The Representation and Approximations of Outer Generalized Inverses

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Abstract

We present a unified representation theorem for the class of all outer generalized inverses of a bounded linear operator. Using this representation we develop a few specific expressions and computational procedures for the set of outer generalized inverses. The obtained result is a generalization of the well-known representation theorem of the Moore-Penrose inverse as well as a generalization of the well-known results for the Drazin inverse and the generalized inverse $A^{(2)}_{T,S}$. Also, as corollaries we get corresponding results for reflexive generalized inverses.

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1 Introduction

Let X and Y be arbitrary Banach spaces and let $\mathcal{L}(X,Y)$ be the set of all bounded linear operators from X to Y. For $A \in \mathcal{L}(X,Y)$, we use $\mathcal{N}(A)$ to denote the null space, and $\mathcal{R}(A)$ to denote the range of A. An operator $G \in \mathcal{L}(Y,X)$ is an outer generalized inverse of A if $GAG = G$. If $T = \mathcal{R}(G)$ and $S = \mathcal{N}(G)$, then G is known as the generalized inverse $A^{(2)}_{T,S}$. It is easy to verify that for given subspaces T of X and S of Y there exists the generalized inverse $A^{(2)}_{T,S}$ if and only if T and S, respectively, are closed and complemented subspaces of X and Y, the restriction $A|_T : T \to A(T)$ is invertible and $A(T) \oplus S = Y$. In this case the generalized inverse $A^{(2)}_{T,S}$ is unique.

Special generalized inverses, such as the Moore-Penrose inverse, the ordinary and the generalized Drazin inverse (in the case when any of them exists) are outer generalized inverses with particular choices of T and S.

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Outer generalized inverses have many applications, for example, applications in the iterative methods for solving nonlinear equations [1], [17] and applications to statistics [9]. In particular, outer generalized inverses play an important role in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverses [16], [22]. The researchers have proposed many numerical methods for computing the $A^{(2)}_{T,S}$ generalized inverse in the literature (see [7], [20], [24], [28], [29], [30]).

For the sake of completeness, we briefly describe known representations of the Moore-Penrose inverse, the Drazin inverse and the $A^{(2)}_{T,S}$ generalized inverse. In this paper $A^{\dagger}$, A^D, A^d, A^g repsectively, denote the Moore-Penrose inverse the Drazin ivnerse, the generalized Drazin inverse and the group inverse of A. By $\mathrm{ind}(A)$ we denote the Drazin index of the operator A.

Theorem 1.1 [10] *Let X and Y be Hilbert spaces. Suppose that $T \in \mathcal{L}(X,Y)$ has closed range and let $\tilde{T} = T^*T|_{\mathcal{R}(T^*)} : \mathcal{R}(T^*) \to \mathcal{R}(T^*)$. Then $T^{\dagger} = \tilde{T}^{-1}T^*$.*

Theorem 1.2 [26] *Let X be a Banach space and $A \in \mathcal{L}(X)$ with $\mathrm{ind}(A) = k$. Then*

$$A^D = \tilde{A}^{-1}A^k,$$

where $\tilde{A} = A^{k+1}|_{\mathcal{R}(A^k)} : \mathcal{R}(A^k) \to \mathcal{R}(A^k)$.

Theorem 1.3 [24], [7] *Let X and Y be Banach spaces, let T and S be closed subspaces of X and Y, respectively, such that for an operator $A \in \mathcal{L}(X,Y)$ the generalized inverse $A^{(2)}_{T,S}$ exists. Let $G \in \mathcal{L}(Y,X)$ be an arbitrary operator which satisfies $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. Then $\mathrm{ind}(GA) = \mathrm{ind}(AG) = 1$ and*

$$A^{(2)}_{T,S} = (GA)^g G = G(AG)^g = \tilde{A}^{-1}G,$$

where $\tilde{A} = A|_T : T \to A(T)$.

In this paper we introduce a unified representation theorem for the set of all outer generalized inverses of bounded linear operators on Banach spaces. Using this representation, we develop a number of iterative methods for generating outer generalized inverses. Moreover we obtain known representations and approximations of the Drazin inverse [4], [5], [27], [23], [26], the generalized inverse $A^{(2)}_{T,S}$ [15], [24], [28], [29], [30] and the weighted Moore-Penrose inverse [25]. As a partial result we get an improvement of the hyper-power iterative method, which is investigated in [19]. Also, a limit representation of the outer generalized inverses, introduced in [18], can be considered as a partial case of the introduced approximations.

2 Representation of outer generalized inverses

In this section we establish a unified representation theorem for outer generalized inverses of operators on Banach spaces. Recall that $A \in \mathcal{L}(X, Y)$ is relatively regular if A has an inner generalized inverse B, i.e. $ABA = A$. It is well-known that A is relatively regular if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, are closed and complemented subspaces of Y and X.

An operator $B \in \mathcal{L}(Y, X)$ is called a reflexive generalized inverse of A, if B is both inner and outer generalized inverse of A.

If $A \in \mathcal{L}(X)$ and 0 is not the accumulation point of the spectrum $\sigma(A)$, then there exists the generalized Drazin inverse of A, denoted by A^d [13]. For the most important properties see [13] and [8].

Theorem 2.1 *Let X and Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$ be arbitrary. Then the following hold:*

(a) *The set of outer generalized inverses A can be represented in the following way:*

$$A\{2\} = \{\left((WA)|_{\mathcal{R}(W)}\right)^{-1} W \ : \ W \in \mathcal{L}(Y, X), \ \mathcal{R}(W) \text{ is closed and } (WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W) \text{ is invertible}\}. \tag{2.1}$$

(b) *If A is relatively regular, then the class of all reflexive generalized inverses of A can be represented as follows:*

$$A\{1,2\} = \{\left((WA)|_{\mathcal{R}(W)}\right)^{-1} W : W \in \mathcal{L}(Y, X), \ \mathcal{R}(W) \text{ is closed, } (WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W) \text{ is invertible and } \mathcal{R}(W) \oplus \mathcal{N}(A) = X\}.$$

(c) *If $A \in \mathcal{L}(X)$, 0 is not the point of accumulation of the spectrum $\sigma(A)$ and E is the spectral idempotent of A corresponding to the point $\{0\}$, then for any $W \in \mathcal{L}(X)$ such that $\mathcal{R}(W) = \mathcal{N}(E)$ and $\mathcal{N}(W) = \mathcal{R}(E)$, the following holds*

$$A^d = \left((WA)|_{\mathcal{R}(W)}\right)^{-1} W.$$

Proof. (a) If G has the form (2.1), i.e. $G = \left((WA)|_{\mathcal{R}(W)}\right)^{-1} W$ for a suitable choice of $W \in \mathcal{L}(Y, X)$, we conclude that G is bounded. Now we verify that

$$GAG = \left((WA)|_{\mathcal{R}(W)}\right)^{-1} WA \left((WA)|_{\mathcal{R}(W)}\right)^{-1} W = \left(WA|_{\mathcal{R}(W)}\right)^{-1} W = G$$

On the other hand, let G be any outer generalized inverse of A. We prove that there exists an appropriate operator W, such that $\mathcal{R}(W)$ is closed, the operator $(WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W)$ is invertible and $G = \left(WA|_{\mathcal{R}(W)}\right)^{-1} W$. Indeed, we can use $W = G$. In this case WA is a projection onto $\mathcal{R}(W)$ and

$$(WA)|_{\mathcal{R}(W)} = I|_{\mathcal{R}(W)}$$

is invertible. Also,

$$\left((WA)|_{\mathcal{R}(W)}\right)^{-1}W = \left(I|_{\mathcal{R}(W)}\right)^{-1}W = W.$$

(b) Let A be relatively regular and let G be any reflexive generalized inverse of A. We take $W = G$. Then $\mathcal{R}(W) \oplus \mathcal{N}(A) = X$, since WA is the projection from X onto $\mathcal{R}(W)$ parallel to $\mathcal{N}(A)$. We simply verify that $G = ((WA)|_{\mathcal{R}(W)})^{-1}W$.

On the other hand, assume that $\mathcal{R}(W)$ is closed, $X = \mathcal{R}(W) \oplus \mathcal{N}(A)$ and $(WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W)$ is invertible. Let $G = \left((WA)|_{\mathcal{R}(W)}\right)^{-1}W$. From the part (a) we know that G is an outer generalized inverse of A. If $x \in X$, then $x = u + v$, where $u \in \mathcal{R}(W)$ and $v \in \mathcal{N}(A)$. Then

$$\left((WA)|_{\mathcal{R}(W)}\right)^{-1}WAx = \left((WA)|_{\mathcal{R}(W)}\right)^{-1}WAu = u = P_{\mathcal{R}(W),\mathcal{N}(A)}x.$$

Hence,

$$A\left((WA)|_{\mathcal{R}(W)}\right)^{-1}WA = AP_{\mathcal{R}(W),\mathcal{N}(A)} = A$$

and G is a reflexive generalized inverse of A.

(c) If E is the spectral idempotent of A corresponding to $\{0\}$, then $X = \mathcal{N}(E) \oplus \mathcal{R}(E)$,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(E) \\ \mathcal{R}(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(E) \\ \mathcal{R}(E) \end{bmatrix},$$

where $A_1 = A|_{\mathcal{N}(E)} : \mathcal{N}(E) \to \mathcal{N}(E)$ is invertible and $A_2 = A|_{\mathcal{R}(E)} : \mathcal{R}(E) \to \mathcal{R}(E)$ is quasinilpotent. In this case the generalized Drazin inverse of A is

$$A^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

If $\mathcal{R}(W) = \mathcal{N}(E)$ and $\mathcal{N}(W) = \mathcal{R}(E)$, then we conclude that W must have the form

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(E) \\ \mathcal{R}(E) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(E) \\ \mathcal{R}(E) \end{bmatrix},$$

where $W_1 = W|_{\mathcal{N}(E)} : \mathcal{N}(E) \to \mathcal{N}(E)$ is invertible. Now,

$$WA = \begin{bmatrix} W_1A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $W_1A_1 = (WA)|_{\mathcal{R}(W)}$ is invertible. It is easy to verify that

$$\left((WA)|_{\mathcal{R}(W)}\right)^{-1}W = A_1^{-1}W_1^{-1}W = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = A^d. \qquad \square$$

The invertibility of $(WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W)$ is a very strong condition, as we can see from the following corollary.

Corollary 2.1 *If A and W are given as in Theorem* 2.1 (a), *then $\mathcal{R}(W)$ and $\mathcal{N}(W)$, respectively, are complemented subspaces of X and Y. Moreover,* $\operatorname{ind}(\tilde{A}) = 1$ *and* $\sigma(\tilde{A}) = \sigma(WA) \setminus \{0\}$, *where we denote* $\tilde{A} = (WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W)$.

Proof. Let $G = ((WA)|_{\mathcal{R}(W)})^{-1}W$. Since A is an inner generalized inverse of G, we get that GA is a projection from X onto $\mathcal{R}(G) = \mathcal{R}(W)$, and also $I - AG$ is a projection from Y onto $\mathcal{N}(G) = \mathcal{N}(W)$. Now, $X = \mathcal{R}(W) \oplus T$ and $Y = A(\mathcal{R}(W)) \oplus \mathcal{N}(W)$, where $T = \mathcal{N}(GA)$. We have the following matrix decomposition:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(W) \\ T \end{bmatrix} \to \begin{bmatrix} A(\mathcal{R}(W)) \\ \mathcal{N}(W) \end{bmatrix},$$

where A_1 is invertible. Also

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(\mathcal{R}(W)) \\ \mathcal{N}(W) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(W) \\ T \end{bmatrix},$$

where W_1 is invertible. Now,

$$WA = \begin{bmatrix} W_1A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix},$$

$\operatorname{ind}(WA) = 1$ and $\sigma(\tilde{A}) = \sigma(WA) \setminus \{0\}$. □

Corollary 2.2 (a) *If $A \in \mathcal{L}(X, Y)$ is arbitrary and if T, S are subspaces of X and Y such that the generalized inverse $A^{(2)}_{T,S}$ exists, then we have to take $\mathcal{R}(W) = T$ and $\mathcal{N}(W) = S$ in* (2.1) *to obtain the known representation in Theorem* 1.3.

(b) *If $A \in \mathcal{L}(X)$, $k = \operatorname{ind}(A) < \infty$, then we can take $W \in \mathcal{L}(X)$ such that $\mathcal{R}(W) = \mathcal{R}(A^k)$ and $\mathcal{N}(W) = \mathcal{N}(A^k)$ then use Theorem* 2.1 (c) *to get*

$$A^D = \left((WA)|_{\mathcal{R}(W)}\right)^{-1} W.$$

(c) *If X and Y are Hilbert spaces, $A \in \mathcal{L}(X, Y)$ and $\mathcal{R}(A)$ is closed, then we can take W satisfying $\mathcal{R}(W) = \mathcal{R}(A^*)$ and $\mathcal{N}(W) = \mathcal{N}(A^*)$, then use Theorem* 2.1 (a) *or* (b) *to get*

$$A^\dagger = \left((WA)|_{\mathcal{R}(W)}\right)^{-1} W.$$

Remark 2.1 (a) *In the case* (c) *of Corollary* 2.2 *we can take $W = A^*$ and get the result of Theorem* 1.1.

(b) *The choice $W = A^l$, $l \geq \operatorname{ind}(A)$, in the case* (b) *of Corollary* 2.2 *gives us the result of Theorem* 1.2.

We also formulate a result for complex matrices, which is important for practical applications.

Corollary 2.3 *Let $A \in \mathcal{C}_r^{m\times n}$. Then the class of all reflexive and the class of all outer generalized inverses of A can be represented as follows:*

$$A\{1,2\} = \{\left((WA)|_{\mathcal{R}(W)}\right)^{-1} W : \ W \in \mathcal{C}^{n\times m}, \\ \operatorname{rank}(WAW) = \operatorname{rank}(W) = \operatorname{rank}(A)\}$$

and

$$A\{2\} = \{\left(WA|_{\mathcal{R}(W)}\right)^{-1} W : \ W \in \mathcal{C}^{n\times m}, \\ \operatorname{rank}(WAW) = \operatorname{rank}(W) \leq \operatorname{rank}(A)\}.$$

Now we are in a position to establish the general representation theorem.

Theorem 2.2 *Let $A \in \mathcal{L}(X,Y)$ and let $W \in \mathcal{L}(Y,X)$ be arbitrary such that $\mathcal{R}(W)$ is closed and $\tilde{A} = (WA)|_{\mathcal{R}(W)} : \mathcal{R}(W) \to \mathcal{R}(W)$ is invertible. Let Ω be an open subset of the set $\mathcal{C} \setminus \{0\}$ such that $\sigma(\tilde{A}) \subset \Omega$. Let $(S_n)_n$ be any family of complex analytic functions on Ω with $\lim\limits_{n\to\infty} S_n(z) = \frac{1}{z}$ uniformly on compact subsets of Ω. Then*

$$G = \lim_{n\to\infty} S_n(\tilde{A})W = \tilde{A}^{-1}W$$

is an outer generalized inverse of A.

Furthermore, for any $\epsilon > 0$, there is an operator norm $\|\cdot\|_*$ such that

$$\frac{\|S_n(\tilde{A})W - G\|_*}{\|G\|_*} \leq \max_{z\in\sigma(\tilde{A})} |S_n(z)z - 1| + O(\epsilon). \tag{2.2}$$

Proof. Using the well-known properties of the functional calculus (see, for example, [21][Theorem 10.27]), we have

$$\lim_{n\to\infty} S_n(\tilde{A}) = \tilde{A}^{-1}.$$

It follows from Theorem 2.1 that

$$\lim_{n\to\infty} S_n(\tilde{A})W = \tilde{A}^{-1}W = G \in A\{2\}.$$

To obtain the error bound we note that $W = \tilde{A}G$ and therefore

$$S_n(\tilde{A})W - G = \left[S_n(\tilde{A})\tilde{A} - I\right] G.$$

Also, for any B and $\epsilon > 0$, there is an operator norm $\|\cdot\|_*$ such that $\|B\|_* \leq \rho(B) + \epsilon$ (see [3][page 77]), where $\rho(B)$ is the spectral radius of B. Thus

$$\begin{aligned} \|S_n(\tilde{A})W - G\|_* &= \|\left[S_n(\tilde{A})\tilde{A} - I\right] G\|_* \\ &\leq \|S_n(\tilde{A})\tilde{A} - I\|_* \|G\|_* \\ &\leq \left[\max_{z \in \sigma(\tilde{A})} |S_n(z)z - 1| + O(\epsilon)\right] \|G\|_*. \end{aligned}$$

Now (2.2) follows immediatelly. $\square$

In order to make use of this general error estimate above on specific approximation procedures it will be convenient to have a lower and upper bounds for $\sigma(\tilde{A})$. This is given in the following theorem.

Theorem 2.3 *Let A and W be as in Theorem 2.1, and denote $\tilde{A} = (WA)|_{\mathcal{R}(W)}$. Then for each $\lambda \in \sigma(\tilde{A})$ we have*

$$\frac{1}{\|(WA)^g\|} \leq |\lambda| \leq \|WA\|.$$

Proof. From Corollary 2.1 we know that $(WA)^g$ exists. Let $\lambda \in \sigma(\tilde{A})$. Since $\|WA\| \geq \|(WA)|_{\mathcal{R}(W)}\|$, it follows that $|\lambda| \leq \|WA\|$ holds. On the other hand, we know that $\frac{1}{\lambda} \in \sigma(\tilde{A}^{-1})$ and

$$\frac{1}{|\lambda|} \leq \rho(\tilde{A}^{-1}) \leq \|(WA)^g\|,$$

which completes the proof. $\square$

3 Approximations of outer generalized inverses

In this section we present several corollaries which illustrate the use of Theorem 2.2 in developing specific representations and computational procedures for outer generalized inverses of operators on Banach spaces. We also find corresponding error bounds.

The section is divided into six subsections.

(a) The following well-known summability method is called the Euler-Knopp method [11]. A series $\sum_{n=0}^{\infty} a_n$ is said to be Euler-Knopp summable with parameter $\alpha > 0$ to the value a if the sequence defined by

$$S_n = \alpha \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} (1-\alpha)^{k-j} \alpha^j a_j$$

converges to a. If $a_j = (1-z)^j$ for $j = 0, 1, \ldots$, then we obtain as the Euler-Knopp transform of the series $\sum_{n=0}^{\infty}(1-z)^n$, the sequence given by

$$S_n(z) = \alpha \sum_{k=0}^{n} (1-\alpha z)^k.$$

We use the notation $\mathcal{C}^+ = \{z \in \mathcal{C} : \operatorname{Re} z > 0\}$.

Theorem 3.1 *Let A and W be given such that the conditions of Theorem 2.1 are satisfied. Suppose that $\sigma(WA)\setminus\{0\} \subset \mathcal{C}^+$, $m = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(WA)\setminus\{0\}\}$, $\rho = \rho(WA)$ and $0 < \alpha < \frac{2m}{\rho^2+m^2}$. Consider the sequence*

$$A_0 = \alpha W, \quad A_{n+1} = (I - \alpha WA)A_n + \alpha W, \quad n \geq 0. \tag{3.1}$$

Then

$$\lim_{n\to\infty} A_n = G \in A\{2\}.$$

Moreover, there exists some β, $0 < \beta < 1$, such that for any $\epsilon > 0$ there exists the operator norm $\|\cdot\|_*$ such that

$$\frac{\|A_n - G\|_*}{\|G\|_*} \leq \beta^{n+1} + O(\epsilon). \tag{3.2}$$

If the spectrum of WA is real and nonnegative, then α can be chosen as $0 < \alpha < 2/\|WA\|$.

Proof. Since $S_n(z) = \alpha\sum_{k=0}^{n}(1-\alpha z)^k$, we get that $\lim\limits_{n\to\infty} S_n(z) = 1/z$ uniformly on compact subsets of the set

$$E_\alpha = \{z : |1-\alpha z| < 1\} = \left\{z = x+iy : 0 < x < 2/\alpha, |y| < \sqrt{(2/\alpha)x - x^2}\right\}.$$

Particularly, if K is a compact subset of E_α, then $\max\{|1-\alpha z| : z \in K\} = \beta_K < 1$.

We need to find sufficient conditions for α such that $\sigma(WA)\setminus\{0\} \subset E_\alpha$. Let $M = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(WA)\setminus\{0\}\}$. Now α can be chosen according to the condition

$$0 < \alpha < \min\left\{\frac{2m}{\rho^2+m^2}, \frac{2M}{\rho^2+M^2}\right\}.$$

Notice that $0 < m \leq M \leq \rho$. Also, the function $t \mapsto \frac{2t}{\rho^2+t^2}$ is increasing in $t \in (0, \rho]$. Hence, the sufficient condition for $\sigma(\tilde{A}) \subset E_\alpha$ is given by

$$0 < \alpha < \frac{2m}{\rho^2+m^2}.$$

Moreover, if the set $\sigma(WA)$ is real and nonnegative, then $E_\alpha = (0, 2/\alpha)$. In this case it is enough to take $0 < \alpha < 2/\|WA\|$.

From the proof of Corollary 2.1 we know the following:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(W) \\ T \end{bmatrix} \to \begin{bmatrix} A(\mathcal{R}(W)) \\ \mathcal{N}(W) \end{bmatrix},$$

where A_1 is invertible,

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(\mathcal{R}(W)) \\ \mathcal{N}(W) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(W) \\ T \end{bmatrix},$$

where W_1 is invertible and consequently

$$WA = \begin{bmatrix} W_1A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\tilde{A} = (WA)|_{\mathcal{R}(W)} = W_1A_1 : \mathcal{R}(W) \to \mathcal{R}(W)$, it is easy to see that

$$(I - \alpha WA)W = (I - \alpha\tilde{A})W.$$

Consequently, using Theorem 2.2 we get

$$S_n(WA)W = S_n(\tilde{A})W \to G \in A\{2\} \quad (n \to \infty)$$

i.e.

$$G = \alpha \sum_{n=0}^{\infty} (I - \alpha WA)^n W. \tag{3.3}$$

If we set $A_n = S_n(WA)W$, then it is easy to verify that

$$A_n = \alpha \sum_{k=0}^{n} (I - \alpha WA)^k W, \tag{3.4}$$

and

$$(I - \alpha WA)A_n = \alpha \sum_{k=1}^{n+1} (I - \alpha WA)^k W = A_{n+1} - \alpha W.$$

From (3.3) and (3.4) we obtain (3.1) and $\lim_{n\to\infty} A_n = G$.

For the error bound we note that the sequence of functions $(S_n(z))_n$ satisfies

$$S_{n+1}(z)z - 1 = (1 - \alpha z)(S_n(z)z - 1).$$

Thus

$$|S_n(z)z - 1| = |1 - \alpha z|^{n+1}.$$

Notice that α is chosen according to the condition

$$\max_{z \in \sigma(\tilde{A})} |1 - \alpha z| = \beta < 1.$$

Hence, for all $z \in \sigma(\tilde{A})$ the following holds

$$|S_n(z)z - 1| \leq \beta^{n+1} \to 0 \quad (n \to \infty). \tag{3.5}$$

If $\epsilon > 0$ is arbitrary, from (3.5) and Theorem 2.2 it follows that there exists an operator norm $\|\cdot\|_*$ such that (3.2) holds. $\square$

Remark 3.1 *Since $\rho \leq \|WA\|$, we can take the following sufficient condition for α in Theorem* 3.1*:*

$$0 < \alpha < \frac{2m}{\|WA\|^2 + m^2}$$

(b) To develop another iterative method, we regard $1/z$ as the root of the function

$$s(y) = y^{-1} - z.$$

The Newton-Raphson method [11] can be used to approximate this root. This is done by generating a sequence $(y_n)_n$ where

$$y_{n+1} = y_n - s(y_n)/s'(y_n) = y_n(2 - zy_n),$$

for suitable y_0.

Theorem 3.2 *Let A and W be given such that the conditions from Theorem* 2.1 *are satisfied. Suppose that $\sigma(WA)\setminus\{0\} \subset \mathcal{C}^+$, $m = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(WA)\setminus\{0\}\}$, $\rho = \rho(WA)$ and $0 < \alpha < \frac{2m}{\rho^2+m^2}$. Consider the sequence*

$$A_0 = \alpha W, \quad A_{n+1} = A_n(2I - AA_n), \quad n \geq 0. \tag{3.6}$$

Then

$$\lim_{n\to\infty} A_n = G \in A\{2\}.$$

Moreover, there exists some β, $0 < \beta < 1$, such that for any $\epsilon > 0$ there exists the operator norm $\|\cdot\|_*$ such that

$$\frac{\|A_n - G\|_*}{\|G\|_*} \leq \beta^{2^n} + O(\epsilon). \tag{3.7}$$

If the spectrum of WA is real and nonnegative, then α can be chosen as $0 < \alpha < 2/\|WA\|$.

Proof. For $\alpha > 0$ we define a sequence of complex functions $(S_n(z))_n$ by

$$S_0(z) = \alpha, \quad S_{n+1}(z) = S_n(z)\left[2 - zS_n(z)\right]. \tag{3.8}$$

Clearly the sequence of functions in (3.8) satisfies

$$zS_{n+1}(z) - 1 = -\left[zS_n(z) - 1\right]^2.$$

Iterating this equality we obtain the following

$$|zS_n(z) - 1| = |\alpha z - 1|^{2^n}, \tag{3.9}$$

Again, let

$$E_\alpha = \{z = x + iy : 0 < x < 2/\alpha,\ |y| < \sqrt{(2/\alpha)x - x^2}\}.$$

In the same way as in Theorem 3.1 we can prove that our choice of α implies $\sigma(\tilde{A}) \subset E_\alpha$. Obviously, $\beta = \sup\limits_{z\in\sigma(\tilde{A})} |1 - \alpha z| < 1$. Hence, for all $z \in \sigma(\tilde{A})$ we get

$$|zS_n(z) - 1| \le \beta^{2^n} \to 0 \quad (n \to \infty)$$

and $\lim\limits_{n\to\infty} S_n(z) = 1/z$ uniformly on compact subsets of E_α.

The sequence $(S_n(\tilde{A}))_n$ defined by

$$S_0(\tilde{A}) = \alpha I, \quad S_{n+1}(\tilde{A}) = S_n(\tilde{A})\left[2I - \tilde{A}S_n(\tilde{A})\right]$$

has the property that $\lim\limits_{n\to\infty} S_n(\tilde{A})W = G \in A\{2\}$. Using Corollary 2.1 it is easy to prove $S_n(\tilde{A})W = S_n(WA)W$. If we set $A_n = S_n(WA)W$, then we have (3.6) immediately.

The error estimate (3.7) follows in the same way as in Theorem 3.1. □

(c) Choosing $S_n = (1/n + z)^{-1}$, we have $\lim\limits_{n\to\infty} S_n(z) = 1/z$ uniformly on compact subsets of $\mathcal{C} \setminus \{0\}$. Hence we have the following result.

Theorem 3.3 *Let A and W be the same as in Theorem 2.1. Then*

$$\lim_{n\to\infty} \left(\frac{1}{n}I + WA\right)^{-1} W = G \in A\{2\}. \tag{3.10}$$

Also, there exists the norm $\|\cdot\|_*$ such that the following norm estimate holds:

$$\frac{\|(\frac{1}{n}I + WA)^{-1}W - G\|_*}{\|G\|_*} \le \frac{\frac{1}{n}\|(WA)^g\|}{1 - \frac{1}{n}\|(WA)^g\|} + O(\epsilon) \tag{3.11}$$

for sufficiently large n.

Proof. We set $S_n(z) = (1/n + z)^{-1}$. Using Theorem 3.1 we get

$$\lim_{n\to\infty} S_n(\tilde{A})W = G \in A\{2\}.$$

Again, we easily verify that $((1/n)I + WA)^{-1}W = ((1/n)I + \tilde{A})^{-1}W$. Note that for $z \in \sigma(\tilde{A})$ we can deduce that

$$|zS_n(z) - 1| = \frac{\frac{1}{n}}{|z + \frac{1}{n}|} \le \frac{\frac{1}{n}}{|z| - \frac{1}{n}}$$

for sufficiently large n. Using Theorem 2.3 for any $z \in \sigma(\tilde{A})$ we can obtain

$$|zS_n(z) - 1| \leq \frac{\frac{1}{n}}{\frac{1}{\|(WA)^g\|} - \frac{1}{n}} = \frac{\frac{1}{n}\|(WA)^g\|}{1 - \frac{1}{n}\|(WA)^g\|}$$

for sufficiently large n. The rest of the proof follows from Theorem 2.2. $\square$

Remark 3.2 *In the partial case* $\mathcal{R}(W) = T$, $\mathcal{N}(W) = S$ *we obtain the well-known limit expression of the generalized inverse* $A^{(2)}_{T,S}$ *originated by Wei in* [24], *that is of the form* (3.10).

(d) So far we have considered results based on approximating the function $1/z$. Now we will use Theorem 2.2 and the Newton-Gregory interpolation [11], [12] of the function $f(z) = 1/z$ to generate iterative methods for computing the generalized inverse $G \in A\{2\}$ and obtain corresponding asymptotic error bounds.

The unique polynomial, generated by the Newton-Gregory interpolation formula, which interpolates the function $f(z) = \frac{1}{z}$ at the points $z = 1, 2, \ldots, n+1$, is equal to

$$p_n(z) = \sum_{j=0}^{n} \binom{z-1}{j} \Delta^j f(1), \tag{3.12}$$

where Δ is the forward difference operator defined by

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^j f(z) = \Delta(\Delta^{j-1} f(z)),$$

and

$$\binom{z}{0} = 1, \quad \binom{z-1}{j} = \frac{(z-1)(z-2)\cdots(z-j)}{j}.$$

It is not hard to see that if $f(z) = 1/z$, then $\Delta^j f(1) = \frac{(-1)^j}{j+1}$. A routine calculation now shows that the interpolation polynomial is given by

$$p_n(z) = \sum_{j=0}^{n} \frac{1}{j+1} \prod_{l=0}^{j-1} \left(1 - \frac{z}{l+1}\right), \tag{3.13}$$

where the product from 0 to -1 is, by convention, taken to be -1. Groetsch in [11] shown that

$$1 - zp_n(z) = \prod_{l=0}^{n} \left(1 - \frac{z}{l+1}\right).$$

Lemma 3.1 *The polynomials* $p_n(z)$ *satisfy*

$$\lim_{n\to\infty} p_n(z) = \frac{1}{z}, \tag{3.14}$$

uniformly on compact subsets of $\mathcal{C}^+$.

Proof. Let K be a compact subset of $\mathcal{C}^+$. Then $m = \inf\{x : z = x + iy \in K\} > 0$. There exists some positive integer j_K such that for all $j \geq j_K$ and all $z = x + iy \in K$ the following is satisfied:

$$0 < x < 2(j+1) \quad \text{and} \quad |y| < \sqrt{2(j+1)x - x^2}.$$

Hence, for all $z \in K$ and all $j \geq j_k$ we have $0 \leq \left|1 - \frac{z}{j+1}\right| < 1$. Notice that $\sum\limits_{j=0}^{\infty} \frac{|z|}{j+1} \geq \sum\limits_{j=0}^{\infty} \frac{m}{j+1} = +\infty$ and consequently $\sum\limits_{j=0}^{\infty} \frac{|z|}{j+1} = +\infty$ uniformly on K. Since $0 \leq \left|1 - \frac{z}{j+1}\right| < 1$ for all $z \in K$, we conclude $\sum\limits_{j=0}^{\infty} \ln\left|1 - \frac{z}{j+1}\right| = -\infty$ uniformly on K. Consequently, $\lim\limits_{n\to\infty} \prod\limits_{j=0}^{n} \left|1 - \frac{z}{j+1}\right| = 0$ uniformly on K. □

Theorem 3.4 *Let A and W satisfy the conditions from Theorem* 3.1 *and suppose that $\sigma(\tilde{A}) \subset \mathcal{C}^+$. Consider the sequence*

$$A_0 = W, \quad A_{n+1} = A_n + \frac{W}{n+2}(I - AA_n) \tag{3.15}$$

Then $\lim\limits_{n\to\infty} A_n = G \in A\{2\}$. Moreover, there exist some constant c and the norm $\|\cdot\|_*$ such that the following holds:

$$\frac{\|A_n - G\|_*}{\|G\|_*} \leq c(n+2)^{-m} + O(\epsilon), \tag{3.16}$$

for n sufficiently large, where $m = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(\tilde{A})\}$.

Proof. It follows from Theorem 2.2 and Lemma 3.1 that $\lim\limits_{n\to\infty} p_n(\tilde{A})W = G \in A\{2\}$, where $\tilde{A} = (WA)_{\mathcal{R}(W)}$. In order to phrase this result in a form which is more convenient for computation we note that $p_0(z) = 1$ and

$$\begin{aligned} p_{n+1}(z) &= p_n(z) + \frac{1}{n+2}\prod_{l=0}^{n}\left(1 - \frac{z}{l+1}\right) \\ &= p_n(z) + \frac{1}{n+2}\left[1 - zp_n(z)\right]. \end{aligned}$$

Consider now an arbitrary operator $W \in \mathcal{L}(Y, X)$ which satisfies conditions of Theorem 2.1, the corresponding outer generalized inverse inverse $G \in \mathcal{L}(Y, X)$ of $A \in \mathcal{L}(X, Y)$. and the following sequence of operators

$$A_n = p_n(\tilde{A})W.$$

Let $z = x + iy \in \sigma(\tilde{A})$. Then

$$0 < m \leq \operatorname{Re} z \leq |z| \leq \|WA\|.$$

We take $L = [\|WA\|]$. Then for $l \geq L$ we have

$$\left|1 - \frac{z}{l+1}\right|^2 \leq e^{-(l+1)^{-1}2x} e^{(l+1)^{-2}(x^2+y^2)},$$

implying

$$\left|1 - \frac{z}{l+1}\right| \leq e^{-x(l+1)^{-1}} e^{2^{-1}(l+1)^{-1}(x^2+y^2)}.$$

Let us denote $S = \sum\limits_{n=0}^{\infty} n^{-2}$. Then we get

$$\prod_{l=L}^{n} \left|1 - \frac{z}{l+1}\right| \leq e^{-x \sum_{l=L}^{n}(l+1)^{-1}} e^{2^{-1}(x^2+y^2)\sum_{l=L}^{n}(l+1)^{-2}}.$$

Since

$$\sum_{l=L}^{n} \frac{1}{l+1} \geq \int_{L+1}^{n+2} \frac{dt}{t} = \ln(n+2) - \ln(L+1),$$

we obtain

$$\prod_{l=L}^{n} \left|1 - \frac{z}{l+1}\right| \leq (n+2)^{-x}(L+1)^x e^{2^{-1}|z|^2 S} \leq (n+2)^{-x}(1+\|WA\|)^x e^{2^{-1}\|WA\|^2 S}.$$

Let c denote

$$c = \max_{z=x+iy\in\sigma(\tilde{A})} (1+\|WA\|)^x \prod_{l=0}^{L-1} \left|1 - \frac{z}{l+1}\right|.$$

Then we get

$$|1 - zp_n(z)| \leq c(n+2)^{-x} \leq c(n+2)^{-m}.$$

Now (3.16) follows from Theorem 2.2. $\square$

Remark 3.3 *An iterative method for the approximation of the Moore-Penrose inverse, which is based on the interpolation of the function* $1/x$ *and Theorem* 1.1 *is derived in* [10]*. Also, in the partial case* $\mathcal{R}(W) = T$, $\mathcal{N}(W) = S$ *we get corresponding iterative method for computing the generalized inverse* $A^{(2)}_{T,S}$*, which is introduced in* [30].

(e) We now take the natural step of approximating the generalized inverse $G \in A\{2\}$ by the Hermite interpolation [12] of the function $1/z$ and its asymptotic error bound.

We seek the unique Hermite interpolation polynomial $q_n(z)$ of degree $2n+1$ which satisfies

$$q_n(i) = 1/i, \quad q_n'(i) = -1/i^2, \quad (i = 1, 2, \ldots, n+1).$$

The Hermite interpolation formula yields the representation

$$q_n(z) = \sum_{i=0}^{n} [2(i+1) - z] \prod_{l=1}^{i} \left(\frac{1-z}{1+l} \right)^2.$$

Here the product from 1 to 0 is, by convention, taken to be 1. An easy inductive argument gives

$$1 - zq_n(z) = \prod_{l=0}^{n} \left(1 - \frac{z}{l+1} \right)^2. \tag{3.17}$$

We obtain the following result.

Theorem 3.5 *Let A and W be the same as in Theorem* 3.1*, such that $\sigma(\tilde{A}) \subset \mathcal{C}^+$. Consider the sequence*

$$A_0 = (2I - WA)W, \; A_{n+1} = A_n + \frac{1}{n+2}\left(2I - \frac{1}{n+2}WA\right) W(I - AA_n). \tag{3.18}$$

Then $\lim_{n\to\infty} A_n = G \in A\{2\}$.

Moreover, there exist the same constant c as in Theorem 3.4 *and the norm $\|\cdot\|_*$ such that the following holds:*

$$\frac{\|A_n - G\|_*}{\|G\|_*} \le c(n+2)^{-m} + O(\epsilon) \tag{3.19}$$

for sufficiently large n, where $m = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(\tilde{A})\}$.

Proof. It follows from Theorem 2.2 and Lemma 3.1 that

$$G = \lim_{n\to\infty} q_n(\tilde{A})W \in A\{2\},$$

where $\tilde{A} = (WA)|_{\mathcal{R}(W)}$. One can verify the following

$$\begin{aligned} q_0(z) &= 2 - z \\ q_{n+1}(z) &= q_n(z) + [2(n+2) - z] \prod_{l=1}^{n+1} \left(\frac{l-z}{l+1} \right)^2 \\ &= q_n(z) + [2(n+2) - z] \prod_{l=0}^{n} \left(\frac{1+l-z}{1+l} \right)^2 \frac{1}{(n+2)^2}. \end{aligned}$$

Using (3.17) we have

$$q_{n+1}(z) = q_n(z) + \frac{1}{n+2}\left(2 - \frac{z}{n+2}\right)[1 - zq_n(z)]$$

An application of Theorem 2.2 gives $G = \lim_{n\to\infty} q_n(\tilde{A})W \in A\{2\}$. Using

$$A_n = q_n(\tilde{A})W = q_n(WA)W$$

we obtain iterations (3.18) and $\lim_{n\to\infty} A_n = G$. The norm estimate follows in the same way as in Theorem 3.4. □

(f) Now we shall consider the hyper power method. Let $p \geq 1$ be an arbitrary integer.

Theorem 3.6 *Let A and W satisfy the same conditions as in Theorem 2.1 and suppose that $\sigma(WA) \setminus \{0\} \subset \mathcal{C}^+$. Let $m = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(WA) \setminus \{0\}\}$, $\rho = \rho(WA)$ and $0 < \alpha < \frac{2m}{\rho^2+m^2}$. Consider the sequence*

$$A_0 = \alpha W, \quad A_{n+1} = A_n \sum_{k=0}^{p-1} (I - AA_n)^k, \ n \geq 0. \tag{3.20}$$

Then

$$\lim_{n\to\infty} A_n = G \in A\{2\}.$$

Moreover, there exists some β, $0 < \beta < 1$, such that for any $\epsilon > 0$ there exists the operator norm $\|\cdot\|_*$ on $\mathcal{L}(Y, X)$ such that

$$\frac{\|A_n W - G\|_*}{\|G\|_*} \leq \beta^{p^n} + O(\epsilon).$$

If the spectrum of WA is real and nonnegative, then α can be chosen as $0 < \alpha < 2/\|WA\|$.

Proof. Define the sequence $(S_n(z))_n$ in the following way:

$$S_0(z) = \alpha, \quad S_{n+1}(z) = S_n(z) \sum_{k=0}^{p-1} (1 - zs_n(z))^k, \quad n \geq 0.$$

Then $(S_n(z))_n$ is a sequence of compelx analytic functions satisfying

$$|zS_{n+1}(z) - 1| = |zS_n(z) - 1|^p$$

and consequently

$$|zS_n(z) - 1| = |\alpha z - 1|^{p^n}.$$

Using the proof of Theorem 3.1 we get that $\lim_{n\to\infty} S_n(z) = 1/z$ uniformly on compact subsets of the set E_α, where E_α is the same as in Theorem 3.1. Hence, $\lim_{n\to\infty} S_n(\tilde{A}) = \tilde{A}^{-1}$. Using Corollary 2.1 we get that $S_n(\tilde{A})W = S_n(WA)W$. Consequently, $\lim_{n\to\infty} A_n = G \in A\{2\}$. The error esitmate follows in the same way as in Theorem 3.1. □

Remark 3.4 *Similar results for refelexive generalized inverses of complex matrices are obtained in [19] and [14].*

4 Numerical results

In this section several illustrative numerical examples are presented.

Example 4.1 *Consider the matrix*

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 5 & 6 \end{bmatrix}$$

of rank 3, and choose the matrix

$$W = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 2 & -1 \end{bmatrix}$$

of rank 2*. Since* $\operatorname{rank}(WAW) = \operatorname{rank}(W) = 2 < \operatorname{rank}(A)$*, conditions of Corollary* 2.3 *are satisfied. We generate the following approximations of outer generalized inverses of* A *corresponding to* W*. By* $x0$ *and* $x1$ *we denote two successive approximations of the outer generalized inverse of* A*.*

Applying the method from Section 3 (a) with $\alpha = 2/[\|WA\| * 100]$*, we get the following outer generalized inverse after 2871 iterations with the precision* $\|x1 - x0\| = 0.4531203356671671 \times 10^{-7}$ *and* $\|x1.a.x1 - x1\| = 0.9981538491324644 \times 10^{-5}$

$$\begin{bmatrix} 1.001646030447 \times 10^{-6} & 0.052631282753692 & 1.001646030447 \times 10^{-6} & 0.052631282753692 \\ -0.99999437232006 & 0.47368254642908 & -0.99999437232006 & 0.47368254642908 \\ 0.9999963756120 & -0.36841998092172 & 0.9999963756120 & -0.36841998092172 \end{bmatrix}.$$

An application of the method from Section 3 (b) gives us the following result after 13 iterative steps with the precision $\|x1.a.x1 - x1\| = 2.08301031776417 \times 10^{-14}$ *and* $\|x1 - x0\| = 2.08459513025796 \times 10^{-14}$ *after 14 iterative steps :*

$$\begin{bmatrix} -1.619079761754 \times 10^{-15} & 0.05263157894736889 & -6.24611582074 \times 10^{-15} & 0.052631578947370625 \\ -0.9999999999999929 & 0.4736842105263136 & -0.9999999999999724 & 0.47368421052630616 \\ 0.999999999999994 & -0.368421052631577 & 0.9999999999999768 & -0.36842105263157077 \end{bmatrix}.$$

The result of the method from Section 3 (d) is the following outer generalized inverse of A

$$\begin{bmatrix} -9.327375025388 \times 10^{-7} & 0.052631855277063 & -9.327375025388 \times 10^{-7} & 0.052631855277063 \\ -1.00000525026419 & 0.4736857647560667 & -1.00000525026419 & 0.4736857647560667 \\ 1.0000033847891943 & -0.36842205420194 & 1.0000033847891943 & -0.36842205420194 \end{bmatrix},$$

with the precision $\|x1.a.x1 - x1\| = 0.931529764160 \times 10^{-5}$ *and* $\|x1 - x0\| = 7.2087836129227 \times 10^{-7}$ *after 46 iterations.*
The result of the method from Section 3 (e) is the following outer generalized inverse of A

$$\begin{bmatrix} 8.1321254458607 \times 10^{-10} & 0.05263157870672663 & 8.13212544586 \times 10^{-10} & 0.05263157870672663 \\ -0.9999999954278058 & 0.4736842091737521 & -0.9999999954278058 & 0.473684209173752 \\ 0.9999999970542309 & -0.3684210517602989 & 0.9999999970542309 & -0.3684210517602989 \end{bmatrix},$$

with the precision $\|x1.a.x1 - x1\| = 8.1105234276314 \times 10^{-9}$ *and* $\|x1 - x0\| = 3.633031520066 \times 10^{-9}$ *after 19 iterations.*
Finally, the method from Section 3 (f) produces the following outer generalized inverse of A

$$\begin{bmatrix} -2.23454067749\times10^{-16} & 0.05263157894736856 & -2.23454067749\times10^{-16} & 0.05263157894736856 \\ -0.999999999999998 & 0.4736842105263149 & -0.999999999999998 & 0.4736842105263149 \\ 0.9999999999999984 & -0.368421052631578 & 0.9999999999999984 & -0.368421052631578 \end{bmatrix},$$

generated with the precision $\|x1.a.x1 - x1\| = 1.325796832150478 \times 10^{-15}$ *and* $\|x1 - x0\| = 2.20212991054386 \times 10^{-15}$ *after 4 iterations.*

Example 4.2 *Consider now the singular M-matrix from* [28]*:*

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

Let us choose the following matrix W:

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

One can verify that $\mathrm{rank}(W) = 2 = \mathrm{rank}(WAW) < \mathrm{rank}(A) = 5$. *Applying the method from Section 3 (a) with* $\alpha = 2/[\|WA\| * 100]$ *we obtain the following approximation of outer generalized inverse of A with the the precision* $\|x1.a.x1 - x1\| = 9.96295860900683 \times 10^{-6}$ *and* $\|x1 - x0\| = 8.201745924803316 \times 10^{-8}$*, after 1446 iterations:*

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ -0.499996477537388 & 0.499996477537388 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.499996477537388 & -0.499996477537388 & -0.499996477537388 & 0.499996477537388 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0.499996477537399 & -0.499996477537399 & 0. & 0. & 0. & 0. \end{bmatrix}.$$

Also, it is an exercise to verify that $\mathrm{ind}(A) = 2$. *Now, using* $W = A^2$ *and* $\alpha = 2/[\|WA\| * 100]$, *the method from Section 3 (a) produces the following approximation of the Drazin inverse from* [27] *after 20705 iterative step, with the precision*

$\|x1.a.x1 - x1\| = 0.9998751461651 \times 10^{-5}$ *and* $\|x1 - x0\| = 0.57285891017949 \times 10^{-8}$*:*

$$\begin{bmatrix} 0.249999999999996 & -0.249999999999996 & 0. & 0. & 0. & 0. \\ -0.249999999999996 & 0.249999999999996 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0.249999999999996 & -0.2499999738304 & 0. & 0. \\ 0. & 0. & -0.249999999999996 & 0.2499999738304 & 0. & 0. \\ 0. & 0. & -0.416663131549148 & -0.583329798215805 & 0.666663131549191 & 0.333329798215859 \\ 0. & 0. & -0.583329798215805 & -0.416663131549148 & 0.333329798215859 & 0.666663131549191 \end{bmatrix}.$$

Using the method from Section 3 (b) with $\alpha = 2/[\|WA\| * 100]$ *we obtain the following approximation of outer inverse of* A *after 12 iterations and with the precision* $\|x1.a.x1 - x1\| = 3.611212654972683 \times 10^{-15}$, $\|x1 - x0\| = 7.219341585552812 \times 10^{-8}$*:*

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ -0.49999999999999 & 0.49999999999999 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.49999999999999 & -0.49999999999999 & -0.49999999999999 & 0.49999999999999 & 0. & 0 \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0.49999999999999 & -0.49999999999999 & 0. & 0. & 0. & 0. \end{bmatrix}.$$

Using $W = A^2$*, the same method produces well known approximation of the Drazin inverse from* [27] *after 27 iterations with the precision* $\|x1.a.x1 - x1\| = 0.$ *and* $\|x1 - x0\| = 1.110223024625156 \times 10^{-16}$*:*

$$\begin{bmatrix} 0.25 & -0.25 & 0. & 0. & 0. & 0. \\ -0.25 & 0.25 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0.25 & -0.25 & 0. & 0. \\ 0. & 0. & -0.25 & 0.25 & 0. & 0. \\ 0. & 0. & -0.4166666666667 & -0.5833333333333 & 0.6666666666667 & 0.3333333333333 \\ 0. & 0. & -0.5833333333333 & -0.4166666666667 & 0.3333333333333 & 0.6666666666667 \end{bmatrix}.$$

Using the method from Section 3 (d) we obtain the following approximation of outer generalized inverse of A *after 2 iterations and with the precision* $\|x1.a.x1 - x1\| = \|x1 - x0\| = 0.$*:*

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ -0.5 & 0.5 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.5 & -0.5 & -0.5 & 0.5 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0.5 & -0.5 & 0. & 0. & 0. & 0. \end{bmatrix}.$$

Using $W = A^2$ *and the method from Section 3 (d) we obtain well known approximation of the Drazin inverse from [27] after 27 iterations with the precision*

$\|x1.a.x1 - x1\| = 0.$ *and* $\|x1 - x0\| = 1.110223024625156 \times 10^{-16}$*:*

$$\begin{bmatrix} 0.25 & -0.25 & 0. & 0. & 0. & 0. \\ -0.25 & 0.25 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0.25 & -0.25 & 0. & 0. \\ 0. & 0. & -0.25 & 0.25 & 0. & 0. \\ 0. & 0. & -0.4166666666667 & -0.5833333333333 & 0.6666666666667 & 0.3333333333333 \\ 0. & 0. & -0.5833333333333 & -0.4166666666667 & 0.3333333333333 & 0.6666666666667 \end{bmatrix}.$$

Using the method from Section 3 (e) we get the exact outer generalized inverse corresponding to A and W after two iterative steps:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the exact Drazin inverse of A:

$$\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{5}{12} & -\frac{7}{12} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{7}{12} & -\frac{5}{12} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Finally, the method from Section s (f) gives the solution

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ -0.5 & 0.5 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ -0.5 & -0.5 & -0.5 & 0.5 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0.5 & -0.5 & 0. & 0. & 0. & 0. \end{bmatrix}$$

immediately after the first iterative step with the precision $\|x1.a.x1 - x1\| = \|x1 - x0\| = 0.$

In the case $W = A^2$ the Drazin inverse of A is produced after 7 iterations, with the precision $\|x1.a.x1 - x1 = 0.$ *and* $\|x1 - x0\| = 2.473830024735911 \times 10^{-15}$

$$\begin{bmatrix} 0.25 & -0.25 & 0. & 0. & 0. & 0. \\ -0.25 & 0.25 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0.25 & -0.25 & 0. & 0. \\ 0. & 0. & -0.25 & 0.25 & 0. & 0. \\ 0. & 0. & -0.41666666666667 & -0.58333333333333 & 0.66666666666667 & 0.33333333333333 \\ 0. & 0. & -0.58333333333333 & -0.41666666666667 & 0.33333333333333 & 0.66666666666667 \end{bmatrix}$$

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