Iterative methods for computing generalized inverses

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Abstract

In this article we extend the method of Yong-Lin Chen (*Iterative methods for computing the generalized inverses* $A^{(2)}_{T,S}$ *of a matrix* A, Appl. Math. Comput. **75** (2-3) (1996)) to inifinite dimensional settings. Precisely, we construct an iterative method for computing outer generalized inverses of operators on Banach spaces, and also for computing the generalized Drazin inverse of Banach algebra elements.

1 Introduction

In [4] the following iterative method

$$X_{k+1} = X_k + \beta Y(I - AX_k), \quad k = 0, 1, 2, \dots, \quad \beta \in \mathbb{C} \setminus \{0\}, \tag{1.1}$$

is used to compute the generalized inverse $A^{(2)}_{T,S}$ of a given matrix A. Some additional condition for Y and X_0 are assumed.

Recently, outer generalized inverses with prescribed range and null-space are extended to Banach space operators (see [5]). Our purpose is to extend the method (1.1) to outer generalized inverses of Banach space operators. Some technical difficulties forces us to prove separately the result for the generalized Drazin inverse of Banach algebra elements.

Key words and phrases: Outer generalized inverses, prescribed idempotents, Drazin inverse.

2000 *Mathematics Subject Classification*: 46H05, 47A05.

[1]Supported by grant no. 144003 of the Ministry of Science, Republic of Serbia.

For $K \subset \mathbb{C}$ we use $\operatorname{acc} K$ to denote the set of all accumulation points of K. If $z \in \mathbb{C}$ and $r > 0$, then $K(z, r) = \{w \in \mathbb{C} : |z - w| < r\}$.

Let $\mathcal{A}$ denote a complex Banach algebra with the unit 1. Recall that the generalized Drazin inverse of $a \in \mathcal{A}$ is denoted by a^d, and it is the unique element of $\mathcal{A}$ satisfying (see [7]):

$$a^d a a^d = a^d, \quad a a^d = a^d a, \quad a(1 - a a^d) \text{ is quasinilpotent.}$$

Moreover, a^d exists if and only if 0 is not the accumulation point of the spectrum of a, denoted by $\sigma(a)$. If $a(1 - aa^d)$ is nilpotent, then the generalized Drazin inverse reduces to the ordinary Drazin inverse.

Let X, Y be Banach spaces, $\mathcal{L}(X, Y)$ be the set of all bounded operators from X to Y, and let T and S be closed subspaces of X and Y respectively. For $A \in \mathcal{L}(X, Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the range and the null-space of A. Suppose that there exists an operator $B \in \mathcal{L}(Y, X)$, such that $BAB = B$, $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$. Then B is usually denoted by $A^{(2)}_{T,S}$. It is easy to see that for given A, T and S as above, there exists the $A^{(2)}_{T,S}$ if and only if: the restriction $A|_T : T \to A(T)$ is invertible, $A(T)$ is closed, $A(T) \oplus S = Y$ and T is complemented in X. In this case $A^{(2)}_{T,S}$ is unique.

As a special case, the (ordinary and generalized) Drazin inverse of $A \in \mathcal{L}(X)$ can be represented as an $A^{(2)}_{T,S}$ inverse, for a special choice of subspaces T and S. Precisely, if $0 \notin \operatorname{acc} \sigma(A)$ and P is the spectral idempotent of A corresponding to $\{0\}$, then $\mathcal{R}(A^d) = \mathcal{N}(P)$ and $\mathcal{N}(A^d) = \mathcal{N}(P)$. Hence, $A^d = A^{(2)}_{\mathcal{N}(P), \mathcal{R}(P)}$.

We assume that the reader is familiar with the basic knowledge of generalized inverses (see at least one of [1, 2, 3, 6]), as well as with the properties of the generalized Drazin inverse (see [7]).

2 Outer generalized inverses of Banach space operators

If T and S are subspaces such that $A^{(2)}_{T,S}$ exists, take $T_1 = \mathcal{N}(I - A^{(2)}_{T,S} A)$ and then $T \oplus T_1 = X$ holds. It is easy to verify that A has the following matrix decomposition:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} A(T) \\ S \end{bmatrix},$$

where $A_1 = A|_T : T \to A(T)$ is invertible. Then

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

We state the result for operators on Banach spaces.

Theorem 2.1. *Let $A \in \mathcal{L}(X, Y)$ and T, S be given such that there exists $A^{(2)}_{T,S} = A' \in \mathcal{L}(Y, X)$. Let $Y \in \mathcal{L}(Y, X)$ have the properties $\mathcal{R}(Y) \subset T$ and $\mathcal{N}(Y) \supset S$.*

Define the sequence $(X_k)_k$ in $\mathcal{L}(Y,X)$ in the following way:

$$X_{k+1} = X_k + \beta Y(I - AX_k), \quad k = 0,1,2,\dots, \quad \beta \in \mathbb{C} \setminus \{0\},$$

where $X_0 \in \mathcal{L}(Y,X)$ satisfies $\mathcal{R}(X_0) \subset T$. Consider the following statements:
(a) $X_k \to A'$;
(b) $(P - \beta YA)^k \to 0$, *where P is the projection of X onto T parallel to T_1;*
(c) $(I_T - \beta YA|_T)^n \to 0$.

Then (b) $\iff$ (c) $\implies$ (a). If $X_0 - A'$ is onto (i.e. it is not the right topological divisor of 0), then (a) $\iff$ (b).

If (b) or (c) is satisfied, then the following hold:
(1) $\sigma(P - \beta YA) = \sigma(I_T - \beta YA|_T) \cup \{0\} \subset K(0,1)$;
(2) $I - P + \beta YA$ *is invertible;*
(3) $A' = \beta(I - P + \beta YA)^{-1}Y$.

Proof. From $\mathcal{R}(Y) \subset T$ and $\mathcal{N}(Y) \supset S$ it follows that Y has the following matrix form:

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

Let

$$P = A'A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix},$$

$$Q = AA' = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to \begin{bmatrix} A(T) \\ S \end{bmatrix}.$$

Notice that $PY = Y = YQ$. Since $\mathcal{R}(X_0) \subset T$ we get $PX_0 = X_0$. By induction on k it is easy to verify $PX_k = X_k$ for all $k = 0,1,2,\dots$. Also, $PA' = A'$. Now we have

$$\begin{aligned} X_{k+1} - A' &= X_k - A' + \beta Y(I - AX_k) \\ &= P(X_k - A') + \beta Y(Q - QAX_k) \\ &= P(X_k - A') + \beta Y(AA'AA' - AA'AX_k) \\ &= P(X_k - A') + \beta Y(AA')A(A' - X_k) \\ &= (P - \beta YA)(X_k - A') \\ &= (P - \beta YA)^{k+1}(X_0 - A'). \end{aligned}$$

Obviously, if $(P - \beta YA)^k \to 0$, then $X_k \to A'$, so (b) $\implies$ (a) is proved.

Notice that

$$P - \beta YA = \begin{bmatrix} I_T - \beta Y_1 A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

hence

$$(P - \beta YA)^n = \begin{bmatrix} (I_T - \beta Y_1 A_1)^n & 0 \\ 0 & 0 \end{bmatrix}.$$

Since

$$(P - \beta YA)^n|_T = (I_T - \beta Y_1 A_1)^n = (I_T - \beta YA|_T)^n,$$

we conclude that $(P-\beta YA)^n \to 0$ if and only if $(I_T - \beta Y_1 A_1)^n \to 0$, which proves (b) $\Longleftrightarrow$ (c).

If X_0 is taken such that $X_0 - A'$ is not the right topological divisor of 0, then $(a) \Longleftrightarrow (b)$ obviously holds.

Since
$$\sigma(P-\beta YA) = \sigma(I_T - \beta Y_1 A_1) \cup \{0\} \subset K(0,1),$$
we get $\sigma(\beta Y_1 A_1) \subset K(1,1)$, so $\beta Y_1 A_1$ is invertible. Moreover, $\beta \neq 0$ and A_1 is invertible. Consequently, Y_1 is invertible. Now,
$$I - P + \beta YA = \begin{bmatrix} \beta Y_1 A_1 & 0 \\ 0 & I_{T_1} \end{bmatrix}$$
is invertible and
$$(I-P+\beta YA)^{-1} = \begin{bmatrix} \frac{1}{\beta} A_1^{-1} Y_1^{-1} & 0 \\ 0 & I_{T_1} \end{bmatrix}.$$
It is easy to verify
$$\beta(I-P+\beta YA)^{-1} Y = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = A'.$$

□

3 The generalized Drazin inverse of Banach algebra elements

In this section we prove the analogous result for the generalized Drazin inverse in Banach algebras. Let $a \in \mathcal{A}$ such that a^d exist. Denote by p the spectral idempotent of a corresponding to $\{0\}$. Then $p = 1 - aa^d$, $a(1-p)$ is invertible in the algebra $(1-p)\mathcal{A}(1-p)$ and $a^d = [a(1-p)]^{-1}_{(1-p)\mathcal{A}(1-p)}$.

We need the following auxiliary result.

Lemma 3.1. *If $a, p \in \mathcal{A}$, such that $p^2 = p$ and $ap = pa$. Then a is invertible in $\mathcal{A}$ if and only if ap is invertible in $p\mathcal{A}p$ and $a(1-p)$ is invertible in $(1-p)\mathcal{A}(1-p)$. In this case*
$$a^{-1} = [ap]^{-1}_{p\mathcal{A}p} + [a(1-p)]^{-1}_{(1-p)\mathcal{A}(1-p)}.$$

Now, we prove the convergence of the iterative method (1.1) for computing the generalized Drazin inverse in Banach algebras.

Theorem 3.1. *Let $a \in \mathcal{A}$, $0 \notin \operatorname{acc} \sigma(a)$, $x_0, y \in \mathcal{A}$, and let p be the spectral idempotent of a corresponding to $\{0\}$. Let $(1-p)x_0 = x_0$, $(1-p)y = y(1-p) = y$, $\beta \in \mathbb{C} \setminus \{0\}$. Define the sequence $(x_k)_k$ in $\mathcal{A}$ in the following way:*
$$x_{k+1} = x_k + \beta y(1 - ax_k), \quad k = 0, 1, 2, \ldots$$

If $(1-p-\beta ya)^k \to 0$*, then* $x_k \to a^d$ *(and the opposite implication holds if* x_0-a^d *is not the right topological divisor of* 0*). In this case* $\sigma(1-p-\beta ya) \subset K(0,1)$*,* $p+\beta ya$ *is invertible and*

$$a^d = \beta(p+\beta ya)^{-1}y.$$

Proof. Recall that $1-p = aa^d$. Form $aa^d x_0 = x_0$ and $aa^d y = y$, by induction on k we prove $aa^d x_k = x_k$ for all $k = 0, 1, 2, \ldots$. Now we compute

$$\begin{aligned} x_{k+1} - a^d &= x_k - a^d + \beta y(aa^d)(1-ax_k) \\ &= aa^d(x_k - a^d) + \beta y(aa^d aa^d - aa^d a x_k) \\ &= aa^d(x_k - a^d) + \beta y(aa^d)a(a^d - x_k) \\ &= (aa^d - \beta ya)(x_k - a^d) \\ &= (aa^d - \beta ya)^{k+1}(x_0 - a^d). \end{aligned}$$

Obviously, if $(aa^d - \beta ya)^{k+1} \to 0$, then $x_k \to a^d$. In this case $\sigma(aa^d - \beta ya) \subset K(0,1)$. By the spectral mapping theorem we get $\sigma(1 - aa^d + \beta ya) \subset K(1,1)$, hence $1 - aa^d + \beta ya = p + \beta ya$ is invertible.

Notice that aa^d commutes with $1 - aa^d + \beta ya$. Using Lemma 3.1 and $y(1 - aa^d) = 0$, we compute

$$\begin{aligned} (1 - aa^d + \beta ya)^{-1} &= (1 - aa^d + \beta ya)^{-1}aa^d + (1 - aa^d + \beta ya)^{-1}(1 - aa^d) \\ &= [(1 - aa^d + \beta ya)(aa^d)]^{-1}_{aa^d\mathcal{A}aa^d} + [(1 - aa^d + \beta ya)(1 - aa^d)]^{-1}_{(1-aa^d)\mathcal{A}(1-aa^d)} \\ &= [\beta ya]^{-1}_{aa^d\mathcal{A}aa^d} + 1 - aa^d. \end{aligned}$$

Now

$$\begin{aligned} &\Big[(aa^d)\beta[(\beta ya)^{-1}_{aa^d\mathcal{A}aa^d} + 1 - aa^d]y(aa^d)\Big]\Big[(aa^d)a(aa^d)\Big] \\ &\quad = (aa^d)(\beta ya)^{-1}_{aa^d\mathcal{A}aa^d}(\beta ya)(aa^d) = aa^d. \end{aligned}$$

We have just proved that $(aa^d)\beta(1 - aa^d + \beta ya)^{-1}y(aa^d)$ is the inverse of $a(aa^d)$ in the algebra $aa^d\mathcal{A}aa^d$. Since aa^d commutes with $1 - aa^d + \beta ya$, and $aa^d y = yaa^d = y$, we obtain that $(1 - aa^d + \beta ya)^{-1}y \in aa^d\mathcal{A}aa^d$. It follows that

$$a^d = \beta(p+\beta ya)^{-1}y.$$

holds.

$\square$

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