

# OPERATORS WITH EQUAL PROJECTIONS RELATED TO THEIR GENERALIZED INVERSES

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## Abstract

In this article we characterize operators on Banach spaces which have the same projections related to their outer or inner generalized inverses. As corollaries, we obtain well-known results for the Drazin inverse of bounded operators.

*Keywords:* Outer and inner generalized inverses, inner generalized inverses, projections, perturbations of generalized inverses.

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## 1 Motivation and introduction

In [2] Castro González, Koliha and Wei characterized matrices with the same eigenprojections, i.e. the same projections corresponding to the Drazin inverses of these matrices. They extended these results to closed operators on Banach spaces in [3]. Results of this type are used to prove error bounds for perturbations of operators with the same eigenprojections.

In this paper we investigate outer inverses of bounded operators with prescribed range and kernel on Banach spaces, as well as inner generalized inverses of these operators. Since the ordinary and the generalized Drazin inverse are outer generalized inverses with the particular choice of range and kernel, present results extend the main results from [2] and results from [3] restricted to a bounded case. Moreover, present results are applicable to the second important generalized inverse: the Moore-Penrose inverse of a bounded closed range operator between Hilbert spaces.

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Let  $X$  and  $Y$  denote arbitrary Banach spaces and let  $\mathcal{L}(X, Y)$  denote the set of all bounded operators from  $X$  to  $Y$ . Also,  $\mathcal{L}(X) = \mathcal{L}(X, X)$ . For  $A \in \mathcal{L}(X, Y)$  we use  $\mathcal{N}(A)$  to denote the kernel and  $\mathcal{R}(A)$  to denote the range of  $A$ .

We assume that the reader is familiar with the concept of the Moore-Penrose, Drazin and generalized Drazin inverse (see [1], [5], [6]). We use  $A^\dagger$ ,  $A^D$  and  $A^d$ , respectively, to denote these generalized inverses of  $A$ , in the case when any one of them exists.

We begin with outer and inner generalized inverses of bounded operators. Mostly, the technique of operator matrices is used (see, for example, [4]).

If there exists some operator  $A' \in \mathcal{L}(Y, X)$  satisfying  $A'A' = A'$ , then  $A'$  is called an outer generalized inverse of  $A$ . If  $T = \mathcal{R}(A')$  and  $S = \mathcal{N}(A')$ , then  $A'$  is well-known as the  $A_{T,S}^{(2)}$  generalized inverse of  $A$ . It can easily be deduced that for given subspaces  $T$  of  $X$  and  $S$  of  $Y$ , there exists the generalized inverse  $A_{T,S}^{(2)}$  of  $A$  if and only if the following is satisfied:  $T$ ,  $S$  and  $A(T)$  are closed complemented subspaces of  $X$ ,  $Y$  and  $Y$  respectively, the restriction  $A_1 = A|_T : T \rightarrow A(T)$  is invertible and  $A(T) \oplus S = Y$ . In this case the generalized inverse  $A_{T,S}^{(2)}$  is unique and the notation is justified. Moreover the following holds  $T = \mathcal{R}(A_{T,S}^{(2)}) = \mathcal{R}(A_{T,S}^{(2)}A)$ . Hence we denote  $T_1 = \mathcal{N}(A_{T,S}^{(2)}A) \subset X$  and  $S_1 = A(T) \subset Y$ . Now we have  $X = T \oplus T_1$  and  $Y = S_1 \oplus S$ . The matrix form of  $A$  follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix},$$

where  $A_1 \in \mathcal{L}(T, S_1)$  is invertible. Now it is easy to verify that

$$A_{T,S}^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} S_1 \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

In the rest of the article we use notations from previous matrix forms, whenever the generalized inverse  $A_{T,S}^{(2)}$  exists. It is well-known that  $A^\dagger$ ,  $A^D$  and  $A^d$  are outer generalized inverses with prescribed range and kernel, for particularly chosen complemented subspaces  $T$  and  $S$ .

Suppose that for  $A, B \in \mathcal{L}(X, Y)$  and subspaces  $T_1X$  and  $S_1Y$  there exists generalized inverses  $A_{T,S}^{(2)}$  and  $B_{T,S}^{(2)}$ . We say that  $A$  and  $B$  have the same projections related to generalized inverses  $A_{T,S}^{(2)}$  and  $B_{T,S}^{(2)}$ , provided that  $AA_{T,S}^{(2)} = BB_{T,S}^{(2)}$  and  $A_{T,S}^{(2)}A = B_{T,S}^{(2)}B$ . Notice that a generalization such as: use  $B_{M,N}^{(2)}$  for some other subspaces  $M$  and  $N$  does not work. We see that the following equality must hold:  $M = \mathcal{R}(B_{M,N}^{(2)}) = \mathcal{R}(B_{M,N}^{(2)}B)$  (since we want equal projections)  $= \mathcal{R}(A_{T,S}^{(2)}A) = T$ ; similarly  $N = S$ .

Things look similar in the case of inner generalized inverses. We say that  $A \in \mathcal{L}(X, Y)$  is relatively regular, or  $g$ -invertible, if there exists an operator  $A' \in \mathcal{L}(Y, X)$  satisfying  $AA'A = A$ . Recall that  $A$  is relatively regular if and only if  $\mathcal{R}(A)$  and  $\mathcal{N}(A)$ , respectively, are closed and complemented subspaces of  $Y$  and  $X$ . In this

case  $AA'$  is a projection from  $Y$  onto  $\mathcal{R}(A)$  and  $I - A'A$  is a projection from  $X$  onto  $\mathcal{N}(A)$ . Define  $T = \mathcal{R}(A'A)$  and  $S = \mathcal{N}(AA')$ . Now we have decomposition of spaces  $X = T \oplus \mathcal{N}(A)$  and  $Y = \mathcal{R}(A) \oplus S$  and consequently the matrix form of  $A$ :

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix},$$

where  $A_1 \in \mathcal{L}(T, \mathcal{R}(A))$  is invertible. Since  $AA'$  is the projection from  $Y$  onto  $\mathcal{R}(A)$  parallel to  $S$ , and  $A'A$  is the projection from  $X$  onto  $T$  parallel to  $\mathcal{N}(A)$ , we conclude that  $A'$  must have the matrix form:

$$A' = A_{T,S,M}^{(1)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & M \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix},$$

for arbitrary  $M \in \mathcal{L}(S, \mathcal{N}(A))$ . Obviously,  $A_{T,S,M}^{(1)}$  is uniquely determined by subspaces  $T$ ,  $S$ , respectively, complemented to  $\mathcal{N}(A)$  and  $\mathcal{R}(A)$ , and by an arbitrary operator  $M \in \mathcal{L}(S, \mathcal{N}(A))$ . Hence, the notation is justified. Moreover,  $A'$  is an outer generalized inverse also, if and only if  $M = 0$ .

It is well-known that  $A^\dagger$  is a particular inner generalized inverse for a particular choice of subspaces  $T$  and  $S$  and for  $M = 0$ .

If  $A, B \in \mathcal{L}(X, Y)$  are relatively regular with corresponding inner generalized inverses  $A'$  and  $B'$ , we discuss the situation  $AA' = BB'$  and  $A'A = B'B$ .

In Section 2 we consider the equality of projections related to outer generalized inverses of given operators. In Section 3 we consider the similar problem for projections related to inner generalized inverses of given operators.

## 2 Projections related to outer generalized inverses

In this section we first prove our main result, characterizing operators with equal projections related to their outer generalized inverses with prescribed range and kernel. All notations from previous section are retained.

**Theorem 2.1** *Suppose that for  $A \in \mathcal{L}(X, Y)$  and closed subspaces  $T \subset X$  and  $S \subset Y$  there exists the generalized inverse  $A_{T,S}^{(2)} \in \mathcal{L}(Y, X)$ . Then the following statements are equivalent:*

(a) *There exists the generalized inverse  $B_{T,S}^{(2)} \in \mathcal{L}(Y, X)$ , satisfying  $AA_{T,S}^{(2)} = BB_{T,S}^{(2)}$  and  $A_{T,S}^{(2)}A = B_{T,S}^{(2)}B$ .*

(b)  *$BA_{T,S}^{(2)}A = AA_{T,S}^{(2)}B$  and there exists the generalized inverse  $(BA_{T,S}^{(2)}A)_{T,S}^{(2)}$ .*

(c)  *$BA_{T,S}^{(2)}A = AA_{T,S}^{(2)}B$  and  $I + A_{T,S}^{(2)}(B - A)$  is invertible.*

Moreover, if previous statements are valid, then

$$B_{T,S}^{(2)} = [I + A_{T,S}^{(2)}(B - A)]^{-1}A_{T,S}^{(2)}.$$

*Proof.* (a) $\Rightarrow$ (b): The following subspace equality  $S_1 = A(T) = \mathcal{R}(AA_{T,S}^{(2)}) = \mathcal{R}(BB_{T,S}^{(2)}) = B(T)$  is obvious and similarly  $T_1 = \mathcal{N}(A_{T,S}^{(2)}A) = \mathcal{N}(B_{T,S}^{(2)}B)$ . Now, matrix forms for  $A$  and  $A_{T,S}^{(2)}$  from the previous section hold. Moreover, we have  $X = T \oplus T_1$  and  $Y = S_1 \oplus S$  and

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix},$$

where  $B_1 \in \mathcal{L}(T, S_1)$  is invertible, and

$$B_{T,S}^{(2)} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} S_1 \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

Now we compute

$$BA_{T,S}^{(2)}A = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} = AA_{T,S}^{(2)}B.$$

Since  $B_{T,S}^{(2)} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ , we easily verify that  $B_{T,S}^{(2)}$  is a reflexive generalized inverse of  $BA_{T,S}^{(2)}A$  and precisely  $(BA_{T,S}^{(2)}A)_{T,S}^{(2)} = B_{T,S}^{(2)}$ .

(b) $\Rightarrow$ (a): Here  $A$  and  $A_{T,S}^{(2)}$  have the same matrix forms as above. Suppose that  $B$  has the form

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix}.$$

From  $BA_{T,S}^{(2)}A = AA_{T,S}^{(2)}B$  we conclude that  $B_3$  and  $B_4$  vanish and consequently

$$BA_{T,S}^{(2)}A = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix}.$$

If

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = (BA_{T,S}^{(2)}A)_{T,S}^{(2)} : \begin{bmatrix} S_1 \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ T_1 \end{bmatrix},$$

from  $\mathcal{N}(C) = S$  we conclude  $C_2 = 0$  and  $C_4 = 0$ , and from  $\mathcal{R}(C) = T$  we conclude that  $C_3 = 0$  and  $C_1$  is invertible. Finally, from  $CBA_{T,S}^{(2)}AC = C$  we get  $C_1B_1C_1 = C_1$  and  $B_1 = C_1^{-1}$ . Now it is easy to prove  $C = B_{T,S}^{(2)}$  and the rest of the part (a) follows immediately.

(a) $\Rightarrow$ (c) Again, we have matrix forms of  $A, A_{T,S}^{(2)}, B, B_{T,S}^{(2)}$  with respect to the same decompositions  $X = T \oplus T_1$  and  $Y = S_1 \oplus S$ . Notice that

$$I + A_{T,S}^{(2)}(B - A) = \begin{bmatrix} A_1^{-1}B_1 & 0 \\ 0 & I \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix}$$

is invertible.

(c) $\Rightarrow$ (a) From  $BA_{T,S}^{(2)}A = AA_{T,S}^{(2)}B$  we conclude that  $B$  must have the form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} S_1 \\ S \end{bmatrix}$$

for some operators  $B_1$  and  $B_2$ . Notice that

$$I + A_{T,S}^{(2)}(B - A) = \begin{bmatrix} A_1^{-1}B_1 & 0 \\ 0 & I \end{bmatrix}$$

is invertible, implying that  $B_1$  is invertible. Define  $C = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ . Now it is easy to see that  $C = B_{T,S}^{(2)}$  and the rest of the part (a) follows easily.

Now, if statements (a)-(c) are valid, then it is easy to get the final result:

$$B_{T,S}^{(2)} = [I + A_{T,S}^{(2)}(B - A)]^{-1}A_{T,S}^{(2)}.$$

We connect present result with the notion of a  $\{T, S\}$ -splitting of an operator. Let  $A \in \mathcal{L}(X, Y)$  and  $T, S$  be subspaces of  $X$  and  $Y$  such that there exists the generalized inverse  $A_{T,S}^{(2)} \in \mathcal{L}(Y, X)$  exists. Recall [4] that  $A = B - U$  is called the  $\{T, S\}$ -splitting of  $A$ , provided that there exists the generalized inverse  $B_{T,S}^{(2)}$ . See also [7], [8] and [9] for finite dimensional settings and interesting applications.

Now, we see that all conditions (a)-(c) of our Theorem 2.1 imply the existence of the generalized inverse  $B_{T,S}^{(2)}$ . Hence, if the conditions of Theorem 2.1 are valid, then  $A = B - (A - B)$  can be considered as a  $\{T, S\}$ -splitting of  $A$  and all corresponding results from [4] are valid.

Recall the generalization of the condition number:

$$\kappa_{T,S}(A) = \|A\| \|A_{T,S}^{(2)}\|$$

whenever the generalized inverse  $A_{T,S}^{(2)}$  exists.

We need a part of Theorem 3.1 proved in [4].

**Lemma 2.1** *Let  $A \in \mathcal{L}(X, Y)$  be given, and closed subspaces  $T$  and  $S$ , respectively, such that there exists the generalized inverse  $A_{T,S}^{(2)}$ . If  $A = B - U$  is a  $\{T, S\}$ -splitting of  $A$ , then the following results hold:*

- (a)  $A_{T,S}^{(2)} - B_{T,S}^{(2)} = B_{T,S}^{(2)}UA_{T,S}^{(2)} = A_{T,S}^{(2)}UB_{T,S}^{(2)}$ .
- (b)  $A_{T,S}^{(2)} = (I - B_{T,S}^{(2)}U)^{-1}B_{T,S}^{(2)} = B_{T,S}^{(2)}(I - UB_{T,S}^{(2)})^{-1}$ .
- (c)  $B_{T,S}^{(2)} = (I + A_{T,S}^{(2)}U)^{-1}A_{T,S}^{(2)} = A_{T,S}^{(2)}(I + UA_{T,S}^{(2)})^{-1}$ .

(d) If  $\|A_{T,S}^{(2)}U\| < 1$ , then

$$\|B_{T,S}^{(2)} - A_{T,S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}U\| \|A_{T,S}^{(2)}\|}{1 - \|A_{T,S}^{(2)}U\|} \leq \kappa_{T,S}(A) \frac{\|A_{T,S}^{(2)}U\|}{\|A\|(1 - \|A_{T,S}^{(2)}U\|)}.$$

This result has a deeper form if we assume that  $A$  and  $B$  have equal projections related to their outer generalized inverses with prescribed range and kernel.

**Theorem 2.2** *Let all conditions (a)-(c) of Theorem 2.1 be valid. Moreover, if  $\|A_{T,S}^{(2)}U\| < 1$  is satisfied, where  $U = B - A$ , then the following hold:*

$$\begin{aligned} \text{(a)} \quad & \frac{\|A_{T,S}^{(2)}U\|}{\kappa_{T,S}(A)(1 + \|A_{T,S}^{(2)}U\|)} \leq \frac{\|B_{T,S}^{(2)} - A_{T,S}^{(2)}\|}{\|A_{T,S}^{(2)}\|} \leq \frac{\|A_{T,S}^{(2)}U\|}{1 - \|A_{T,S}^{(2)}U\|} \leq \frac{\kappa_{T,S}(A)\|U\|/\|A\|}{1 - \kappa_{T,S}(A)\|U\|/\|A\|}. \\ \text{(b)} \quad & \frac{\|A_{T,S}^{(2)}\|}{1 + \|A_{T,S}^{(2)}U\|} \leq \|B_{T,S}^{(2)}\| \leq \frac{\|A_{T,S}^{(2)}\|}{1 - \|A_{T,S}^{(2)}U\|}. \end{aligned}$$

*Proof.* (a) Using Lemma 2.1 we compute:

$$\begin{aligned} A_{T,S}^{(2)}U &= A_{T,S}^{(2)}(B - A) = A_{T,S}^{(2)}B - A_{T,S}^{(2)}A \\ &= A_{T,S}^{(2)}B - B_{T,S}^{(2)}B = (A_{T,S}^{(2)} - B_{T,S}^{(2)})B = A_{T,S}^{(2)}UB_{T,S}^{(2)}B \\ &= A_{T,S}^{(2)}UA_{T,S}^{(2)}A = A_{T,S}^{(2)}UA_{T,S}^{(2)}(I + A_{T,S}^{(2)}U)^{-1}(I + A_{T,S}^{(2)}U)A \\ &= A_{T,S}^{(2)}UB_{T,S}^{(2)}(I + A_{T,S}^{(2)}U)A \\ &= (A_{T,S}^{(2)} - B_{T,S}^{(2)})(I + A_{T,S}^{(2)}U)A \end{aligned}$$

Thus, the first inequality of (a) follows. Other inequalities follow immediately from Lemma 2.1 (d).

(b) This part follows from Lemma 2.1 (a).

### 3 Projections related to inner generalized inverses

In this section we obtain results concerning the equality of projections related to inner generalized inverses. Notations from the first Section are retained.

**Theorem 3.1** *Let  $A \in \mathcal{L}(X, Y)$  be relatively regular such that there exists the generalized inverse  $A_{T,S,M}^{(1)}$  and let  $B \in \mathcal{L}(X, Y)$ . The following statements are equivalent.*

- (a) *There exists an inner generalized inverse  $B' \in \mathcal{L}(Y, X)$  of  $B$  satisfying  $AA_{T,S,M}^{(1)} = BB'$  and  $A_{T,S,M}^{(1)}A = B'B$ .*
- (b)  *$BA_{T,S,M}^{(1)}A = AA_{T,S,M}^{(1)}B$  and there exists a generalized inverse  $(BA_{T,S,M}^{(1)}A)_{T,S,N}^{(1)}$  for some  $N \in \mathcal{L}(S, \mathcal{N}(A))$ .*

Moreover, if previous statements hold, then  $I + A_{T,S,M}^{(1)}(B - A)$  is invertible and the generalized inverse  $B'$  has the form

$$B' = B_{T,S,N}^{(1)} = (I + A_{T,S,M}^{(1)}(B - A))^{-1} A_{T,S,M}^{(1)}$$

for some  $N \in \mathcal{L}(S, \mathcal{N}(A))$ .

*Proof.* (a) $\Rightarrow$ (b): Recall the notations for  $A$  and  $A_{T,S,M}^{(1)}$ . From the equality of corresponding projections we get

$$\mathcal{R}(A) = \mathcal{R}(AA_{T,S,M}^{(1)}) = \mathcal{R}(BB') = \mathcal{R}(B)$$

and similarly  $\mathcal{N}(A) = \mathcal{N}(B)$ . We conclude that  $B$  has the matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix},$$

where  $B_1$  is invertible. Any inner generalized inverse of  $B$  must have the form

$$\begin{bmatrix} B_1^{-1} & K \\ L & N \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix}.$$

Again, using the equality of corresponding projections, we conclude that

$$B' = B_{T,S,N}^{(1)} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & N \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix}.$$

Now we easily get

$$BA_{T,S,M}^{(1)}A = AA_{T,S,M}^{(1)}B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix}$$

and also

$$(BA_{T,S,M}^{(1)}A)_{T,S,N}^{(1)} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & N \end{bmatrix}$$

is an appropriate inner generalized inverse for any  $N \in \mathcal{L}(S, \mathcal{N}(A))$ .

(b) $\Rightarrow$ (a): Now,  $A$  and  $A_{T,S,M}^{(1)}$  have the same matrix forms as above. Let  $B$  has the form

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix}.$$

From  $BA_{T,S,M}^{(1)}A = AA_{T,S,M}^{(1)}B$  we conclude  $B_2 = 0$  and  $B_3 = 0$ . The existence of the generalized inverse  $(BA_{T,S,M}^{(1)}A)_{T,S,N}^{(1)}$  for some  $N \in \mathcal{L}(S, \mathcal{N}(A))$  means that  $X = T \oplus \mathcal{N}(BA_{T,S,M}^{(1)}A)$  and  $Y = \mathcal{R}(BA_{T,S,M}^{(1)}A) \oplus S$ . Since  $\mathcal{N}(BA_{T,S,M}^{(1)}A) = \mathcal{N}(B_1) \oplus \mathcal{N}(A)$  and  $\mathcal{N}(B_1) \subset T$ , we conclude  $\mathcal{N}(B_1) = \{0\}$ . From  $\mathcal{R}(BA_{T,S,M}^{(1)}A) = \mathcal{R}(B_1)$

and  $\mathcal{R}(B_1) \subset \mathcal{R}(A)$ , we conclude that  $\mathcal{R}(B_1) = \mathcal{R}(A)$ . Hence,  $B_1$  is invertible. We conclude that

$$(BA_{T,S,M}^{(1)}A)_{T,S,N}^{(1)} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & N \end{bmatrix} = B_{T,S,N}^{(1)}$$

is the required inner generalized inverse of  $B$ .

If (a) and (b) hold, then

$$I + A_{T,S,M}^{(1)}(B - A) = \begin{bmatrix} A_1^{-1}B_1 & 0 \\ 0 & I \end{bmatrix}$$

is invertible. The equality

$$B' = B_{T,S,N}^{(1)} = (I + A_{T,S,M}^{(1)}(B - A))^{-1}A_{T,S,M}^{(1)}$$

for some  $N \in \mathcal{L}(S, \mathcal{N}(A))$  is obvious.

If for  $A \in \mathcal{L}(X, Y)$  there exists the generalized inverse  $A_{T,S,M}^{(1)}$ , then the following generalization of the condition number can be defined:

$$\kappa_{T,S,M}(A) = \|A\| \|A_{T,S,M}^{(1)}\|.$$

The notion of proper splitting of operator is very closed for the equality of projections related to inner generalized inverses. Recall that for a relatively regular operator  $A \in \mathcal{L}(X, Y)$  and operators  $U, V \in \mathcal{L}(X, Y)$ ,  $A = U - V$  is called the proper splitting of  $A$ , provided that  $\mathcal{R}(A) = \mathcal{R}(U)$  and  $\mathcal{N}(A) = \mathcal{N}(U)$  [4].

We need the following result proved in Theorem 2.1 from [4].

**Lemma 3.1** *Let  $A \in \mathcal{L}(X, Y)$  be relatively regular and let  $A = U - V$  be a proper splitting of  $A$ .*

(a) *If  $A^{(1)} = A_{T,S,M}^{(1)}$  is an inner generalized inverse of  $A$  for some  $M : S \rightarrow \mathcal{N}(A)$ , then there exists an inner generalized inverse of  $U$  which has the form  $U_{T,S,N}^{(1)}$  for some  $N : S \rightarrow \mathcal{N}(A)$ . In particular, there exists the inner generalized inverse  $U_{T,S,M}^{(1)}$  of  $U$ . (b)  $A_{T,S,K}^{(1)} - U_{T,S,K}^{(1)} = U_{T,S,N}^{(1)}VA_{T,S,M}^{(1)} = A_{T,S,M}^{(1)}VU_{T,S,N}^{(1)}$  for arbitrary  $K, M, N : S \rightarrow \mathcal{N}(A)$ .*

(c)  $A_{T,S,M}^{(1)} = (I - U_{T,S,M}^{(1)}V)^{-1}U_{T,S,M}^{(1)} = U_{T,S,M}^{(1)}(I - VU_{T,S,M}^{(1)})^{-1}$  for arbitrary  $M : S \rightarrow \mathcal{N}(A)$ .

(d)  $U_{T,S,M}^{(1)} = (I + A_{T,S,M}^{(1)}V)^{-1}A_{T,S,M}^{(1)} = A_{T,S,M}^{(1)}(I + VA_{T,S,M}^{(1)})^{-1}$  for arbitrary  $M : S \rightarrow \mathcal{N}(A)$ .

(e) *If  $\|A_{T,S,M}^{(1)}V\| < 1$  for some  $M : S \rightarrow \mathcal{N}(A)$ , then*

$$\begin{aligned} \|U_{T,S,M}^{(1)} - A_{T,S,M}^{(1)}\| &\leq \frac{\|A_{T,S,M}^{(1)}V\| \|A_{T,S,M}^{(1)}\|}{1 - \|A_{T,S,M}^{(1)}V\|} \\ &\leq \kappa_{T,S,M}(A) \frac{\|A_{T,S,M}^{(1)}V\|}{\|A\|(1 - \|A_{T,S,M}^{(1)}V\|)}. \end{aligned}$$



*Remark.* Notice that the statement of Lemma 2.1 (a) is formulated incorrectly in [4], where it is stated that any inner generalized inverse of  $U$  must have the proposed form. This fact is already explained in the proof of our Theorem 3.1 (a) $\Rightarrow$ (b).

Now we can prove the following perturbation result in the same way as Lemma 2.2

**Theorem 3.2** *Let all conditions (a)-(b) of Theorem 3.1 be valid. Moreover, if  $\|A_{T,S,M}^{(1)}U\| < 1$  is satisfied, where  $U = B - A$ , then the following hold:*

$$(a) \frac{\|A_{T,S,M}^{(1)}U\|}{\kappa_{T,S,M}(A)(1+\|A_{T,S,M}^{(1)}\| \|U\|)} \leq \frac{\|B_{T,S,M}^{(1)} - A_{T,S,M}^{(1)}\|}{\|A_{T,S,M}^{(1)}\|} \leq \frac{\|A_{T,S,M}^{(1)}U\|}{1 - \|A_{T,S,M}^{(1)}U\|} \leq \frac{\kappa_{T,S,M}(A)\|U\|/\|A\|}{1 - \kappa_{T,S,M}(A)\|U\|/\|A\|}.$$

$$(b) \frac{\|A_{T,S,M}^{(1)}\|}{1 + \|A_{T,S,M}^{(1)}U\|} \leq \|B_{T,S,M}^{(1)}\| \leq \frac{\|A_{T,S,M}^{(1)}\|}{1 - \|A_{T,S,M}^{(1)}U\|}.$$

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