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CHARACTERIZING HERMITIAN, NORMAL AND EP OPERATORS

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Abstract

In this paper further characterizations of Hermitian, normal and EP operators on Hilbert spaces are established. Thus the recent results of O. M. Baksalary and G. Trenkler (Linear Multilin. Algebra, to appear) are extended to the infinite dimensional setting with proofs based on operator matrices.

1 Introduction and preliminaries

In this paper we use H to denote arbitrary Hilbert space and $\mathcal{L}(H)$ the space of all bounded linear operators on H. For $A \in \mathcal{L}(H)$ let $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, denote the range and the null space of A. Recall that A is Hermitian if $A = A^*$, and A is normal if $AA^* = A^*A$. The Moore–Penrose

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inverse of a closed range operator A is the unique operator $A^\dagger \in \mathcal{L}(H)$ satisfying the Penrose equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (A^\dagger A)^* = A^\dagger A, \quad (AA^\dagger)^* = AA^\dagger.$$

The operator A is EP if $AA^\dagger = A^\dagger A$; EP stands for 'equal projection' as in this case $\mathcal{R}(A^*) = \mathcal{R}(A)$. It is well known that if A is normal with closed range, then A is EP. The converse is not true even in a finite dimensional space.

There are many characterization of Hermitian, normal and EP operators (see, for example [1, 2, 5, 7, 8, 9, 11, 12, 14, 16, 17, 18, 19]. In this note we extend the results obtained for complex matrices by O. M. Baksalary and G. Trenkler [1] to closed range operators on an arbitrary Hilbert space.

We assume a basic familiarity with the properties of generalized inverses, as in [3, 4, 6, 13]. In particular we recall that the group inverse of $A \in \mathcal{L}(H)$ is the unique operator $A^\# \in \mathcal{L}(H)$ such that

$$AA^\# = A^\# A, \quad AA^\# A = A, \quad A^\# AA^\# = A^\#.$$

The *ascent* and *descent* of $A \in \mathcal{L}(H)$ are defined by

$$\operatorname{asc} A = \inf\{p : \mathcal{N}(A^p) = \mathcal{N}(A^{p+1})\}, \quad \operatorname{dsc} A = \inf\{p : \mathcal{R}(A^p) = \mathcal{R}(A^{p+1})\};$$

if they are finite, they are equal, and their common value is $\operatorname{ind}(A)$, the index of A. An operator $A \in \mathcal{L}(H)$ is group invertible if and only if $\operatorname{ind}(A) \le 1$.

Our techniques are based on properties of operator matrices. We will need the following result, a special case of the result obtained in [10] for a generalized Drazin inverse.

Lemma 1.1. *Let X and Y be Banach spaces, $A_1 \in \mathcal{L}(X)$ and $A_2 \in \mathcal{L}(Y, X)$. Then*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}$$

is group invertible if and only if A_1 is group invertible. In this case

$$A^\# = \begin{bmatrix} {A_1}^\# & ({A_1}^\#)^2 A_2 \\ 0 & 0 \end{bmatrix}. \tag{1.1}$$

Proof. Relies on a direct verification of the equations defining the group inverse. □

In the paper we will make use of the following two operator matrix representations. Recall that an operator $A \in \mathcal{L}(H)$ is *positive*, written $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. We shall write $A > 0$ if A is positive and invertible.

Lemma 1.2. *A closed range operator $A \in \mathcal{L}(H)$ has the following matrix representations with respect to the orthogonal sums of closed subspaces $H = \mathcal{R}(A^*) \oplus \mathcal{N}(A) = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

(a) *We have*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{1.2}$$

where $B = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ onto itself and $B > 0$. Also,

$$A^\dagger = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}.$$

Moreover, if $\operatorname{ind}(A) \leq 1$, *then A_1 is invertible and*

$$A^\# = \begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix}.$$

(b) *Alternatively,*

$$A = \begin{bmatrix} A_3 & 0 \\ A_4 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}, \tag{1.3}$$

where $C = A_3^*A_3 + A_4^*A_4$ maps $\mathcal{R}(A^*)$ onto itself and $C > 0$. Also,

$$A^\dagger = \begin{bmatrix} C^{-1}A_3^* & C^{-1}A_4^* \\ 0 & 0 \end{bmatrix}.$$

Moreover, if $\operatorname{ind}(A) \leq 1$, *then A_3 is invertible and*

$$A^\# = \begin{bmatrix} A_3^{-1} & 0 \\ A_4A_3^{-2} & 0 \end{bmatrix}.$$

Proof. (a) The proof of the matrix form for A is straightforward; B is invertible as it maps $\mathcal{R}(A)$ bijectively onto itself while $\mathcal{R}(A)$ is a Banach space. Since $A^* = \left[\begin{smallmatrix} A_1^* & 0 \\ A_2^* & 0 \end{smallmatrix}\right]$, we get that $AA^* = \left[\begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Now, $(AA^*)^\dagger = (AA^*)^\# = \left[\begin{smallmatrix} B^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. From $A^\dagger = A^*(AA^*)^\dagger$ we obtain the matrix form for $A^\dagger$. If $\operatorname{ind}(A) \leq 1$, we apply Lemma 1.1 to get the expression for $A^\#$. The group inverse of A_1 is in fact equal to A_1^{-1} since $\mathcal{N}(A_1) \cap \mathcal{R}(A_1) = \{0\}$, and A_1 is surjective.

(b) We apply the results of the preceding part of the proof to A^*, and then take the Hilbert space adjoint of the operator matrix. $\square$

We can compare our operator representation (1.2) with the matrix representation due to Hartwig and Spindelböck [15] used as the main tool by Baksalary and Trenkler in [1]:

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$

where U is unitary, Σ is the diagonal matrix of the nonzero singular values of A, and K, L satisfy the condition $KK^* + LL^* = I_r$, with r the rank of A. Then

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*, \quad A^\# = U \begin{bmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ 0 & 0 \end{bmatrix} U^*.$$

The purpose of this paper is to derive infinite dimensional analogues of some results for complex matrices obtained by Cheng and Tian [7], Baksalary and Trenkler [1] and others. Our characterizations of the Hermitian, normal and EP operators are based on the following lemma which follows easily from the representations of Lemma 1.2.

Lemma 1.3. *Let $A \in \mathcal{L}(H)$ have a closed range. Then relative to the representations* (1.2) *and* (1.3) *of the preceding lemma,*

(i) *A is Hermitian if and only if $A_2 = 0$ and A_1 is Hermitian* (*or $A_4 = 0$ and A_3 is Hermitian*).

(ii) *A is normal if and only if $A_2 = 0$ and A_1 is normal* (*or $A_4 = 0$ and A_3 is normal*).

(iii) *A is EP if and only if $A_2 = 0$* (*or $A_4 = 0$*)*. In this case $A_1 = A_3$ and*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}, \quad A^\dagger = A^\# = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

In order to illustrate differences between the finite and the infinite dimensional case, we consider the following example (see also [10]).

Example 1.1. Let $A \in \mathcal{L}(\ell^2)$ be the left shift on the real ℓ^2 space, that is, let $A(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Then $A^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ and $A^\dagger = A^*$. In this case $AA^\dagger = I$ and $A^\dagger A(x_1, x_2, \dots) = (0, x_2, x_3, \dots)$. Let $B = A^*$. Then $B^* = B^\dagger = A$. In this case $(BB^*)^\dagger = BB^*$. Operators A and B are neither normal, nor EP, but they satisfy the following equalities (which in the finite dimensional case would ensure the normality or the EP property):

(i) $AAA^* = AA^*A$;

(ii) $BB^*B = B^*BB$;

(iii) $BB^*B^* = B^*BB^*$.

We observe that the ascent of A and descent of B are infinite.

2 Characterizations of Hermitian operators

We start with a characterization of a Hermitian operator with closed range. First we recall that any closed range Hermitian operator $A \in \mathcal{L}(H)$ is group invertible, that is, $\operatorname{ind}(A) \le 1$. Every closed range Hermitian operator is EP.

The following conditions characterizing a Hermitian operator extend those for a finite dimensional operator provided the range of the operator is closed.

Theorem 2.1. *Let $A \in \mathcal{L}(H)$ be a closed range operator. Then the following are equivalent*:

$$\text{(i) } A \textit{ is Hermitian;} \quad \text{(ii) } AAA^\dagger = A^*; \quad \text{(iii) } AA^*A^\dagger = A.$$

Proof. The conditions (ii) and (iii) hold for a closed range hermitian operator as in this case $A^* = A$ and $AA^\dagger = A^\dagger A$.

(ii) $\Longrightarrow$ (i): From (ii) we obtain that $A^2 = A^*A$. Using the representation (1.2),

$$\begin{bmatrix} A_1^2 & A_1A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*A_1 & A_1^*A_2 \\ A_2^*A_1 & A_2^*A_2 \end{bmatrix}.$$

Hence, $A_2^*A_2 = 0$ and consequently $A_2 = 0$. Since $AA^\dagger = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, from (ii) we obtain $A_1 = A_1^*$, which implies $A = A^*$ by Lemma 1.3 (i).

(iii) $\Longrightarrow$ (i): We use the representation (1.3). From (iii) we obtain $A^\dagger AA^*A^\dagger A = A^\dagger A^2$, which is equivalent to

$$\begin{bmatrix} A_3^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_3 & 0 \\ 0 & 0 \end{bmatrix};$$

hence $A_3^* = A_3$. By (iii) again, $A^\dagger AA^*A^\dagger = A^\dagger A$, which is equivalent to

$$\begin{bmatrix} A_3C^{-1}A_3 & A_3C^{-1}A_4^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $A_3C^{-1}A_3 = I$ and $A_3C^{-1}A_4^* = 0$. The first equation implies that A_3 is invertible, and so $A_4 = 0$ by the second. By Lemma 1.3 (i), A is Hermitian. $\square$

The following theorem assumes that the given operator is of closed range as well as of index not exceeding 1.

Theorem 2.2. *Let* $A \in \mathcal{L}(H)$ *have a closed range. Then* A *is Hermitian if and only if* $\operatorname{ind}(A) \le 1$, *and any one the following equivalent conditions hold*:

$$\begin{array}{rlrl} \text{(i)} & A^*AA^{\#} = A; & \text{(ii)} & A^*A^*A^{\#} = A^*; \\ \text{(iii)} & A^*A^\dagger A^\dagger = A^{\#}; & \text{(iv)} & A^*A^\dagger A^{\#} = A^\dagger; \\ \text{(v)} & A^*A^\dagger A^{\#} = A^{\#}; & \text{(vi)} & A^*A^{\#}A^{\#} = A^{\#}; \\ \text{(vii)} & A^{\#}A^*A^{\#} = A^\dagger. & & \end{array}$$

Proof. If A is Hermitian, then (i)–(vii) follow from the equations $A^* = A$, $AA^\dagger = A^\dagger A$, and $A^{\#} = A^\dagger$.

Conversely we prove that each of equations (i)–(vii) implies that A is Hermitian. According to Lemma 1.3, it is enough to show that $A_2 = 0$ and A_1 is Hermitian (or $A_4 = 0$ and A_3 Hermitian). Recall that A_1 and A_3 are invertible.

(i) Postmultiplying (i) by A we get $A^*A = A^2$; by the representation (1.2),

$$\begin{bmatrix} A_1^*A_1 & A_1^*A_2 \\ A_2^*A_1 & A_2^*A_2 \end{bmatrix} = \begin{bmatrix} A_1^2 & A_1A_2 \\ 0 & 0 \end{bmatrix}.$$

Hence $A_2^*A_2 = 0$ and $A_1^*A_1 = A_1^2$. Then $A_2 = 0$ and $A_1^* = A_1$.

(ii) Using the representation (1.2) we obtain

$$\begin{bmatrix} (A_1^*)^2A_1^{-1} & (A_1^*)^2A_1^{-2}A_2 \\ A_2^* & A_2^*A_1^{-2}A_2 \end{bmatrix} = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix}.$$

Then $(A_1^*)^2A_1^{-1} = A_1^*$ and $(A_1^*)^2A_1^{-2}A_2 = 0$. Hence, $A_2 = 0$ and $A_1^* = A_1$.

(iii) First we observe that (iii) implies $A^*A^\dagger A^\dagger A = A^{\#}A$. Using the representation (1.3), we obtain

$$\begin{bmatrix} A_3^*C^{-1}A_3^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_4A_3^{-1} & 0 \end{bmatrix}.$$

Thus $A_4A_3^{-1} = 0$, that is, $A_4 = 0$, and $A_3^*C^{-1}A_3^* = I$, which implies $A_3^* = A_3$.

(iv) By the representation (1.3),

$$\begin{bmatrix} A_3^*A_3^{-2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C^{-1}A_3^* & C^{-1}A_4^* \\ 0 & 0 \end{bmatrix}.$$

Then $C^{-1}A_4^* = 0$ implies $A_4 = 0$, and $A_3^* A_3^{-2} = C^{-1}A_3^*$ implies $A_3^* = A_3$.

(v) From the representation (1.3) we get

$$\begin{bmatrix} A_3^* A_3^{-2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_3^{-1} & 0 \\ A_4 A_3^{-2} & 0 \end{bmatrix};$$

then $A_4 A_3^{-2} = 0$, which implies $A_4 = 0$, and $A_3^* A_3^{-2} = A_3^{-1}$, which implies $A_1^* = A_1$.

(vi) Using the representation (1.2), we get the equality

$$\begin{bmatrix} A_1^* A_1^{-2} & A_1^* A_1^{-3} A_2 \\ A_2^* A_1^{-2} & A_2^* A_1^{-3} A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-1} & A_1^{-2} A_2 \\ 0 & 0 \end{bmatrix}$$

from which it follows that $A_2^* A_1^{-2} = 0$ and $A_1^* A_1^{-2} = A_1^{-1}$. Hence $A_2 = 0$ and $A_1 = A_1^*$.

(vii) From (vii) we get $AA^{\#} A^* A^{\#} = AA^{\dagger}$. By (1.2) this yields

$$\begin{bmatrix} A_1^{-1} B A_1^{-1} & A_1^* A_1^{-2} A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

From $A_1^* A_1^{-2} A_2 = 0$ we get $A_2 = 0$, and from $A_1^{-1} B A_1^{-1} = I$ we get $A_1^* = A_1$. □

3 Characterizations of normal operators

First we recall that a normal operator A with closed range is EP; hence $A^{\dagger} = A^{\#}$ and $A^{\dagger} A = AA^{\dagger}$. Further, from $\operatorname{ind}(A) \leq 1$ it follows that $\operatorname{asc}(A)$ and $\operatorname{dsc}(A)$ are finite.

Theorem 3.1. *A closed range operator $A \in \mathcal{L}(H)$ is normal if and only if one of the following equivalent conditions holds.*

(i) $\operatorname{asc}(A) < \infty$ *and* $AAA^* = AA^*A$;

(ii) $\operatorname{dsc}(A) < \infty$ *and* $AA^*A = A^*AA$;

(iii) $\operatorname{asc}(A) < \infty$ *and* $AA^*A^{\dagger} = A^{\dagger}AA^*$;

(iv) $AA^*A^{\dagger} = A^*$;

(v) $A^{\dagger}A^*A = A^*$.

Proof. If A is normal, then conditions (i)–(v) clearly hold.

Next we prove that any one of the conditions (i)–(v) implies that A is normal. Recall that by Lemma 1.3 it is enough to show that $A_2 = 0$ and A_1 is normal (or $A_4 = 0$ and A_3 is normal).

(i) First assume that A is surjective. Since $\operatorname{asc}(A) < \infty$ it follows that $\operatorname{asc}(A) = 0$ and A is invertible. Now, from (i) it follows that A is normal. In general the representation (1.2) yields

$$\begin{bmatrix} A_1B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} BA_1 & BA_2 \\ 0 & 0 \end{bmatrix}.$$

Hence, we get that $BA_2 = 0$ and $A_1B = BA_1$. From the first relation we obtain $A_2 = 0$, implying that A_1 is right invertible since $B = A_1A_1^*$ is invertible. Also, $\operatorname{asc}(A) < \infty$ obviously implies $\operatorname{asc}(A_1) < \infty$. From $A_1B = BA_1$ we obtain $A_1A_1A_1^* = A_1A_1^*A_1$. Since A_1 is right invertible, it is surjective, and from the first part of the proof it follows that A_1 is normal. See Example 1.1 (i) to see that the assumption $\operatorname{asc}(A) < \infty$ is crucial.

(ii) This implication is analogous to the previous one. See Example 1.1 (ii) to see the importance of $\operatorname{dsc}(A) < \infty$.

(iii) From (iii) and the representation (1.2) we obtain

$$\begin{bmatrix} BA_1^*B^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix}.$$

Hence, $A_2 = 0$ and $BA_1^*B^{-1} = A_1^*$. Since $B = A_1A_1^*$ is invertible, A_1 is right invertible. From $\operatorname{asc}(A) < \infty$ we get that $\operatorname{asc}(A_1) < \infty$ and consequently A_1 is invertible. Now, from $BA_1^*B^{-1} = A_1^*$ it follows that A_1 is normal. Example 1.1 (iii) shows that the statement (iii) of this theorem does not hold without the assumption $\operatorname{asc}(A) < \infty$.

(iv) In view of the representation (1.2), equation (iv) implies

$$\begin{bmatrix} BA_1^*B^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix}.$$

We get that $A_2 = 0$ and $BA_1^*B^{-1} = A_1^*$, implying that A_1 is normal.

(v) Follows from the previous implication, taking adjoints of operators in (iv). $\square$

The next theorem deals with the case of a closed range operator with index not exceeding 1; this means that the operator is both Moore–Penrose and group invertible; in general, $A^\dagger \neq A^\#$.

Theorem 3.2. *A closed range operator $A \in \mathcal{L}(H)$ is normal if and only if* $\operatorname{ind}(A) \le 1$ *and one of the following equivalent conditions holds.*

(i) $AA^*A^\# = A^*A^\#A$; (ii) $AA^*A^\# = A^\#AA^*$;

(iii) $AA^\#A^* = A^\#A^*A$; (iv) $A^*AA^\# = A^\#A^*A$;

(v) $A^*A^*A^\# = A^*A^\#A^*$; (vi) $A^*A^\dagger A^\# = A^\#A^*A^\dagger$;

(vii) $A^*A^\# = A^\#A^*$; (viii) $A^*A^\#A^* = A^\#A^*A^*$;

(ix) $A^*A^\#A^\dagger = A^\dagger A^*A^\#$; (x) $A^*A^\#A^\# = A^\#A^*A^\#$;

(xi) $A^\dagger A^*A^\# = A^\#A^\dagger A^*$; (xii) $A^\dagger A^\#A^* = A^\#A^*A^\dagger$;

(xiii) $A^\#A^*A^\# = A^\#A^\#A^*$; (xiv) $A^*A^\dagger = A^\#A^*$;

(xv) $A^*A^\# = A^\dagger A^*$; (xvi) $AA^*A^\# = A^*$;

(xvii) $A^\#A^*A = A^*$.

Proof. If A is normal, then conditions (i)–(xvii) hold since A is EP and commutes with A^* and $A^\dagger$, while $A^\dagger = A^\#$.

To prove the reverse implications, we use the representations (1.2) or (1.3). In view of Lemma 1.3 it is enough to prove that $A_2 = 0$ and A_1 is normal (or $A_4 = 0$ and A_3 is normal).

(i) Using the representation (1.2), we see that (i) implies

$$\begin{bmatrix} BA_1^{-1} & BA_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* & A_1^*A_1^{-1}A_2 \\ A_2^* & A_2^*A_1^{-1}A_2 \end{bmatrix}.$$

It follows that $A_2 = 0$ and $BA_1^{-1} = A_1^*$. Hence, A_1 is normal.

(ii) By (1.2),

$$\begin{bmatrix} BA_1^{-1} & BA_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1}A_1^* + A_1^{-2}A_2A_2^* & 0 \\ 0 & 0 \end{bmatrix}.$$

We obtain $BA_1^{-2}A_2 = 0$ and $BA_1^{-1} = A_1^{-1}A_1^* + A_1^{-2}A_2A_2^*$. By the first relation, $A_2 = 0$, and by the second, A_1 is normal.

(iii) When we use the representation (1.3), we get

$$\begin{bmatrix} A_3^* & A_4^* \\ A_4A_3^{-1}A_3^* & A_4A_3^{-1}A_4^* \end{bmatrix} = \begin{bmatrix} A_3^{-1}C & 0 \\ A_4A_3^{-2}C & 0 \end{bmatrix}.$$

We obtain $A_4 = 0$ and $A_3^* = A_3^{-1}C$, implying that A_3 is normal.

(iv) Again, using the representation (1.3), we obtain

$$\begin{bmatrix} CA_3^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_3^{-1}C & 0 \\ A_4A_3^{-1}C & 0 \end{bmatrix}.$$

We obtain $A_4A_3^{-1}C = 0$ and $CA_3^{-1} = A_3^{-1}C$. From the first relation we get $A_4 = 0$ and from the second we conclude that A_3 is normal.

(v) Using the representation (1.3), we get

$$\begin{bmatrix} A_3^*CA_3^{-2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} CA_3^{-2}A_3^* & CA_3^{-2}A_4^* \\ 0 & 0 \end{bmatrix}.$$

Thus $CA_3^{-2}A_4^* = 0$ and $A_3^*CA_3^{-2} = CA_3^{-2}A_3^*$. Consequently, $A_4 = 0$ and A_3 is normal.

(vi) Using again the representation (1.3), we obtain

$$\begin{bmatrix} A_3^*A_3^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_3^{-1}A_3^* & A_3^{-1}A_4^* \\ A_4A_3^{-1}A_3^* & A_4A_3^{-1}A_4^* \end{bmatrix}.$$

Then $A_3^{-1}A_4^* = 0$ and $A_3^*A_3^{-1} = A_3^{-1}A_3^*$, that is, $A_4 = 0$ and A_3 is normal.

(vii) From (1.2) we obtain the equality

$$\begin{bmatrix} A_1^*A_1^{-1} & A_1^*A_1^{-2}A_2 \\ A_2^*A_1^{-1} & A_2^*A_1^{-2}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-2}B & 0 \\ 0 & 0 \end{bmatrix}.$$

From $A_2^*A_1^{-1} = 0$ and $A_1^*A_1^{-1} = A_1^{-2}B$ we conclude that $A_2 = 0$ and A_1 is normal.

(viii) Using (1.2) we get

$$\begin{bmatrix} A_1^*A_1^{-2}B & 0 \\ A_2^*A_1^{-2}B & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-2}BA_1^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $A_2^*A_1^{-2}B = 0$ which implies $A_2 = 0$, and $A_1^*A_1^{-2}B = A_1^{-2}BA_1^*$, which implies that A_1 is normal.

(ix) From (ix) we get the equality

$$\begin{bmatrix} A_1^*A_1^{-2} & 0 \\ A_2^*A_1^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1}A_1^*A_1^{-1} & A_1^*B^{-1}A_1^*A_1^{-2}A_2 \\ A_2^*B^{-1}A_1^*A_1^{-1} & A_2^*B^{-1}A_1^*A_1^{-2}A_2 \end{bmatrix}.$$

It follows that $A_1^*B^{-1}A_1^*A_1^{-2}A_2 = 0$ and $A_1^*A_1^{-2} = A_1^*B^{-1}A_1^*A_1^{-1}$. We deduce that $A_2 = 0$ and A_1 is normal.

(x) Using the representation (1.2), we get the equality

$$\begin{bmatrix} A_1^*A_1^{-2} & A_1^*A_1^{-3}A_2 \\ A_2^*A_1^{-2} & A_2^*A_1^{-3}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-2}BA_1^{-1} & A_1^{-2}BA_1^{-2}A_2 \\ 0 & 0 \end{bmatrix}.$$

Then $A_2^*A_1^{-2} = 0$ and $A_1^*A_1^{-2} = A_1^{-2}BA_1^{-1}$. Therefore $A_2 = 0$ and A_1 is normal.

(xi) From (1.2) we get

$$\begin{bmatrix} A_1^*B^{-1}A_1^*A_1^{-1} & A_1^*B^{-1}A_1^*A_1^{-2}A_2 \\ A_2^*B^{-1}A_1^*A_1^{-1} & A_2^*B^{-1}A_1^*A_1^{-2}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-2}A_1^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $A_2^*B^{-1}A_1^*A_1^{-1} = 0$ and $A_1^*B^{-1}A_1^*A_1^{-1} = A_1^{-2}A_1^*$, which implies that $A_2 = 0$ and A_1 is normal.

(xii) Using (1.2) again we obtain the equality

$$\begin{bmatrix} A_1^*B^{-1}A_1^{-2}B & 0 \\ A_2^*B^{-1}A_1^{-2}B & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-2}BA_1^*B^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $A_2^*B^{-1}A_1^{-2}B = 0$ and $A_1^*B^{-1}A_1^{-2}B = A_1^{-2}BA_1^*B^{-1}$, from which it follows that $A_2 = 0$ and A_1 is normal.

(xiii) The representation (1.2) gives

$$\begin{bmatrix} A_1^{-2}BA_1^{-1} & A_1^{-2}BA_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1{}^{-3}B & 0 \\ 0 & 0 \end{bmatrix}.$$

We obtain $A_1^{-2}BA_1^{-2}A_2 = 0$ and $A_1^{-2}BA_1^{-1} = A_1^{-3}B$, implying $A_2 = 0$ and the normality of A_1.

(xiv) Applying (1.2) again, we have

$$\begin{bmatrix} (A_1^*)^2B^{-1} & 0 \\ A_2^*A_1^*B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-2}B & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $A_2^*A_1^*B^{-1} = 0$ and $(A_1^*)^2B^{-1} = A^{-2}B$. This implies that $A_2 = 0$ and A_1 is normal.

(xv) Follows from the previous implication taking adjoints of operators in (xiv), and using the relations $(A^*)^\dagger = (A^\dagger)^*$ and $(A^\#)^* = (A^*)^\#$.

(xvi) On application of (1.2) we get

$$\begin{bmatrix} BA_1^{-1} & BA_1^{-1}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix}.$$

From $BA_1^{-2}A_2 = 0$ and $BA_1^{-1} = A_1^*$ we obtain that $A_2 = 0$ and A_1 is normal.

(xvii) Follows from the previous implication by taking adjoints of operators in (xvi). $\square$

4 Characterization of EP operators

In the next theorem we deal with closed range operators A of index not exceeding 1. These operators are simultaneously Moore–Penrose and group invertible, but the two inverses are in general different.

Theorem 4.1. *A closed range operator* $A \in \mathcal{L}(H)$ *is EP if and only if* $\operatorname{ind}(A) \leq 1$ *and one of the following equivalent conditions holds.*

(i) $AA^\dagger A^\# = A^\dagger A^\# A$;	(ii) $AA^\dagger A^\# = A^\# AA^\dagger$;
(iii) $AA^\# A^* = A^* AA^\#$;	(iv) $AA^\# A^\dagger = A^\dagger AA^\#$;
(v) $AA^\# A^\dagger = A^\# A^\dagger A$;	(vi) $A^\dagger AA^\# = A^\# A^\dagger A$;
(vii) $(A^\dagger)^2 A^\# = A^\dagger A^\# A^\dagger$;	(viii) $(A^\dagger)^2 A^\# = A^\# (A^\dagger)^2$;
(ix) $A^\dagger A^\# = A^\# A^\dagger$;	(x) $A^\dagger A^\# A^\dagger = A^\# (A^\dagger)^2$;
(xi) $A^\dagger (A^\#)^2 = A^\# A^\dagger A^\#$;	(xii) $A^\dagger (A^\#)^2 = (A^\#)^2 A^\dagger$;
(xiii) $(A^\#)^2 A^\dagger = A^\# A^\dagger A^\#$;	(xiv) $AA^\# = A^\dagger A$;
(xv) $A^* A^\dagger = A^* A^\#$;	(xvi) $A^\dagger A^* = A^\# A^*$;
(xvii) $A^\dagger A^\dagger = A^\dagger A^\#$;	(xviii) $A^\dagger A^\dagger = A^\dagger A^\#$;
(xix) $(A^\dagger)^2 = (A^\#)^2$;	(xx) $A^\dagger A^\# = (A^\#)^2$;
(xxi) $A^\dagger A^\# = (A^\#)^2$;	(xxii) $A(A^\dagger)^2 = A^\#$;
(xxiii) $AA^\# A^\dagger = A^\#$;	(xxiv) $A^* AA^\# = A^*$;
(xxv) $A^\dagger AA^\# = A^\dagger$;	(xxvi) $A^\# A^\dagger A = A^\dagger$.

Proof. If A is EP, then it commutes with its Moore–Penrose inverse and $A^\dagger = A^\#$. It is not difficult to verify that conditions (i)–(xxvi) hold.

To prove any of the reverse implications, we use the representations of Lemma 1.2. By Lemma 1.3 it is enough to prove that $A_2 = 0$ or $A_4 = 0$. For the sake of brevity we display each of the equations (i)–(xxvi) in a matrix form using either the representation (1.2) or (1.3). We let the reader verify that in each case we have $A_1 = 0$ or $A_4 = 0$.

$$\begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* B^{-1} & A_1^* B^{-1} A_1^{-1} A_2 \\ A_2^* B^{-1} & A_2^* B^{-1} A_1^{-1} A_2 \end{bmatrix}, \tag{i}$$

$$\begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{ii}$$

$$\begin{bmatrix} A_1^* + A_1^{-1}A_2A_2^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* & A_1^*A_1^{-1}A_2 \\ A_2^* & A_2^*A_1^{-1}A_2 \end{bmatrix}, \tag{iii}$$

$$\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1} & A_1^*B^{-1}A_1^{-1}A_2 \\ A_2^*B^{-1} & A_2^*B^{-1}A_1^{-1}A_2 \end{bmatrix}, \tag{iv}$$

$$\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix}, \tag{v}$$

$$\begin{bmatrix} A_3^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_3^{-1} & 0 \\ A_3^{-2}A_4 & 0 \end{bmatrix}, \tag{vi}$$

$$\begin{bmatrix} C^{-1}A_3^*A_3^{-2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_3^{-2}C^{-1}A_3^* & A_3^{-2}C^{-1}A_4^* \\ 0 & 0 \end{bmatrix}, \tag{vii}$$

$$\begin{bmatrix} A_1^*B^{-1}A_1^{-1} & A_1^*B^{-1}A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1}A_1^*B^{-1} & 0 \\ 0 & 0 \end{bmatrix} \text{ (premultiply by } A), \tag{viii}$$

$$\begin{bmatrix} A_1^*B^{-1}A_1^{-1} & A_1^*B^{-1}A_1^{-2}A_2 \\ A_2^*B^{-1}A_1^{-1} & A_2^*B^{-1}A_1^{-2}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{bmatrix}, \tag{ix}$$

take the adjoint of (vii), (x)

$$\begin{bmatrix} A_1^*B^{-1}A_1^{-2} & A_1^*B^{-1}A_1^{-3}A_2 \\ A_2^*B^{-1}A_1^{-2} & A_2^*B^{-1}A_1^{-3}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-3} & A_1^{-4}A_2 \\ 0 & 0 \end{bmatrix}, \tag{xi}$$

$$\begin{bmatrix} A_1^*B^{-1}A_1^{-2} & A_1^*B^{-1}A_1^{-3}A_2 \\ A_2^*B^{-1}A_1^{-2} & A_2^*B^{-1}A_1^{-3}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-3} & 0 \\ 0 & 0 \end{bmatrix}, \tag{xii}$$

take the adjoint of (xi), (xiii)

$$\begin{bmatrix} I & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1}A_1 & A_1^*B^{-1}A_2 \\ A_2^*B^{-1}A_1 & A_2^*B^{-1}A_2 \end{bmatrix}, \tag{xiv}$$

$$\begin{bmatrix} (A_1^*)^2B^{-1} & 0 \\ A_2^*A_1^*B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_1^*A_1^{-1} & A_1^*A_1^{-2}A_2 \\ A_2^*A_1^{-1} & A_2^*A_1^{-2}A_2 \end{bmatrix}, \tag{xv}$$

take the adjoint of (xv), (xvi)

$$\begin{bmatrix} A_1^*B^{-1}A_1^*B^{-1} & 0 \\ A_2^*B^{-1}A_1^*B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1}A_1^{-1} & A_1^*B^{-1}A_1^{-2}A_2 \\ A_2^*B^{-1}A_1^{-1} & A_2^*B^{-1}A_1^{-2}A_2 \end{bmatrix}, \tag{xvii}$$

take the adjoint of (xvii), (xviii)

$$\begin{bmatrix} A_1^*B^{-1}A_1^*B^{-1} & 0 \\ A_2^*B^{-1}A_1^*B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-2} & A_1^{-3}A_2 \\ 0 & 0 \end{bmatrix}, \tag{xix}$$

$$\begin{bmatrix} A_1^*B^{-1}A_1^{-1} & A_1^*B^{-1}A_1^{-2}A_2 \\ A_2^*B^{-1}A_1{}^{-1}1 & A_2^*B^{-1}A_1^{-2}A_2 \end{bmatrix} = \begin{bmatrix} A_1^{-2} & A_1^{-3}A_2 \\ 0 & 0 \end{bmatrix}, \tag{xx}$$

$$\text{take the adjoint of (xx)}, \tag{xxi}$$

$$\begin{bmatrix} A_1^*B^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{-1} & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix}, \tag{xxii}$$

$$\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^* & 0 \end{bmatrix}, \tag{xxiii}$$

$$\begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_1^* & A_1^*A_1^{-1}A_2 \\ A_2^* & A_2^*A_1^{-1}A_2 \end{bmatrix}, \tag{xxiv}$$

$$\begin{bmatrix} A_1^*B^{-1} & A_1^*B^{-1}A_1^{-1}A_2 \\ A_2^*B^{-1} & A_2^*B^{-1}A_1^{-1}A_2 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}, \tag{xxv}$$

$$\begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}. \tag{xxvi}$$

This completes the proof. □

In closing we observe that from our characterizations of closed range Hermitian, normal and EP operators on a Hilbert space we recover the results for finite complex matrices obtained by Baksalary and Trenkler [1], Cheng and Tian [7], Hartwig and Spindelböck [15], and many others, as well as some of the results obtained recently in [9] by the first author.

References

[1] O. M. Baksalary and G. Trenkler, *Characterizations of EP, normal and Hermitian matrices*, Linear Multilin. Algebra, to appear.

[2] T. S. Baskett and I. J. Katz, *Theorems on products of EP_r matrices*, Linear Algebra Appl. **2** (1969), 87–103.

[3] A. Ben-Israel and T. N. E. Greville, *Generalized inverses: theory and applications*, 2nd ed., Springer, 2003.

[4] S. R. Caradus, *Generalized inverses and operator theory*, Queen's papers in pure and applied mathematics 50, Queen's University, Kingston, 1978.

[5] S. L. Campbell and C. D. Meyer Jr., *EP operators and generalized inverse*, Canadian Math. Bull. **18** (1975), 327–333.

[6] S. L. Campbell and C. D. Meyer Jr., *Generalized inverses of linear transformations*, reprint, Dover Publications, New York, 1991.

[7] S. Cheng and Y. Tian, *Two sets of new characterizations for normal and EP matrices*, Linear Algebra Appl. **375** (2003), 181–195.

[8] D. S. Djordjević, *Products of EP operators on Hilbert spaces*, Proc. Amer. Math. Soc. **129** (2000), 1727–1731.

[9] D. S. Djordjević,*Characterizations of normal, hyponormal and EP operators*, J. Math. Anal. Appl., to appear.

[10] D. S. Djordjević and P. S. Stanimirović, *Generalized Drazin inverse and generalized resolvent*, Czechoslovak Math. J. **51 (126)** (2001), 617–634.

[11] L. Elsner and K. D. Ikramov, *Normal matrices: an update*, Linear Algebra Appl. **285** (1998), 291–303.

[12] R. Grone, C. R. Johnson, E. M. Sa and H. Wolkowicz, *Normal matrices*, Linear Algebra Appl. **87** (1987), 213–225.

[13] R. E. Harte, *Invertibility and singularity for bounded linear operators*, Marcel Dekker, New York, 1988.

[14] R. E. Hartwig and I. J. Katz, *On products of EP matrices*, Linear Algebra Appl. **252** (1997), 339–345.

[15] R. E. Hartwig and K. Spindelböck, *Matrices for which A^* and $A^\dagger$ commute*, Linear Miltilin. Algebra **14** (1984), 241–256.

[16] I. J. Katz, *Weigmann type theorems for EP_r matrices*, Duke Math. J. **32** (1965), 423–428.

[17] J. J. Koliha, *A simple proof of the product theorem for EP matrices*, Linear Algebra Appl. **294** (1999), 213–215.

[18] J. J. Koliha, *Elements of C*-algebras commuting with their Moore-Penrose inverse*, Studia Math. **139** (2000), 81–90.

[19] G. Lešnjak, *Semigroups of EP linear transformations*, Linear Algebra Appl. **304** (2000), 109–118.

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