Condition number of the W–weighted Drazin inverse

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Abstract

In this paper we get the explicit condition number formulas for the W–weighted Drazin inverse of a rectangular matrix using the Schur decomposition and the spectral norm. We characterize the spectral norm and the Frobenius norm of the relative condition number of the W–weighted Drazin inverse, and the level-2 condition number of the W–weighted Drazin inverse. The sensitivity for the W–weighted Drazin inverse solution of linear systems is presented. We also present the structured perturbation of the W–weighted Drazin inverse.

Key words and phrases: Condition number, weighted Drazin inverse, Schur decomposition.

2000 *Mathematics subject classification*: 15A09, 15A12.

1 Introduction

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. By $rank(A)$, $A^\top$, A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ we denote the rank, the transpose, the conjugate transpose, the range (column space) and the null space, respectively, of $A \in \mathbb{C}^{m\times n}$.

Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then $A^{D,W} = X \in \mathbb{C}^{m\times n}$ is the W-weighted Drazin inverse of A if (see [7])

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA.$$

where $k = ind(AW)$, the index of AW, is the smallest nonnegative integer k for which $rank\left[(AW)^k\right] = rank\left[(AW)^{k+1}\right]$. If $A \in \mathbb{C}^{n\times n}$ and $W = I_n$, then $X = A^D$, where A^D is the ordinary Drazin inverse of A.

The W-weighted Drazin inverse of A has the following properties:

$$A^{D,W} = [(AW)^D]^2 A = A[(WA)^D]^2$$

*The authors are supported by the Ministry of Science, Republic of Serbia, grant no. 144003.

$$\mathcal{R}(A^{D,W}) = \mathcal{R}((AW)^k), \quad \mathcal{N}(A^{D,W}) = \mathcal{N}((WA)^k),$$
$$rank((AW)^k) = rank((WA)^k),$$

where $k = \max\{ind(AW), ind(WA)\}$. Some interesting properties of the Drazin and the W-weighted Drazin inverse can be found in [4].

J. Chen and Z. Xu (see [2]) characterized the condition number of the Drazin inverse and singular linear systems for restrained matrices, by using the Schur decomposition and the spectral norm instead of the P-norm, where P is a transformation matrix of the Jordan canonical form of A. Note that, in general, the computation of the Jordan canonical form is an ill-posed problem. Their results generalize some early work including [10, 12], because of well-posed properties of the Schur decomposition. In [1, 5, 9] the authors established some results for the condition number of the W–weighted Drazin inverse and the W–weighted Drazin inverse solution of a linear system, by using a special norm called PQ-norm. The definition of the PQ-norm depends on Jordan canonical form of A. The results obtained in [1] are extended to linear bounded operators between Hilbert spaces in [6]. In this paper, we establish the condition number of the W–weighted Drazin inverse of a rectangular matrix by the Schur decomposition and the familiar 2-norm instead of the PQ-norm in [1].

2 Representation of the W-Drazin inverse

We recall the next theorm.

Lemma 2.1. (Schur decomposition)[3] *If $A \in \mathbb{C}^{n\times n}$, then there exists an unitary $U \in \mathbb{C}^{n\times n}$ such that*

$$U^*AU = T = D + N,$$

where $D = diag(\lambda_1, \ldots, \lambda_n)$, and $N \in C^{n\times n}$ is strictly upper triangular.

Furthermore, U can be chosen so that the eigenvalues λ_i appear in any order along the diagonal.

Let $A \in \mathbb{C}^{n\times n}$ satisfies the following condition:

$$rank(A^k) = r, \quad ind(A) = k, \quad \mathcal{R}(A^k) = \mathcal{R}((A^k)^*), \tag{1}$$

and the Schur decomposition of A can be written as follows

$$A = U \begin{bmatrix} B & D \\ 0 & C \end{bmatrix} U^*, \tag{2}$$

where U is unitary, B is $r \times r$ upper triangular and nonsingular matrix, and $C = [c_{i,j}]$ is strictly upper triangular, i.e. $c_{i,j} = 0$ whenever $1 \leq j \leq i \leq n-r$.

In [2] J. Chen and Z. Xu used the Schur decomposition of a restrained matrix A to get its expression of the Drazin inverse in the next theorem.

Theorem 2.1. [2] *Let $A \in \mathbb{C}^{n\times n}$. If A fulfills the condition* (1)*, then the Schur decomposition of A has the form as follows*

$$A = U \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} U^*, \tag{3}$$

where U is unitary, B is an $r \times r$ upper triangular and nonsingular matrix, C is strictly upper triangular. Then

$$A^D = U \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{4}$$

Then we obtain the following theorem.

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. Then we have*

$$A = U \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} V^*, \qquad W = V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^*$$

$$A^{D,W} = U \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*, \tag{5}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, A_1 and W_1 are nonsingular matrices, A_2W_2 and W_2A_2 are strictly upper triangular matrices.

Proof. We have $rank((WA)^k) = rank((AW)^k) = r$. From Theorem 2.1, we have the Schur decomposition of AW and WA:

$$AW = U \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} U^*, \qquad WA = V \begin{bmatrix} D & 0 \\ 0 & F \end{bmatrix} V^*, \tag{6}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, B and D are $r \times r$ upper triangular and nonsingular matrices, C and F are strictly upper triangular matrices.

We can represent A and W as

$$A = U \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} V^*, \qquad W = V \begin{bmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{bmatrix} U^*.$$

Since C and F are strictly upper triangular matrices, we obtain $C^k = 0$ and $F^k = 0$. Now, we get

$$(AW)^k A = U \begin{bmatrix} B^k & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} V^* = U \begin{bmatrix} B^k A_1 & B^k A_{12} \\ 0 & 0 \end{bmatrix} V^*$$

and

$$A(WA)^k = U \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} V^* V \begin{bmatrix} D^k & 0 \\ 0 & 0 \end{bmatrix} V^* = U \begin{bmatrix} A_1 D^k & 0 \\ A_{21} D^k & 0 \end{bmatrix} V^*.$$

Using the equation $(AW)^k A = A(WA)^k$, we deduce $B^k A_{12} = 0$ and $A_{21} D^k = 0$. We know that B and D are nonsingular, thus $A_{12} = 0$ and $A_{21} = 0$, i.e.

$$A = U \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} V^*.$$

From

$$AW = U \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} V^* V \begin{bmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{bmatrix} U^* = U \begin{bmatrix} A_1 W_1 & A_1 W_{12} \\ A_2 W_{21} & A_2 W_2 \end{bmatrix} U^*,$$

$$WA = V \begin{bmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{bmatrix} U^* U \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} V^* = V \begin{bmatrix} W_1 A_1 & W_{12} A_2 \\ W_{21} A_1 & W_2 A_2 \end{bmatrix} V^*$$

and (6), we obtain $A_1 W_1 = B$, $W_1 A_1 = D$, $A_2 W_2 = C$, $W_2 A_2 = F$, $A_1 W_{12} = 0$ and $W_{21} A_1 = 0$. Hence, A_1 and W_1 are invertible, $A_2 W_2$ and $W_2 A_2$ are strictly upper triangular matrices, $W_{12} = 0$ and $W_{21} = 0$. So

$$W = V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^*.$$

Finally, by $A^{D,W} = [(AW)^D]^2 A = A[(WA)^D]^2$, we get

$$A^{D,W} = U \begin{bmatrix} B^{-2} A_1 & 0 \\ 0 & 0 \end{bmatrix} V^* = U \begin{bmatrix} A_1 D^{-2} & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

i.e. $B^{-2} A_1 = A_1 D^{-2}$. Thus,

$$A^{D,W} = U \begin{bmatrix} (W_1 A_1 W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

This completes the proof. $\square$

3 Condition numbers of W-Drazin inverse

In this section we consider the following linear system

$$WAWx = b,$$

where $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $ind(AW) = k_1$, $ind(WA) = k_2$, $b \in \mathcal{R}((WA)^{k_2})$ and $x \in \mathcal{R}((AW)^{k_1})$. The W-weighted Drazin-inverse solution x has the form

$$x = A^{D,W}b.$$

The definition of the absolute condition number was introduced by Rice in [8]. If F is a continuously differentiable function

$$F : \mathbb{C}^{m\times n} \times \mathbb{C}^n \longrightarrow \mathbb{C}^m$$

$$(A, x) \longmapsto F(A, x),$$

the absolute condition number of F at x is the scalar $\|F'(x)\|$. The relative condition of F at x is

$$\frac{\|F'(x)\|\|x\|}{\|y\|}.$$

Introduce the following operator:

$$F : \mathbb{C}^{m\times n} \times \mathbb{C}^n \longrightarrow \mathbb{C}^m$$

$$(A, b) \longmapsto F(A, b) = A^{D,W}b = x.$$

We known that the operator F is a differentiable function, when the perturbation E in A fulfils the following condition:

$$\mathcal{R}(EW) \subseteq \mathcal{R}((AW)^k), \quad \mathcal{N}((WA)^k) \subseteq \mathcal{N}(WE), \tag{7}$$

where $k = \max\{k_1, k_2\}$. It is easy to verify that (7) is equivalent to

$$A^{D,W}(WAW)EW = EW, \quad WE(WAW)A^{D,W} = WE. \tag{8}$$

We need the following result.

Lemma 3.1. [11] *Let $A, E \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k = \max\{ind(AW), ind(WA)\}$. If E satisfies the condition (7) and $\|A^{D,W}WEW\|_2 < 1$, then*

$$(A + E)^{D,W} = (I + A^{D,W}WEW)^{-1}A^{D,W} = A^{D,W}(I + WEWA^{D,W})^{-1}.$$

We choose the parameterized weighted Frobenius norm $\|[\alpha WAW, \beta b]\|_{U,Q}^{(F)}$, where U is the same matrix as in (5) and $Q = diag(U, 1)$, because we can choose different parameters α, β for different perturbations.

We get the explicit formula for the condition number of the W-weighted Drazin-inverse solution by means of the 2-norm and Frobenius norm which generalize the main result in [1].

Theorem 3.1. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. If the perturbation E in A fulfills the condition (7), then the absolute condition number of the W–weighted Drazin inverse solution of linear system, with the norm*

$$\|[\alpha WAW, \beta b]\|_{U,Q}^{(F)} = \sqrt{\alpha^2\|WAW\|_F^2 + \beta^2\|b\|_2^2}$$

on the data (A, b) and the norm $\|x\|_2$ on the solution, is

$$C = \|A^{D,W}\|_2\sqrt{\frac{1}{\beta^2} + \frac{\|x\|_2^2}{\alpha^2}},$$

where $Q = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$ *and U is the same matrix as in (5).*

Proof. We know that $F(A, b) = A^{D,W}b$. Under the condition (7), F is a differentiable function and F' is defined as follows

$$F'(A, b)|_{(E,f)} = \lim_{\epsilon\longrightarrow 0}\frac{(A + \epsilon E)^{D,W}(b + \epsilon f) - A^{D,W}b}{\epsilon},$$

where E is the perturbation of A and f is the perturbation of b.

Since E satisfies the condition (7), we have (see [7])

$$(A + \epsilon E)^{D,W} = A^{D,W} - \epsilon A^{D,W}WEWA^{D,W} + O(\epsilon^2),$$

and then we can easily get that

$$F'(A, b)|_{(E,f)} = -A^{D,W}WEWx + A^{D,W}f.$$

Then

$$\begin{aligned}\|F'(A, b)|_{(E,f)}\|_2 &= \|F'(A, b)|_{(E,f)}\|_F \\ &= \|A^{D,W}(WEWx - f)\|_F \\ &\le \|A^{D,W}\|_2(\|WEW\|_F\|x\|_2 + \|f\|_2).\end{aligned}$$

The norm of a linear map $F'(A,b)$ is the supermum of $\|F'(A,b)|_{(E,f)}\|_F$ on the unit ball of $\mathbb{C}^{m\times n}\times\mathbb{C}^n$. Since

$$(\|[\alpha WEW,\beta f]\|_{U,Q}^{(F)})^2=\alpha^2\|WEW\|_F^2+\beta^2\|f\|_2^2$$

we get

$$\begin{aligned}
&\|F'(A,b)\|=\\
&=\sup_{\alpha^2\|WEW\|_F^2+\beta^2\|f\|_2^2=1}\|A^{D,W}(WEWx-f)\|_F\\
&\leq\sup_{\alpha^2\|WEW\|_F^2+\beta^2\|f\|_2^2=1}\|A^{D,W}\|_2(\|WEW\|_F\|x\|_2+\|f\|_2)\\
&=\sup_{\alpha^2\|WEW\|_F^2+\beta^2\|f\|_2^2=1}\|A^{D,W}\|_2\left(\alpha\|WEW\|_F\frac{\|x\|_2}{\alpha}+\beta\|f\|_2\frac{1}{\beta}\right)\\
&=\|A^{D,W}\|_2\sup_{\alpha^2\|WEW\|_F^2+\beta^2\|f\|_2^2=1}(\alpha\|WEW\|_F,\beta\|f\|_2)\cdot\left(\frac{\|x\|_2}{\alpha},\frac{1}{\beta}\right)
\end{aligned}$$

where $(\alpha\|WEW\|_F,\beta\|f\|_2)$ and $\left(\frac{\|x\|_2}{\alpha},\frac{1}{\beta}\right)$ can be consider as vectors in R^2.

Therefore, from the Cauchy–Schwarz inequality, we get:

$$\|F'(A,b)\|\leq\|A^{D,W}\|_2\sqrt{\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}}.$$

Now we show that this upper bound is reachable. There are vectors u i v such that

$$(W_1A_1W_1)^{-1}u=\|(W_1A_1W_1)^{-1}\|_2v=\|A^{D,W}\|_2v,$$

where $\|u\|_2=\|v\|_2=1$.

Let

$$\hat{u}=V\begin{bmatrix}u\\0\end{bmatrix},\qquad\hat{v}=U\begin{bmatrix}v\\0\end{bmatrix}.$$

It is easy to check that

$$\|\hat{u}\|_2=\|\hat{v}\|_2=1.$$

Then

$$\begin{aligned} A^{D,W}\hat{u} &= U\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*V\begin{bmatrix} u \\ 0 \end{bmatrix} \\ &= U\begin{bmatrix} (W_1A_1W_1)^{-1}u \\ 0 \end{bmatrix} \\ &= U\begin{bmatrix} \|(W_1A_1W_1)^{-1}\|_2 v \\ 0 \end{bmatrix} \\ &= \|(W_1A_1W_1)^{-1}\|_2 U\begin{bmatrix} v \\ 0 \end{bmatrix} \\ &= \|A^{D,W}\|_2\hat{v}. \end{aligned}$$

Now we take

$$\eta = \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}, \qquad f = \frac{1}{\beta^2\eta}\hat{u},$$

$$E = -\frac{1}{\alpha^2\eta}U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}x^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

So we have

$$\begin{aligned} EW &= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}x^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \times \\ &\times V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* \\ &= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}x^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \end{aligned}$$

Since

$$A^{D,W}(WAW) = U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

we can verify E fulfills the first equation of condition (8)

$$\begin{aligned} &A^{D,W}(WAW)EW = \\ &= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}x^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= -\frac{1}{\alpha^2\eta}U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}x^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= EW. \end{aligned}$$

In the same way, we have

$$\begin{aligned} WE &= -\frac{1}{\alpha^2\eta} V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* \times \\ &\times\ U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \hat{u} x^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= -\frac{1}{\alpha^2\eta} V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} x^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= -\frac{1}{\alpha^2\eta} \hat{u} x^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \end{aligned}$$

Since

$$(WAW)A^{D,W} = V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

then

$$\begin{aligned} WE(WAW)A^{D,W} &= -\frac{1}{\alpha^2\eta} \hat{u} x^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= -\frac{1}{\alpha^2\eta} \hat{u} x^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= WE. \end{aligned}$$

Hence, E fulfills the condition (8). Now we want to verify the perturbation (E, f) is feasible, that is, $\alpha^2\|WEW\|_F^2 + \beta^2\|f\|_2^2 = 1$. Notice that

$$x = A^{D,W} b = U \begin{bmatrix} (W_1 A_1 W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* b,$$

and then

$$\alpha^2\|WEW\|_F^2 + \beta^2\|f\|_2^2$$

$$
\begin{aligned}
&= \frac{1}{\alpha^2\eta^2}\left\|V\begin{bmatrix} W_1 & 0\\ 0 & W_2\end{bmatrix}\begin{bmatrix} W_1^{-1} & 0\\ 0 & 0\end{bmatrix}V^*\hat{u}x^*U\begin{bmatrix} W_1^{-1} & 0\\ 0 & 0\end{bmatrix}\times\right.\\
&\times \left.\begin{bmatrix} W_1 & 0\\ 0 & W_2\end{bmatrix}U^*\right\|_F^2+\frac{1}{\beta^2\eta^2}\|\hat{u}\|_2^2\\
&= \frac{1}{\alpha^2\eta^2}\left\|\hat{u}x^*U\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}U^*\right\|_F^2+\frac{1}{\beta^2\eta^2}\\
&= \frac{1}{\alpha^2\eta^2}\left\|\hat{u}b^*V\begin{bmatrix} (W_1A_1W_1)^{-1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}U^*\right\|_F^2+\frac{1}{\beta^2\eta^2}\\
&= \frac{1}{\alpha^2\eta^2}\left\|\hat{u}b^*V\begin{bmatrix} (W_1A_1W_1)^{-1} & 0\\ 0 & 0\end{bmatrix}U^*\right\|_F^2+\frac{1}{\beta^2\eta^2}\\
&= \frac{1}{\alpha^2\eta^2}\left\|\hat{u}x^*\right\|_F^2+\frac{1}{\beta^2\eta^2}\\
&= \frac{1}{\alpha^2\eta^2}\|\hat{u}\|_2^2\|x^*\|_2^2+\frac{1}{\beta^2\eta^2}\\
&= \frac{1}{\eta^2}\left(\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}\right)\\
&= 1.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
F'(A,b)|_{(E,f)} &= -A^{D,W}WEWx+A^{D,W}f\\
&= \frac{1}{\alpha^2\eta}A^{D,W}\hat{u}x^*U\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}U^*x+\frac{1}{\beta^2\eta}A^{D,W}\hat{u}\\
&= \frac{1}{\alpha^2\eta}A^{D,W}\hat{u}x^*x+\frac{1}{\beta^2\eta}\|A^{D,W}\|_2\hat{v}\\
&= \frac{1}{\alpha^2\eta}\|x\|_2^2\|A^{D,W}\|_2\hat{v}+\frac{1}{\beta^2\eta}\|A^{D,W}\|_2\hat{v}\\
&= \|A^{D,W}\|_2\eta\hat{v}.
\end{aligned}
$$

Then

$$\|F'(A,b)|_{(E,f)}\|_2=\|A^{D,W}\|_2\sqrt{\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}}.$$

with $\alpha^2\|WEW\|_F^2+\beta^2\|f\|_2^2=1$, implies

$$\|F'(A,b)\|\geq\|A^{D,W}\|_2\sqrt{\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}},$$

and we complete the proof. □

If E satisfies the condition (7), then the 2-norm relative condition number of the W–weighted Drazin inverse is defined as

$$Cond(A) = \lim_{\epsilon \to 0^+} \sup_{\|WEW\|_2 \leq \epsilon \|WAW\|_2} \frac{\|(A+E)^{D,W} - A^{D,W}\|_2}{\epsilon \|A^{D,W}\|_2}$$

and the corresponding condition number for the linear systems $WAWx = b$ is defined as

$$Cond(A,b) = \lim_{\epsilon \to 0^+} \sup_{\substack{\|WEW\|_2 \leq \epsilon \|WAW\|_2 \\ \|f\|_2 \leq \epsilon \|b\|_2}} \frac{\|(A+E)^{D,W}(b+f) - A^{D,W}b\|_2}{\epsilon \|A^{D,W}b\|_2}.$$

The level-2 condition number of W–weighted Drazin inverse is defined as

$$Cond^{[2]}(A) = \lim_{\epsilon \to 0} \sup_{\|WEW\|_2 \leq \epsilon \|WAW\|_2} \frac{|Cond(A+E) - Cond(A)|}{\epsilon\, Cond(A)}$$

and the level-2 corresponding condition number is defined as

$$Cond^{[2]}(A,b) = \lim_{\epsilon \to 0} \sup_{\substack{\|WEW\|_2 \leq \epsilon \|WAW\|_2 \\ \|f\|_2 \leq \epsilon \|b\|_2}} \frac{|Cond(A+E, b+f) - Cond(A,b)|}{\epsilon\, Cond(A,b)}.$$

Theorem 3.2. *Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}((AW)^{k_1 *})$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}((WA)^{k_2 *})$. If the perturbation E in A fulfills the condition* (7)*, then the condition number*

$$Cond(A) = \lim_{\epsilon \to 0^+} \sup_{\|WEW\|_2 \leq \epsilon \|WAW\|_2} \frac{\|(A+E)^{D,W} - A^{D,W}\|_2}{\epsilon \|A^{D,W}\|_2}, \tag{9}$$

satisfies

$$Cond(A) = \|WAW\|_2 \|A^{D,W}\|_2. \tag{10}$$

Proof. By neglecting $\mathcal{O}(\epsilon^2)$ terms in a standard expansion, it follows from Lemma 3.1 that

$$(A+E)^{D,W} - A^{D,W} = -A^{D,W} WEWA^{D,W}.$$

Let $E = \epsilon \|WAW\|_2 \hat{E}$, using $\|WEW\|_2 \leq \epsilon \|WAW\|_2$, we have $\|W\hat{E}W\|_2 \leq 1$. Then

$$\|A^{D,W} W\hat{E}WA^{D,W}\|_2 \leq \|A^{D,W}\|_2 \|W\hat{E}W\|_2 \|A^{D,W}\|_2 \leq \|A^{D,W}\|_2^2.$$

The result is proved if we can show that

$$\sup_{\|W\hat{E}W\|_2 \le 1} \|A^{D,W} W \hat{E} W A^{D,W}\|_2 = \|A^{D,W}\|_2^2.$$

There exists vectors x and y such that $\|x\|_2 = \|y\|_2 = 1$

$$\|(W_1 A_1 W_1)^{-1} y\|_2 = \|x^*(W_1 A_1 W_1)^{-1}\|_2 = \|(W_1 A_1 W_1)^{-1}\|_2.$$

Choose

$$\hat{E} = U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} \begin{bmatrix} x^* & 0 \end{bmatrix} \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

We can verify that

$$\begin{aligned}
\|W\hat{E}W\|_2 &= \left\| V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} \begin{bmatrix} x^* & 0 \end{bmatrix} \right. \\
&\times \left. \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* \right\|_2 \\
&= \left\| V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 \\
&= \left\| V \begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 \\
&= \|yx^*\|_2 \\
&= \|y\|_2 \|x\|_2 \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\|A^{D,W} W\hat{E}W A^{D,W}\|_2 &= \left\| U \begin{bmatrix} (W_1 A_1 W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* V \begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix} U^* U \right. \\
&\times \left. \begin{bmatrix} (W_1 A_1 W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \right\|_2 \\
&= \left\| U \begin{bmatrix} ((W_1 A_1 W_1)^{-1} y)(x^*(W_1 A_1 W_1)^{-1}) & 0 \\ 0 & 0 \end{bmatrix} V^* \right\|_2 \\
&= \|(W_1 A_1 W_1)^{-1} y\|_2 \|x^*(W_1 A_1 W_1)^{-1}\|_2 \\
&= \|(W_1 A_1 W_1)^{-1} y\|_2^2 \\
&= \|A^{D,W}\|_2^2.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
\hat{E}W &= U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix}\begin{bmatrix} x^* & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}U^* \\
&= U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix}U^*,
\end{aligned}$$

and

$$\begin{aligned}
W\hat{E} &= V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}U^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix}\begin{bmatrix} x^* & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* \\
&= V\begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*.
\end{aligned}$$

Now, from

$$\begin{aligned}
A^{D,W}(WAW)\hat{E}W &= U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&= U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&= \hat{E}W,
\end{aligned}$$

and

$$\begin{aligned}
W\hat{E}(WAW)A^{D,W} &= V\begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*V\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}V^* \\
&= V\begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* \\
&= W\hat{E},
\end{aligned}$$

we have that $\hat{E}$ satisfies the condition (7). We complete the proof. □

Then we consider the condition number with the Frobenius norm.

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. If the perturbation E in A fulfills the condition (7), then the condition number*

$$(11) \qquad Cond_F(A) = \lim_{\epsilon\to 0^+} \sup_{\|WEW\|_F \le \epsilon\|WAW\|_F} \frac{\|(A+E)^{D,W} - A^{D,W}\|_F}{\epsilon\|A^{D,W}\|_F},$$

satisfies

$$Cond_F(A) = \frac{\|WAW\|_F \|A^{D,W}\|_2^2}{\|A^{D,W}\|_F}. \tag{12}$$

Proof. Analogously to the proof of Theorem 3.2, we should prove that

$$\sup_{\|W\hat{E}W\|_2 \le 1} \|A^{D,W} W\hat{E}W A^{D,W}\|_F = \|A^{D,W}\|_2^2.$$

Take

$$\hat{E} = U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} \begin{bmatrix} x^* & 0 \end{bmatrix} \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

where $\|x\|_2 = \|y\|_2 = 1$ and $\|(W_1A_1W_1)^{-1}y\|_2 = \|x^*(W_1A_1W_1)^{-1}\|_2 = \|(W_1A_1W_1)^{-1}\|_2$. Thus

$$\begin{aligned}
\|A^{D,W} W\hat{E}W A^{D,W}\|_F &= \left\| U \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*V \begin{bmatrix} yx^* & 0 \\ 0 & 0 \end{bmatrix} U^*U \right. \\
&\times \left. \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \right\|_F \\
&= \left\| U \begin{bmatrix} ((W_1A_1W_1)^{-1}y)(x^*(W_1A_1W_1)^{-1}) & 0 \\ 0 & 0 \end{bmatrix} V^* \right\|_F \\
&= \left\| \begin{bmatrix} ((W_1A_1W_1)^{-1}y)(x^*(W_1A_1W_1)^{-1}) & 0 \\ 0 & 0 \end{bmatrix} \right\|_F \\
&= \|(W_1A_1W_1)^{-1}y\|_2 \|x^*(W_1A_1W_1)^{-1}\|_2 \\
&= \|(W_1A_1W_1)^{-1}y\|_2^2 \\
&= \|A^{D,W}\|_2^2.
\end{aligned}$$

The proof is completed. □

Now we characterize the condition number of linear systems by means of 2-norm.

Theorem 3.4. *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. *If the perturbation* E *in* A *fulfills the condition* (7)*, then the condition number of singular linear systems* $WAWx = b$

$$Cond(A,b) = \lim_{\epsilon\to 0^+} \sup_{\substack{\|WEW\|_2 \le \epsilon\|WAW\|_2 \\ \|f\|_2 \le \epsilon\|b\|_2}} \frac{\|(A+E)^{D,W}(b+f) - A^{D,W}b\|_2}{\epsilon\|A^{D,W}b\|_2}, \tag{13}$$

satisfies

$$Cond(A,b) = \|WAW\|_2 \|A^{D,W}\|_2 + \frac{\|A^{D,W}\|_2 \|b\|_2}{\|A^{D,W} b\|_2}. \tag{14}$$

Proof. From

$$\begin{aligned} (A+E)^{D,W}(b+f) - A^{D,W} b &= [(A+E)^{D,W} - A^{D,W}]b + (A+E)^{D,W} f \\ &= -A^{D,W} W E W A^{D,W} b + (A+E)^{D,W} f \\ &= -A^{D,W} W E W x + A^{D,W} f + \mathcal{O}(\epsilon^2), \end{aligned}$$

we get

$$\begin{aligned} \|(A+E)^{D,W}(b+f) - A^{D,W} b\|_2 &\leq \|A^{D,W}\|_2 \|WEW\|_2 \|x\|_2 + \|A^{D,W}\|_2 \|f\|_2 \\ &\leq \epsilon \|A^{D,W}\|_2 (\|WAW\|_2 \|x\|_2 + \|b\|_2). \end{aligned}$$

Hence,

$$Cond(a,b) \leq \|WAW\|_2 \|A^{D,W}\|_2 + \frac{\|A^{D,W}\|_2 \|b\|_2}{\|A^{D,W} b\|_2}.$$

Now, suppose $y = V \begin{bmatrix} z \\ 0 \end{bmatrix}$, where $\|z\|_2 = 1$, $\|(W_1 A_1 W_1)^{-1} z\|_2 = \|(W_1 A_1 W_1)^{-1}\|_2$. Then we have $\|y\|_2 = 1$ and

$$\begin{aligned} \|A^{D,W} y\|_2 &= \left\| U \begin{bmatrix} (W_1 A_1 W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* V \begin{bmatrix} z \\ 0 \end{bmatrix} \right\|_2 \\ &= \|(W_1 A_1 W_1)^{-1} z\|_2 \\ &= \|A^{D,W}\|_2. \end{aligned}$$

Let

$$f = \epsilon y \|b\|_2, \quad E = -\frac{\epsilon \|WAW\|_2}{\|x\|_2} U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* y x^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

It is easy to verify that $A^{D,W}(WAW)EW = EW$ and $WE(WAW)A^{D,W} = WE$, i.e. we can get that E fulfills the condition (7). Then

$$\|f\|_2 = \epsilon \|b\|_2 \|y\|_2 = \epsilon \|b\|_2$$

and

$$
\begin{aligned}
\|WEW\|_2 &= \frac{\epsilon\|WAW\|_2}{\|x\|_2}\left\|V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}U^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*yx^*U\right. \\
&\times \left.\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}U^*\right\|_2 \\
&= \frac{\epsilon\|WAW\|_2}{\|x\|_2}\left\|V\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}V^*V\begin{bmatrix} z \\ 0 \end{bmatrix}(A^{D,W}b)^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*\right\|_2 \\
&= \frac{\epsilon\|WAW\|_2}{\|x\|_2}\left\|V\begin{bmatrix} z \\ 0 \end{bmatrix}b^*V\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*\right\|_2 \\
&= \frac{\epsilon\|WAW\|_2}{\|x\|_2}\left\|yb^*V\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*\right\|_2 \\
&= \frac{\epsilon\|WAW\|_2}{\|x\|_2}\|yx^*\|_2 \\
&= \frac{\epsilon\|WAW\|_2}{\|x\|_2}\|y\|_2\|x\|_2 \\
&= \epsilon\|WAW\|_2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|(A+E)^{D,W}(b+f)-A^{D,W}b\|_2 &= \|-A^{D,W}WEWx+A^{D,W}f\|_2 \\
&= \left\|\frac{\epsilon\|WAW\|_2}{\|x\|_2}A^{D,W}yx^*x+\epsilon\|b\|_2A^{D,W}y\right\|_2 \\
&= \epsilon(\|WAW\|_2\|x\|_2+\|b\|_2)\|A^{D,W}\|_2
\end{aligned}
$$

The proof is completed. □

Similarly, we can get the next theorem with Frobenius norm.

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. If the perturbation E in A fulfills the condition (7), then the condition number of singular linear systems $WAWx = b$*

(15)

$$
Cond_F(A,b) = \lim_{\epsilon\to 0^+} \sup_{\substack{\|WEW\|_F\le\epsilon\|WAW\|_F \\ \|f\|_F\le\epsilon\|b\|_F}} \frac{\|(A+E)^{D,W}(b+f)-A^{D,W}b\|_F}{\epsilon\|A^{D,W}b\|_F},
$$

satisfies

$$
Cond(A,b)_F = \|WAW\|_F\|A^{D,W}\|_2 + \frac{\|A^{D,W}\|_2\|b\|_2}{\|A^{D,W}b\|_2}. \tag{16}
$$

Proof. Analogously to the proof of Theorem 3.4, we can prove this theorem also. $\square$

The next results show that for the W–weighted Drazin inverse, or for solving a linear system, the sensitivity of the condition number is approximately given by the condition number itself.

Firstly, we need the following lemmas.

Lemma 3.2. *For $\hat{u}, \hat{v}$ in Theorem 2.1, there exists $S \in \mathbb{C}^{m\times n}$ such that*

$$WSW\hat{v} = -\hat{u}, \quad \|WSW\|_2 = 1,$$

where S fulfills condition (7)*.*

Proof. Let

$$S = -U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \hat{u}\hat{v}^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

Then

$$\begin{aligned} WSW\hat{v} &= -V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* U \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \hat{u}\hat{v}^* U \\ &\times \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* V \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* \hat{v} \\ &= -V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^* V \begin{bmatrix} u \\ 0 \end{bmatrix} \hat{v}^* U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} v \\ 0 \end{bmatrix} \\ &= -\hat{u}\hat{v}^*\hat{v} \\ &= -\hat{u}\|\hat{v}\|_2^2 \\ &= -\hat{u}. \end{aligned}$$

Now let us study the 2-norm of WSW

$$\begin{aligned} \|WSW\|_2 &= \left\| \hat{u}\hat{v}^* U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 \\ &= \left\| \hat{u} \begin{bmatrix} v^* & 0 \end{bmatrix} U^* U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \right\|_2 \\ &= \|\hat{u}\hat{v}^*\|_2 \\ &= \|\hat{u}\|_2 \|\hat{v}\|_2 \\ &= 1. \end{aligned}$$

Now we verify S satisfies condition (7). First we know,

$$\begin{aligned} SW &= -U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^* \\ &= -U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \end{aligned}$$

Thus

$$\begin{aligned} A^{D,W}(WAW)SW &= -U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= -U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= SW. \end{aligned}$$

In the same way, we have

$$\begin{aligned} WS &= -V\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} U^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= -V\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \end{aligned}$$

Now

$$\begin{aligned} WS(WAW)A^{D,W} &= -V\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^*\hat{u}\hat{v}^*U\begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} VV^* \\ &= WS, \end{aligned}$$

then S fulfills condition (7). $\square$

Lemma 3.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. When $\epsilon \to 0$, we have*

$$\max_{\|WEW\|_2\leq\epsilon\|WAW\|_2}\left|\|(A+E)^{D,W}\|_2-\|A^{D,W}\|_2\right| = \epsilon\|A^{D,W}\|_2 Cond(A)+\mathcal{O}(\epsilon^2),$$

for E fulfills the condition (7).

Proof. Since E fulfills the condition (7), we have

$$(A+E)^{D,W} = A^{D,W} - A^{D,W}WEWA^{D,W} + \mathcal{O}(\epsilon^2).$$

Now

$$\max_{\|WEW\|_2 \le \epsilon \|WAW\|_2} \Big| \|(A+E)^{D,W}\|_2 - \|A^{D,W}\|_2 \Big| \le \epsilon \|A^{D,W}\|_2 Cond(A) + \mathcal{O}(\epsilon^2).$$

Set $E = \epsilon \|WAW\|_2 S$, where S is defined in Lemma 3.2. Then

$$\begin{aligned}
&\|A^{D,W} - A^{D,W} WEWA^{D,W}\|_2 \\
&\ge \|(A^{D,W} - A^{D,W} WEWA^{D,W})\hat{u}\|_2 \\
&= \|A^{D,W}\hat{u} - A^{D,W} WEWA^{D,W}\hat{u}\|_2 \\
&= \|A^{D,W}\hat{u} - \epsilon \|WAW\|_2 A^{D,W} WSWA^{D,W}\hat{u}\|_2 \\
&= \Big\| \|A^{D,W}\|_2 \hat{v} - \epsilon \|WAW\|_2 \|A^{D,W}\|_2 A^{D,W} WSW\hat{v} \Big\|_2 \\
&= \|A^{D,W}\|_2 \Big\| \hat{v} + \epsilon \|WAW\|_2 A^{D,W}\hat{u} \Big\|_2 \\
&= \|A^{D,W}\|_2 \Big\| \hat{v} + \epsilon \|WAW\|_2 \|A^{D,W}\|_2 \hat{v} \Big\|_2 \\
&= \|A^{D,W}\|_2 \Big(1 + \epsilon \|WAW\|_2 \|A^{D,W}\|_2 \Big).
\end{aligned}$$

□

We now can get easy the following results.

Theorem 3.6. [1] *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. *If the perturbation* E *in* A *fulfills the condition* (7), *then the level-2 condition number*

$$Cond^{[2]}(A) = \lim_{\epsilon\to 0} \sup_{\|WEW\|_2 \le \epsilon \|WAW\|_2} \frac{|Cond(A+E) - Cond(A)|}{\epsilon Cond(A)} \tag{17}$$

satisfies

$$|Cond^{[2]}(A) - Cond(A)| \le 1. \tag{18}$$

Theorem 3.7. [1] *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. *If the perturbation* E *in* A *fulfills the condition* (7), *then the level-2 condition number of singular linear systems* $WAWx = b$

(19)

$$Cond^{[2]}(A,b) = \lim_{\epsilon\to 0} \sup_{\substack{\|WEW\|_2 \le \epsilon \|WAW\|_2 \\ \|f\|_2 \le \epsilon \|b\|_2}} \frac{|Cond(A+E, b+f) - Cond(A,b)|}{\epsilon Cond(A,b)}$$

satisfies

$$\frac{Cond(A,b)}{4} - \frac{1}{2} \le Cond^{[2]}(A,b) \le 3Cond(A,b) + 2. \tag{20}$$

4 Structured perturbation

In this section, we present a structured perturbation of the W–weighted Drazin inverse by means of 2–norm. The notation $|A| \le |B|$ means that $|a_{i,j}| \le |b_{i,j}|$ for $A = (a_{i,j})$ and $B = (b_{i,j})$.

Theorem 4.1. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, $k_1 = ind(AW)$, $k_2 = ind(WA)$, $k = \max\{k_1, k_2\}$, $r = rank((AW)^k)$, $\mathcal{R}((AW)^{k_1}) = \mathcal{R}(((AW)^{k_1})^*)$, $\mathcal{R}((WA)^{k_2}) = \mathcal{R}(((WA)^{k_2})^*)$. If $|U^*EWU| \le |U^*AWU|$, $|V^*WEV| \le |V^*WAV|$ and $\|A^{D,W}\|_2\|WEW\|_2 < 1$, then*

$$(A+E)^{D,W} = (I + A^{D,W}WEW)^{-1}A^{D,W},$$

where U and V are the same matrices as in (5).

Proof. Consider the partition $E = U \begin{bmatrix} E_1 & E_{12} \\ E_{21} & E_2 \end{bmatrix} V^*$. ¿From Theorem 2.2 and $|U^*EWU| \le |U^*AWU|$, we get

$$\left|\begin{bmatrix} E_1W_1 & E_{12}W_2 \\ E_{21}W_1 & E_2W_2 \end{bmatrix}\right| \le \left|\begin{bmatrix} A_1W_1 & 0 \\ 0 & A_2W_2 \end{bmatrix}\right|.$$

It is obvious that $E_{21}W_1 = 0$ and $|E_2W_2| \le |A_2W_2|$. Since W_1 is invertible and A_2W_2 is strictly upper triangular matrix, we have $E_{21} = 0$ and E_2W_2 is strictly upper triangular matrix.

Similarly from $|V^*WEV| \le |V^*WAV|$, we have $E_{12} = 0$ and W_2E_2 is strictly upper triangular matrix.

Now, from $E = U \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} V^*$, we easy obtain the structure of $A+E$

$$A + E = U \begin{bmatrix} A_1 + E_1 & 0 \\ 0 & A_2 + E_2 \end{bmatrix} V^*,$$

and

$$(A+E)W = U \begin{bmatrix} (A_1 + E_1)W_1 & 0 \\ 0 & (A_2 + E_2)W_2 \end{bmatrix} U^*.$$

Since $\|A^{D,W}\|_2\|WEW\|_2 < 1$, then $I + A^{D,W}WEW$ is nonsingular, i.e.

$$I + A^{D,W}WEW = U \begin{bmatrix} W_1^{-1}A_1^{-1}(A_1 + E_1)W_1 & 0 \\ 0 & I \end{bmatrix} U^*$$

is nonsingular. Thus $W_1^{-1}A_1^{-1}(A_1+E_1)W_1$ is nonsingular and A_1+E_1 is also nonsingular, $(A_2+E_2)W_2$ is strictly upper triangular matrix. Hence,

$$\begin{aligned}(A+E)^{D,W} &= \left([(A+E)W]^D\right)^2(A+E)\\ &= U\begin{bmatrix} W_1^{-1}(A_1+E_1)^{-1}W_1^{-1} & 0\\ 0 & 0\end{bmatrix}V^*\\ &= (I+A^{D,W}WEW)^{-1}A^{D,W}.\end{aligned}$$

□

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