sigma \geq \|R\|$ *and* $\sigma > \|H\|$. *If* $W_\sigma = -BH(\sigma^2 - H^2)^{-1}B^*$ *and*

$$A_\sigma = \begin{bmatrix} H & B^* \\ B & W_\sigma \end{bmatrix},$$

then $\|A_\sigma\| \leq \sigma$.

A selfadjoint part of this problem is proved by M. G. Krein (see [9] and [13, Sec. 125]). One special case of the Davis-Kahan-Weinberger theorem was proved by B. Sz.-Nagy and C. Foias (see [14, Theorem 1] and also [3]). Several proofs of Theorem (DKW) are presented in [4, Sec. 3], [5, Theorem 1.2] and [12, Theorem 1].

The boundary case appears if we assume $\|H\| = \|R\| = \mu$. One solution (as a non-selfadjoint extension) is found in [4, Sec. 3]. In this case at least one of $\mu I - H$ and $\mu I + H$ is not invertible, but we can consider their Moore-Penrose inverses (in the case when they exist). Zheng used this idea in [15, Theorem 4.1] and completely solved this problem in finite dimensional settings. Kahan also found one solution of this problem, but he did not publish his results, which appeared in [11, p. 231–233] without any proof. See also results of Fioas and Frazho [6, Chapter IV]. Zhang also proved Theorem (DKW-central) in finite dimensional settings, under the more general assumption $\|H\| \leq \mu$. Finally, we mention that finite-dimensional dilation results of this type have lots of applications in numerical analysis (see [5], [7], [8] and [10]).

In this paper we extend Zheng's results for operators on arbitrary Hilbert spaces.

2. Notations

We use notations in the same way as in [15].

Recall that an operator $T^\dagger \in \mathcal{L}(\mathcal{K},\mathcal{H})$ is the Moore-Penrose inverse of $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$, if the following is satisfied:

$$TT^\dagger T = T,\ T^\dagger TT^\dagger = T^\dagger,\ (TT^\dagger)^* = TT^\dagger,\ (T^\dagger T)^* = T^\dagger T.$$

It is well-known that $T^\dagger$ exists if and only if $\mathcal{R}(T)$ is closed, and in this case $T^\dagger$ is unique [2].

Assume that $T \in \mathcal{L}(\mathcal{H})$ and 0 is not the point of accumulation of the spectrum $\sigma(T)$ of T. If the point $\{0\}$ is the pole of the resolvent $\lambda \mapsto (\lambda - T)^{-1}$, then the order of this pole is the Drazin index (or the index) of T, denoted by $\mathrm{ind}(T)$. Notice that $\mathrm{ind}(T) < \infty$ holds if and only if there exists the Drazin inverse of T, i.e. there exists the unique operator $T^D \in \mathcal{L}(\mathcal{H})$, such that the following hold:

$$T^DTT^D = T^D,\ TT^D = T^DT,\ T^{n+1}T^D = T^n$$

and the least n in the previous definition is equal to $\mathrm{ind}(T)$. If $\mathrm{ind}(T) \le 1$, then T^D is known as the group inverse of T, denoted by $T^\#$. If $\mathrm{ind}(T) = 0$, then T is invertible and $T^{-1} = T^D$.

In this article the group inverse is of special interest. If $\mathrm{ind}(T) \le 1$, then $\mathcal{H} = \mathcal{R}(T) \dot{+} \mathcal{N}(T)$ and this sum is not necessarily orthogonal. Also, T has the matrix form with respect to this decomposition:

$$T = \begin{bmatrix} 0 & 0 \\ 0 & T_1 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(T) \\ \mathcal{R}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(T) \\ \mathcal{R}(T) \end{bmatrix},$$

where $T_1 = T|_{\mathcal{R}(T)} : \mathcal{R}(T) \to \mathcal{R}(T)$ is invertible [2].

In the case when T is selfadjoint and has a closed range, the Moore-Penrose inverse coincides with the group inverse of T. Also, $\mathcal{R}(T)$ is closed if and only if 0 is not the accumulation point of $\sigma(T)$. In this case the decomposition $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$ is orthogonal.

If $T = T^* \in \mathcal{L}(\mathcal{H})$, then we write $T \ge 0$ if and only if $(Tx, x) \ge 0$ for all $x \in \mathcal{H}$, where $(\cdot,\cdot)$ is the inner product in $\mathcal{H}$. Also, $T > 0$ if and only if $T \ge 0$ and T is invertible.

3. Results

The following result is proved in [1].

Lemma 3.1. *Let*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

where $S_{11} = S_{11}^*$, $S_{22} = S_{22}^*$ *and* $\mathcal{R}(S_{11})$ *is closed. Then* $S \geq 0$ *if and only if the following is satisfied:*

$$S_{11} \geq 0,\ S_{11}S_{11}^{\dagger}S_{12} = S_{12} \text{ and } S_{22} - S_{12}^*S_{11}^{\dagger}S_{12} \geq 0.$$

Although the original proof in [1] is given for finite dimensional spaces $\mathcal{H}$ and $\mathcal{K}$, the result is valid in infinite dimensional settings also.

We now prove the first auxiliary result.

Lemma 3.2. *Let* $R = \begin{bmatrix} H \\ B \end{bmatrix} : [\,\mathcal{H}\,] \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$, $H = H^*$ *and* $\rho = \|R\|$. *Then* $\mathcal{N}(\rho - H) \subset \mathcal{N}(B)$, $\mathcal{R}(\rho - H) \supset \mathcal{R}(B^*)$, $\mathcal{N}(\rho + H) \subset \mathcal{N}(B)$ *and* $\mathcal{R}(\rho + H) \supset \mathcal{R}(B^*)$.

Proof. Obviously, $\|H\| \leq \rho$. Let $x \in \mathcal{N}(\rho - H)$ and $\|x\| = 1$. Then

$$\rho^2 \geq \|Rx\|^2 = \|Hx\|^2 + \|Bx\|^2 = \rho^2 + \|Bx\|^2,$$

implying $Bx = 0$. The rest of the proof is similar. Notice that if there exists any $x \in \mathcal{N}(\rho - H)$ and $\|x\| = 1$, then $\|H\| = \rho = \|R\|$. $\square$

The following result represents a pseudo inverse form of solutions of the Davis-Kahan-Weinberger theorem.

Theorem 3.3. *Let* $R = \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} : [\mathcal{H}] \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$, $H = H^*$, $\rho = \|R\|$, $W = W^* \in \mathcal{L}(\mathcal{K})$, $A = \begin{bmatrix} H & B^* \\ B & W \end{bmatrix}$ *and let* $\mathcal{R}(\rho - H)$ *and* $\mathcal{R}(\rho + H)$ *be closed. Then* $\|A\| = \rho$ *if and only if*

$$B(\rho + H)^{\dagger}B^* - \rho \leq W \leq \rho - B(\rho - H)^{\dagger}B^*.$$

Proof. Obviously, $\rho = \|R\| \leq \|A\|$. Since $A = A^*$, in order to prove $\|A\| \leq \rho$, it is enough to prove $\rho - A \geq 0$ and $\rho + A \geq 0$. Notice that

$$\rho - A = \begin{bmatrix} \rho - H & -B^* \\ -B & \rho - W \end{bmatrix}.$$

From Lemma 3.1 we know that $\rho - A \geq 0$ if and only if:

(1) $\rho - H \geq 0$;
(2) $(\rho - H)(\rho - H)^\dagger B^* = B^*$;
(3) $\rho - W - (-B)(\rho - H)^\dagger(-B^*) \geq 0$.

We know that (1) always holds. The condition (2) is equivalent to $\mathcal{R}(B^*) \subset \mathcal{R}(\rho - H)$, which is always true according to Lemma 3.2. Finally, (3) is equivalent to $\rho - B(\rho - H)^\dagger B^* \geq W$.

Similarly, $\rho + A \geq 0$ is equivalent to $B(\rho + H)^\dagger B^* - \rho \leq W$. $\square$

Now we prove the extension of Corollary (DKW-central).

Theorem 3.4. *Let* $R = \begin{bmatrix} H \\ B \end{bmatrix}$, $H = H^*$, $\rho = \|R\|$ *and let* $\mathcal{R}(\rho - H)$ *and* $\mathcal{R}(\rho + H)$ *be closed. If*

$$W = -BH(\rho^2 - H^2)^\dagger B^*$$

and

$$A = \begin{bmatrix} H & B^* \\ B & W \end{bmatrix},$$

then

$$\|A\| = \rho = \|R\|.$$

Proof. The case $\rho = 0$ is trivial. Hence, assume $\rho > 0$. Since the Moore-Penrose inverse of a selfadjoint operator coincides with its group inverse, we conclude that the decomposition $\mathcal{H} = \mathcal{N}(\rho - H) \oplus \mathcal{R}(\rho - H)$ is orthogonal and

$$\rho - H = \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} : \begin{bmatrix} \mathcal{N}(\rho - H) \\ \mathcal{R}(\rho - H) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(\rho - H) \\ \mathcal{R}(\rho - H) \end{bmatrix},$$

where M is invertible and $M > 0$. We conclude that H and $\rho + H$ have the following matrix forms with respect to the same decomposition of $\mathcal{H}$:

$$H = \begin{bmatrix} \rho & 0 \\ 0 & \rho - M \end{bmatrix}, \quad \rho + H = \begin{bmatrix} 2\rho & 0 \\ 0 & 2\rho - M \end{bmatrix}.$$

Since $\rho + H \geq 0$, we conclude $0 < M \leq 2\rho$. From $\operatorname{ind}(\rho + H) \leq 1$ we conclude that $\operatorname{ind}(2\rho - M) \leq 1$. Now, $\mathcal{R}(\rho - H) = \mathcal{N}(2\rho - M) \oplus \mathcal{R}(2\rho - M)$ and this decomposition is orthogonal, since $2\rho - M$ is selfadjoint. Also

$$2\rho - M = \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} : \begin{bmatrix} \mathcal{N}(2\rho - M) \\ \mathcal{R}(2\rho - M) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(2\rho - M) \\ \mathcal{R}(2\rho - M) \end{bmatrix},$$

where N is invertible. Since $M \leq 2\rho$ we conclude $N > 0$. Notice that

$$M = \begin{bmatrix} 2\rho & 0 \\ 0 & 2\rho - N \end{bmatrix},$$

hence from $M > 0$ we get $0 < N < 2\rho$. Finally, we get

$$H = \begin{bmatrix} \rho & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & N - \rho \end{bmatrix}, \quad \rho - H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\rho & 0 \\ 0 & 0 & 2\rho - N \end{bmatrix},$$

$$\rho + H = \begin{bmatrix} 2\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N \end{bmatrix}$$

and conclude $\mathcal{N}(\rho + H) = \mathcal{N}(2\rho - M)$. From Lemma 3.2 we know that $\mathcal{N}(\rho - H) \subset \mathcal{N}(B)$ and $\mathcal{N}(\rho + H) \subset \mathcal{N}(B)$, implying the following decomposition of B:

$$B = [0 \quad 0 \quad B_1] : \begin{bmatrix} \mathcal{N}(\rho - H) \\ \mathcal{N}(\rho + H) \\ \mathcal{R}(2\rho - M) \end{bmatrix} \to \mathcal{K}$$

and also the matrix form of R:

$$R = \begin{bmatrix} \rho & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & H_1 \\ 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(\rho - H) \\ \mathcal{N}(\rho + H) \\ \mathcal{R}(2\rho - M) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(\rho - H) \\ \mathcal{N}(\rho + H) \\ \mathcal{R}(2\rho - M) \\ \mathcal{K} \end{bmatrix},$$

where $H_1 = N - \rho$. Notice that $-\rho < H_1 < \rho$. If $P_{\mathcal{R}(2\rho-M)}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{R}(2\rho - M)$, and $P_{\mathcal{R}(2\rho-M)\oplus\mathcal{K}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{R}(2\rho - M) \oplus \mathcal{K}$, then

$$R_1 = \begin{bmatrix} H_1 \\ B_1 \end{bmatrix} = P_{\mathcal{R}(2\rho-M)\oplus\mathcal{K}} R P_{\mathcal{R}(2\rho-M)},$$

implying $\|R_1\| \le \|R\|$.

Let $W_\rho = -B_1 H_1 (\rho^2 - H_1^2)^{-1} B_1^*$ and

$$A_\rho = \begin{bmatrix} H_1 & B_1^* \\ B_1 & W_\rho \end{bmatrix}.$$

From Lemma (DKW-central) we know that $\|A_\rho\| \le \|R_1\| \le \|R\| = \rho$.

Now we have the matrix form of A:

$$A = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & H_1 & B_1^* \\ 0 & 0 & B_1 & W_\rho \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & A_\rho \end{bmatrix}.$$

It is easy to see that $\|A\| = \rho$.

We only have to prove the equality

$$BH(\rho^2 - H^2)^\dagger B^* = B_1 H_1 (\rho^2 - H_1^2)^{-1} B_1^*.$$

Since ρ and $-\rho$ are not accumulation points of the spectrum $\sigma(H)$, we conclude that ρ^2 is not the accumulation point of H^2. Hence, $(\rho^2 - H^2)^\dagger$ exists. Now we compute

$$\begin{aligned} & BH(\rho^2 - H^2)^\dagger B^* = \\ & = \begin{bmatrix} 0 & 0 & B_1 \end{bmatrix} \begin{bmatrix} \rho & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & H_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N^{-1}(2\rho - N)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ B_1^* \end{bmatrix} \\ & = B_1 H_1 N^{-1} (2\rho - N)^{-1} B_1^* = B_1 H_1 (\rho^2 - H_1^2)^{-1} B_1^*. \quad \square \end{aligned}$$

As a corollary, we get the following result, which can not be verified easily by a direct computation.

Corollary 3.5. *If $\mathcal{R}(\rho - H)$ and $\mathcal{R}(\rho + H)$ are closed, where $\rho = \|R\|$, $R = \begin{bmatrix} H \\ B \end{bmatrix}$ and $H = H^*$, then*

$$B(\rho + H)^\dagger B^* - \rho \leq -BH(\rho^2 - H^2)^\dagger \leq \rho - B(\rho - H)^\dagger B^*.$$

Thus, we extended Zheng's results in [15, Theorem 4.1 and Theorem 4.2].

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