Condition number related to the outer inverse of a complex matrix

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Abstract

In this paper we obtain the formula for computing the condition number of a complex matrix, which is related to the outer generalized inverse of a given matrix. We use the Schur decomposition of a matrix. We characterize the spectral norm and the Frobenius norm of the relative condition number of the generalized inverse, and the level-2 condition number of the generalized inverse. The sensitivity for the generalized Drazin inverse solution of linear systems is presented. We also present the structured perturbation of the generalized inverse.

Key words and phrases: Condition number, outer generalized inverse, Schur decomposition.

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1 Introduction

In this paper we prove the formula for computing the condition number of a given complex matrix, which is related to the outer generalized inverse of a given matrix.

First, we present some introductional material. If A is a complex matrix, then the smallest non-negative integer k, which satisfies $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$, is called the index of A, denoted by $\mathrm{ind}(A)$. If $\mathrm{ind}(A) = k$, then we know that there exists the unique matrix A^D which satisfies the equations:

$$A^k A^D A = A^k,\ A^D A A^D = A^D,\ A A^D = A^D A.$$

The matrix A is the Drazin inverse of A. In the case $\mathrm{ind}(A) = 1$, then A^D is the group inverse of A, which is denoted by A^g. Moreover, $\mathrm{ind}(A) = 0$ if and only if A is invertible, and ind this case $A^{-1} = A^D$.

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More general situation appears if we assume that A is rectangular. For this purpose, Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices. By $rank(A)$, $A^\top$, A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ we denote the rank, transpose, conjugate transpose, range (column space) and null space, respectively, of $A\in\mathbb{C}^{m\times n}$. Let $W\in\mathbb{C}^{n\times m}$. Then $AW\in\mathbb{C}^{m\times m}$ and there exists the Drazin inverse of AW, naturally denoted by $(AW)^D$. Now, the weighted W-Drazin inverse of A is defined as $A^{D,W}=((AW)^D)^2A$. For the properties of the weighted W-Drazin inverse see [16].

Among other very nice properties, the Drazin inverse of a matrix is also its outer inverse. The notion of outer generalized inverses is available for rectangular matrices. If $A\in\mathbb{C}^{m\times n}$, then $B\in\mathbb{C}^{n\times m}$ is an outer generalized inverse of A, if $BAB=B$ holds. The interesting case is $B\neq 0$. In this paper we are focused to those outer generalized inverses of A that have fixed null-space and range. From the well-known rank properties, it follows that $\mathrm{rank}(B)\leq \mathrm{rank}(A)$. Outer generalized inverses have applications it solving singular linear systems [1]. We formalize previous consideration in the following definition.

Definition 1.1. *Let $A\in\mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s\leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. If a matrix $X\in\mathbb{C}^{n\times m}$ satisfies*

$$XAX=X,\quad \mathcal{R}(X)=T,\quad \mathcal{N}(X)=S,$$

then X is called the outer inverse or generalized inverse of A, and the notation $X=A^{(2)}_{T,S}$ is common.

The main characterization of $A^{(2)}_{T,S}$-generalized inverse is given as follows.

Lemma 1.1. [1] *Let $A\in\mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s\leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. Then A has an outer inverse X such that $\mathcal{R}(X)=T$ and $\mathcal{N}(X)=S$ if and only if $AT\oplus S=\mathbb{C}^m$, and in this case $X=A^{(2)}_{T,S}$ is unique.*

We also need the following results.

Lemma 1.2. [18] *Let $A\in\mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s\leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. In addition, suppose $G\in\mathbb{C}^{n\times m}$ such that $\mathcal{R}(G)=T$ and $\mathcal{N}(G)=S$. If A has an outer inverse $A^{(2)}_{T,S}$, then $\mathrm{ind}(AG)=\mathrm{ind}(GA)=1$. Further, we have*

$$A^{(2)}_{T,S}=(GA)^gG=G(AG)^g. \tag{1}$$

Lemma 1.3. [13] *If A satisfies the conditions of Lemma 1.2, then*

$$rank(AG) = rank(GA) = rank(G).$$

If A is square and invertible, then the condition number of A is defined as $k(A) = \|A\| \cdot \|A^{-1}\|$, where $\| \cdot \|$ is some matrix norm. The study of condition numbers is important in the theory of stability of linear systems. If A is rectangular (or even square and singular), then we do not have the condition number of A in the previous sense. But still, we have some generalized inverse of A, say A^-. Now, the "generalized“ condition number of A related to A^- is defined as $\|A\| \cdot \|A^-\|$. Generalized condition numbers have applications in studying singular linear systems.

Higham [12] discussed different condition numbers of regular inverses and nonsingular linear systems. Concerning generalized inverses and singular linear systems there are similar results on these problems. Gratton in [11] investigated an $m \times n$ full rank real matrix A and obtained its condition number for the linear least squares problem. The authors studied the (weighted) least squares solution in [5, 6, 9, 20, 21] and the (W-weighted) Drazin-inverse solution in [4, 8, 19, 21, 22, 25, 26].

The following result is known as the Schur decomposition theorem.

Lemma 1.4. (Schur decomposition)[10] *If $A \in \mathbb{C}^{n \times n}$, then there exists an unitary $U \in \mathbb{C}^{n \times n}$ such that*

$$U^* A U = T = D + N$$

where $D = diag(\lambda_1, \ldots, \lambda_n)$ and $N \in C^{n \times n}$ is strictly upper triangular.

Furthermore, U can be chosen so that the eigenvalues λ_i appear in any order along the diagonal.

In [15] we characterized the condition number related to the W-Drazin inverse and singular linear systems for restrained matrices, using the Schur decomposition and the spectral norm. These results generalize some early work including [2, 3], because of the well-posed properties of the Schur decomposition. In [7, 24] some results are established for the condition number of the generalized inverse and the generalized inverse solution of a linear system, using a special norm called PQ-norm. We mention that the PQ-norm depends on the Jordan canonical form of A. Note that, in general, the computation of the Jordan canonical form is an ill-posed problem. The results obtained in [7] are extended to linear bounded operators between Hilbert spaces in [14]. In this paper we establish the condition number of the generalized inverse of a rectangular matrix by the Schur decomposition and the familiar 2-norm instead of the PQ-norm in [7].

2 Representation of the outer inverse

Let $A \in \mathbb{C}^{n\times n}$ satisfies the following condition:

$$rank(A^k) = r, \quad \text{ind}(A) = k, \quad \mathcal{R}(A^k) = \mathcal{R}(A^{k^*}). \tag{2}$$

In general, the Schur decomposition of A can be written as follows

$$A = U \begin{bmatrix} B & D \\ 0 & C \end{bmatrix} U^* \tag{3}$$

where U is unitary, B is $r \times r$ upper triangular and nonsingular matrix, and $C = [c_{i,j}]$ is strictly upper triangular, i.e. $c_{i,j} = 0$ whenever $1 \le j \le i \le n-r$.

In [3] J. Chen and Z. Xu used the Schur decomposition of a restrained matrix A to get its expression of Drazin inverse. Precisely, they proved the following theorem.

Theorem 2.1. [3] *Let $A \in \mathbb{C}^{n\times n}$. If A fulfills the condition* (2)*, then the Schur decomposition of A has the form as follows*

$$A = U \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} U^*, \tag{4}$$

where U is unitary, B is an $r \times r$ upper triangular and nonsingular matrix, C is strictly upper triangular. Then

$$A^D = U \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{5}$$

We prove the following result.

Theorem 2.2. *Let A, G, T and S be the same as in Lemma 1.2, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. Then we have*

$$A = V \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} U^*, \qquad G = U \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} V^*$$

$$A^{(2)}_{T,S} = U \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*, \tag{6}$$

where U and V are unitary matrices, A_1 and G_1 are nonsingular matrices.

Proof. From Lemma 1.2 and Lemma 1.3, we have $\mathrm{ind}(AG) = \mathrm{ind}(GA) = 1$ and $rank(GA) = rank(AG) = p$. By Theorem 2.1, there is a Schur decomposition of AG and GA as follows:

$$AG = V \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^*, \qquad GA = U \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{7}$$

where $V \in \mathbb{C}^{m\times m}$ and $U \in \mathbb{C}^{n\times n}$ are unitary matrices, C and D are $p \times p$ upper triangular and nonsingular matrices.

We can represent A and W as

$$A = V \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} U^*, \qquad G = U \begin{bmatrix} G_1 & G_{12} \\ G_{21} & G_2 \end{bmatrix} V^*.$$

Then, we get

$$(GA)^g G = U \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} G_1 & G_{12} \\ G_{21} & G_2 \end{bmatrix} V^* = U \begin{bmatrix} C^{-1}G_1 & C^{-1}G_{12} \\ 0 & 0 \end{bmatrix} V^*$$

and

$$G(AG)^g = U \begin{bmatrix} G_1 & G_{12} \\ G_{21} & G_2 \end{bmatrix} V^* V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* = U \begin{bmatrix} G_1 D^{-1} & 0 \\ G_{21}D^{-1} & 0 \end{bmatrix} V^*.$$

Using the equation $A^{(2)}_{T,S} = (GA)^g G = G(AG)^g$, we deduce $C^{-1}G_{12} = 0$ and $G_{21}D^{-1} = 0$. We know that C and D are nonsingular, thus $G_{12} = G_{21} = 0$, i.e.

$$G = U \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} V^*.$$

From

$$AG = V \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} U^* U \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} V^* = V \begin{bmatrix} A_1G_1 & A_{12}G_2 \\ A_{21}G_1 & A_2G_2 \end{bmatrix} V^*,$$

$$GA = U \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} V^* V \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} U^* = U \begin{bmatrix} G_1A_1 & G_1A_{12} \\ G_2A_{21} & G_2A_2 \end{bmatrix} U^*$$

and (7), we obtain $A_1G_1 = D$, $G_1A_1 = C$, $G_1A_{12} = 0$ and $A_{21}G_1 = 0$. Hence, A_1 and G_1 are invertible, $A_{12} = 0$ and $A_{21} = 0$. So

$$A = V \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} U^*.$$

Since $G = GAA^{(2)}_{T,S} = A^{(2)}_{T,S}AG$ and $AA^{(2)}_{T,S} = AG(AG)^g$, with $A^{(2)}_{T,S}A = (GA)^g GA$, we have

$$\begin{aligned} G &= U \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} V^* = U \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} V^* V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= U \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} V^*, \end{aligned}$$

i.e. $G_2 = 0$.

Finally, we get

$$A^{(2)}_{T,S} = (GA)^g G = U \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*.$$

This completes the proof. □

3 Condition number related to the generalized inverse

In this section we consider the following linear system

$$Ax = b, \quad x \in T,$$

where $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. The generalized $A^{(2)}_{T,S}$-inverse solution x has the form

$$x = A^{(2)}_{T,S} b.$$

The definition of the absolute condition number was introduced by Rice in [17]. If F is a continuously differentiable function

$$F : \mathbb{C}^{m\times n} \times \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

$$(A, x) \longmapsto F(A, x),$$

then the absolute condition number of F at x is the scalar $\|F'(x)\|$. The relative condition of F at x is

$$\frac{\|F'(x)\|\|x\|}{\|y\|}.$$

Consider the following operator:

$$F : \mathbb{C}^{m\times n} \times \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

$$(A,b) \longmapsto F(A,b) = A^{(2)}_{T,S}b = x.$$

We know that the operator F is differentiable function, if the perturbation E in A fulfils the following condition:

$$\mathcal{R}(E) \subseteq AT, \quad \mathcal{R}(E^*) \subseteq A^*S^\top. \tag{8}$$

It is easy to verify that (8) is equivalent to

$$AA^{(2)}_{T,S}E = E, \quad EA^{(2)}_{T,S}A = E. \tag{9}$$

We need the following result.

Lemma 3.1. [23] *Let $A, E \in \mathbb{C}^{m\times n}$, and let T, S be subspace of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that $AT \oplus S = \mathbb{C}^m$. If E satisfies the condition* (8) *and $\|EA^{(2)}_{T,S}\|_2 < 1$, then*

$$(A+E)^{(2)}_{T,S} = (I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + EA^{(2)}_{T,S})^{-1}.$$

We choose the parameterized weighted Frobenius norm $\|[\alpha A, \beta b]\|^{(F)}_{U,Q}$, where U is defined as in (6) and $Q = diag(U,1)$, because we can take different parameters α, β for different perturbations.

We get the explicit formula for the condition number of the generalized $A^{(2)}_{T,S}$-inverse solution by means of the 2-norm and Frobenius norm.

Theorem 3.1. *Let A, G, T and S be the same as in Lemma 1.2, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If the perturbation E in A fulfills the condition* (8)*, then the absolute condition number of the generalized $A^{(2)}_{T,S}$-inverse solution of a linear system, with the norm*

$$\|[\alpha A, \beta b]\|^{(F)}_{U,Q} = \sqrt{\alpha^2\|A\|^2_F + \beta^2\|b\|^2_2}$$

on the data (A,b), and the norm $\|x\|_2$ on the solution, is given by

$$C = \|A^{(2)}_{T,S}\|_2\sqrt{\frac{1}{\beta^2} + \frac{\|x\|^2_2}{\alpha^2}},$$

where $Q = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$ and U is the same matrix as in (6).

Proof. We know that $F(A,b) = A^{(2)}_{T,S}b$. Under the condition (8), F is a differentiable function and F' is defined as follows

$$F'(A,b)|_{(E,f)} = \lim_{\epsilon \to 0} \frac{(A+\epsilon E)^{(2)}_{T,S}(b+\epsilon f) - A^{(2)}_{T,S}b}{\epsilon},$$

where E is the perturbation of A and f is the perturbation of b.

Since E satisfies the condition (8), we have

$$(A+\epsilon E)^{(2)}_{T,S} = A^{(2)}_{T,S} - \epsilon A^{(2)}_{T,S} E A^{(2)}_{T,S} + O(\epsilon^2),$$

and then we can easily get that

$$F'(A,b)|_{(E,f)} = -A^{(2)}_{T,S}Ex + A^{(2)}_{T,S}f.$$

Then

$$\begin{aligned} \|F'(A,b)|_{(E,f)}\|_2 &= \|F'(A,b)|_{(E,f)}\|_F \\ &= \|A^{(2)}_{T,S}(Ex-f)\|_F \\ &\leq \|A^{(2)}_{T,S}\|_2(\|E\|_F\|x\|_2 + \|f\|_2). \end{aligned}$$

The norm of a linear map $F'(A,b)$ is the supermum of $\|F'(A,b)|_{(E,f)}\|_F$ on the unit ball of $\mathbb{C}^{m\times n} \times \mathbb{C}^n$. Since

$$(\|[\alpha E, \beta f]\|^{(F)}_{U,Q})^2 = \alpha^2\|E\|_F^2 + \beta^2\|f\|_2^2$$

we get

$$\begin{aligned} &\|F'(A,b)\| = \\ &= \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} \|A^{(2)}_{T,S}(Ex-f)\|_F \\ &\leq \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} \|A^{(2)}_{T,S}\|_2(\|E\|_F\|x\|_2 + \|f\|_2) \\ &= \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} \|A^{(2)}_{T,S}\|_2 \left(\alpha\|E\|_F\frac{\|x\|_2}{\alpha} + \beta\|f\|_2\frac{1}{\beta}\right) \\ &= \|A^{(2)}_{T,S}\|_2 \sup_{\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1} (\alpha\|E\|_F, \beta\|f\|_2) \cdot \left(\frac{\|x\|_2}{\alpha}, \frac{1}{\beta}\right) \end{aligned}$$

where $(\alpha\|E\|_F, \beta\|f\|_2)$ and $\left(\frac{\|x\|_2}{\alpha}, \frac{1}{\beta}\right)$ can be consider as vectors in R^2.

Therefore, from the Cauchy–Schwarz inequality, we get:

$$\|F'(A,b)\| \le \|A^{(2)}_{T,S}\|_2 \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

Now we show that this upper bound is reachable. There are vectors u i v such that

$$A_1^{-1}u = \|A_1^{-1}\|_2 v = \|A^{(2)}_{T,S}\|_2 v,$$

where $\|u\|_2 = \|v\|_2 = 1$.

Let

$$\hat{u} = V \left[\begin{array}{c} u \\ 0 \end{array} \right], \qquad \hat{v} = U \left[\begin{array}{c} v \\ 0 \end{array} \right].$$

It is easy to check that

$$\|\hat{u}\|_2 = \|\hat{v}\|_2 = 1.$$

Then

$$\begin{aligned} A^{(2)}_{T,S}\hat{u} &= U \left[\begin{array}{cc} A_1^{-1} & 0 \\ 0 & 0 \end{array} \right] V^*V \left[\begin{array}{c} u \\ 0 \end{array} \right] = U \left[\begin{array}{c} A_1^{-1}u \\ 0 \end{array} \right] \\ &= U \left[\begin{array}{c} \|A_1^{-1}\|_2 v \\ 0 \end{array} \right] = \|A_1^{-1}\|_2 U \left[\begin{array}{c} v \\ 0 \end{array} \right] \\ &= \|A^{(2)}_{T,S}\|_2 \hat{v}. \end{aligned}$$

Now we take

$$\eta = \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}, \qquad E = -\frac{1}{\alpha^2\eta}\hat{u}x^*, \qquad f = \frac{1}{\beta^2\eta}\hat{u}.$$

So we have

$$\begin{aligned} AA^{(2)}_{T,S}E &= -\frac{1}{\alpha^2\eta} V \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] V^*\hat{u}x^* \\ &= -\frac{1}{\alpha^2\eta} V \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] V^*V \left[\begin{array}{c} u \\ 0 \end{array} \right] x^* \\ &= -\frac{1}{\alpha^2\eta} V \left[\begin{array}{c} u \\ 0 \end{array} \right] x^* \\ &= -\frac{1}{\alpha^2\eta}\hat{u}x^* \\ &= E, \end{aligned}$$

and

$$
\begin{aligned}
EA^{(2)}_{T,S}A &= -\frac{1}{\alpha^2\eta}\hat{u}x^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&= -\frac{1}{\alpha^2\eta}\hat{u}(A^{(2)}_{T,S}b)^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&= -\frac{1}{\alpha^2\eta}\hat{u}b^*V\begin{bmatrix} (A_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix}U^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&= -\frac{1}{\alpha^2\eta}\hat{u}b^*V\begin{bmatrix} (A_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&= -\frac{1}{\alpha^2\eta}\hat{u}x^* \\
&= E.
\end{aligned}
$$

Hence, E fulfills the condition (8). Now we want to verify the perturbation (E,f) is feasible, that is, $\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2=1$. Notice that

$$
x = A^{(2)}_{T,S}b = U\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*b,
$$

and then

$$
\begin{aligned}
&\alpha^2\|E\|_F^2+\beta^2\|f\|_2^2 \\
&\qquad = \frac{1}{\alpha^2\eta^2}\|\hat{u}x^*\|_F^2+\frac{1}{\beta^2\eta^2}\|\hat{u}\|_2^2 \\
&\qquad = \frac{1}{\alpha^2\eta^2}\|\hat{u}\|_2^2\|x^*\|_2^2+\frac{1}{\beta^2\eta^2} \\
&\qquad = \frac{1}{\eta^2}\left(\frac{\|x\|_2^2}{\alpha^2}+\frac{1}{\beta^2}\right) \\
&\qquad = 1.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
F'(A,b)|_{(E,f)} &= -A^{(2)}_{T,S}Ex + A^{(2)}_{T,S}f \\
&= \frac{1}{\alpha^2\eta}A^{(2)}_{T,S}\hat{u}x^*x+\frac{1}{\beta^2\eta}A^{(2)}_{T,S}\hat{u} \\
&= \frac{1}{\alpha^2\eta}\|A^{(2)}_{T,S}\|_2\hat{v}\|x\|_2^2+\frac{1}{\beta^2\eta}\|A^{(2)}_{T,S}\|_2\hat{v} \\
&= \|A^{(2)}_{T,S}\|_2\eta\hat{v}.
\end{aligned}
$$

Then

$$\|F'(A,b)|_{(E,f)}\|_2 = \|A^{(2)}_{T,S}\|_2 \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

with $\alpha^2\|E\|_F^2 + \beta^2\|f\|_2^2 = 1$, implies

$$\|F'(A,b)\| \geq \|A^{(2)}_{T,S}\|_2 \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}},$$

and we complete the proof. □

If E satisfies the condition (8), then the 2-norm relative condition number of the generalized inverse $A^{(2)}_{T,S}$ is defined as

$$Cond(A) = \lim_{\epsilon \to 0^+} \sup_{\|E\|_2 \leq \epsilon \|A\|_2} \frac{\|(A+E)^{(2)}_{T,S} - A^{(2)}_{T,S}\|_2}{\epsilon \|A^{(2)}_{T,S}\|_2}$$

and the corresponding condition number for the linear systems $Ax = b$ is defined as

$$Cond(A,b) = \lim_{\epsilon \to 0^+} \sup_{\substack{\|E\|_2 \leq \epsilon \|A\|_2 \\ \|f\|_2 \leq \epsilon \|b\|_2}} \frac{\|(A+E)^{(2)}_{T,S}(b+f) - A^{(2)}_{T,S}b\|_2}{\epsilon \|A^{(2)}_{T,S}b\|_2}.$$

The level-2 condition number of the generalized $A^{(2)}_{T,S}$-inverse is defined as

$$Cond^{[2]}(A) = \lim_{\epsilon \to 0} \sup_{\|E\|_2 \leq \epsilon \|A\|_2} \frac{|Cond(A+E) - Cond(A)|}{\epsilon Cond(A)}$$

and the level-2 corresponding condition number is defined as

$$Cond^{[2]}(A,b) = \lim_{\epsilon \to 0} \sup_{\substack{\|E\|_2 \leq \epsilon \|A\|_2 \\ \|f\|_2 \leq \epsilon \|b\|_2}} \frac{|Cond(A+E, b+f) - Cond(A,b)|}{\epsilon Cond(A,b)}.$$

In [27], Zhang and Wei found the expressions for 2-norm condition number for $A^{(2)}_{T,S}$ and the generalized inverse $A^{(2)}_{T,S}$ solution, respectively

$$Cond(A) = \|A\|_2 \|A^{(2)}_{T,S}\|_2. \tag{10}$$

$$Cond(A,b) = \|A\|_2 \|A^{(2)}_{T,S}\|_2 + \frac{\|A^{(2)}_{T,S}\|_2 \|b\|_2}{\|A^{(2)}_{T,S}b\|_2}. \tag{11}$$

The following results show that for the generalized $A^{(2)}_{T,S}$-inverse for solving a linear system, the sensitivity of the condition number is approximately given by the condition number itself.

First, we need the following lemmas.

Lemma 3.2. *For $\hat{u}, \hat{v}$ in Theorem 2.1, there exists $S \in \mathbb{C}^{m\times n}$ such that*

$$S\hat{v} = -\hat{u}, \quad \|S\|_2 = 1,$$

where S fulfills condition (8)*.*

Proof. Let $S = -\hat{u}\hat{v}^*$, then $S\hat{v} = -\hat{u}\hat{v}^*\hat{v} = -\hat{u}\|\hat{v}\|_2^2 = -\hat{u}$. Now let us study the 2-norm of S

$$\|S\|_2 = \|\hat{u}\hat{v}^*\|_2 = \|\hat{u}\|_2\|\hat{v}\|_2 = 1.$$

Now we verify that S satisfies condition (8). We have

$$AA^{(2)}_{T,S}S = -V\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}V^*\hat{u}\hat{v}^* = -V\begin{bmatrix} u \\ 0 \end{bmatrix}\hat{v}^* = S,$$

and

$$\begin{aligned} SA^{(2)}_{T,S}A &= -\hat{u}\hat{v}^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* = -\hat{u}\begin{bmatrix} v^* & 0 \end{bmatrix}U^*U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= -\hat{u}\begin{bmatrix} v^* & 0 \end{bmatrix}U^* = -\hat{u}\hat{v}^* = S. \end{aligned}$$

Then S fulfills condition (8). $\square$

Lemma 3.3. *Let A, G, T and S be the same as in Lemma 1.2, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If $\epsilon \to 0$, then*

$$\max_{\|E\|_2\le\epsilon\|A\|_2}\left|\|(A+E)^{(2)}_{T,S}\|_2 - \|A^{(2)}_{T,S}\|_2\right| = \epsilon\|A^{(2)}_{T,S}\|_2 Cond(A) + \mathcal{O}(\epsilon^2),$$

provided that E fulfills the condition (8)*.*

Proof. Since E fulfills the condition (8), we have

$$(A+E)^{(2)}_{T,S} = A^{(2)}_{T,S} - A^{(2)}_{T,S}EA^{(2)}_{T,S} + \mathcal{O}(\epsilon^2).$$

Now

$$\max_{\|E\|_2\le\epsilon\|A\|_2}\left|\|(A+E)^{(2)}_{T,S}\|_2 - \|A\|_2\right| \le \epsilon\|A^{(2)}_{T,S}\|_2 Cond(A) + \mathcal{O}(\epsilon^2).$$

Set $E = \epsilon\|A\|_2 S$, where S is defined in Lemma 3.2. Then

$$\begin{aligned}
&\|A^{(2)}_{T,S} - A^{(2)}_{T,S}EA^{(2)}_{T,S}\|_2 \\
&\geq \|(A^{(2)}_{T,S} - A^{(2)}_{T,S}EA^{(2)}_{T,S})\hat{u}\|_2 \\
&= \|A^{(2)}_{T,S}\hat{u} - A^{(2)}_{T,S}EA^{(2)}_{T,S}\hat{u}\|_2 \\
&= \|A^{(2)}_{T,S}\hat{u} - \epsilon\|A\|_2 A^{(2)}_{T,S}SA^{(2)}_{T,S}\hat{u}\|_2 \\
&= \Big\|\|A^{(2)}_{T,S}\|_2\hat{v} - \epsilon\|A\|_2\|A^{(2)}_{T,S}\|_2 A^{(2)}_{T,S}S\hat{v}\Big\|_2 \\
&= \|A^{(2)}_{T,S}\|_2\Big\|\hat{v} + \epsilon\|A\|_2 A^{(2)}_{T,S}\hat{u}\Big\|_2 \\
&= \|A^{(2)}_{T,S}\|_2\Big\|\hat{v} + \epsilon\|A\|_2\|A^{(2)}_{T,S}\|_2\hat{v}\Big\|_2 \\
&= \|A^{(2)}_{T,S}\|_2\Big(1 + \epsilon\|A\|_2\|A^{(2)}_{T,S}\|_2\Big).
\end{aligned}$$

□

We now can obtain easily the following results from [7].

Corollary 3.1. [7] *Let A, G, T and S be the same as in Lemma 1.2, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If the perturbation E in A fulfills the condition* (8)*, then the level-2 condition number*

$$Cond^{[2]}(A) = \lim_{\epsilon\to 0}\sup_{\|E\|_2\leq\epsilon\|A\|_2}\frac{|Cond(A+E) - Cond(A)|}{\epsilon Cond(A)} \tag{12}$$

satisfies

$$|Cond^{[2]}(A) - Cond(A)| \leq 1. \tag{13}$$

Corollary 3.2. [7] *Let A, G, T and S be the same as in Lemma 1.2, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If the perturbation E in A fulfills the condition* (8)*, then the level-2 condition number of linear systems $Ax = b$, $x \in T$,*

$$Cond^{[2]}(A,b) = \lim_{\epsilon\to 0}\sup_{\substack{\|E\|_2\leq\epsilon\|A\|_2\\ \|f\|_2\leq\epsilon\|b\|_2}}\frac{|Cond(A+E,b+f) - Cond(A,b)|}{\epsilon Cond(A,b)} \tag{14}$$

satisfies

$$\frac{Cond(A,b)}{(1+\zeta)^2} - \frac{1}{1+\zeta} \leq Cond^{[2]}(A,b) \leq 3Cond(A,b) + 2, \tag{15}$$

where $\zeta = \frac{\|b\|_2}{\|AA^{(2)}_{T,S}b\|_2}$.

4 Structured perturbation

In this section we present a structured perturbation of the generalized inverse $A^{(2)}_{T,S}$ by means of 2–norm. The notation $|A| \le |B|$ means that $|a_{i,j}| \le |b_{i,j}|$ for $A = (a_{i,j})$ and $B = (b_{i,j})$.

Theorem 4.1. *Let A, G, T and S be the same as in Lemma 1.2, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If $|V^*EU| \le |V^*AU|$ and $\|A^{(2)}_{T,S}E\|_2 < 1$, then*

$$(A+E)^{(2)}_{T,S} = (I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S},$$

where U and V are the same matrices as in (6).

Proof. Consider the partition $E = V \begin{bmatrix} E_1 & E_{12} \\ E_{21} & E_2 \end{bmatrix} U^*$. From Theorem 2.2 and $|V^*EU| \le |V^*AU|$, we get

$$\left| \begin{bmatrix} E_1 & E_{12} \\ E_{21} & E_2 \end{bmatrix} \right| \le \left| \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \right|.$$

It is obvious that $E_{21} = 0$, $E_{12} = 0$ and $|E_2| \le |A_2|$. Now, from $E = V \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} U^*$, we easy obtain the structure of $A + E$

$$A + E = V \begin{bmatrix} A_1 + E_1 & 0 \\ 0 & A_2 + E_2 \end{bmatrix} U^*.$$

Since $\|A^{(2)}_{T,S}E\|_2 < 1$, we get that $I + A^{(2)}_{T,S}E$ is nonsingular, i.e.

$$I + A^{(2)}_{T,S}E = U \begin{bmatrix} A_1^{-1}(A_1 + E_1) & 0 \\ 0 & I \end{bmatrix} U^*$$

is nonsingular. Thus $A_1^{-1}(A_1 + E_1)$ is nonsingular and $A_1 + E_1$ is also nonsingular. It is not difficult to verify that $(A+E)\mathcal{R}(G) \oplus \mathcal{N}(G) = \mathbb{C}^m$.

Hence, $(A+E)^{(2)}_{T,S}$ exists and

$$\begin{aligned}
(A+E)^{(2)}_{T,S} &= G[(A+E)G]_g \\
&= U\begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} V^*\left(V\begin{bmatrix} (A_1+E_1)G_1 & 0 \\ 0 & 0 \end{bmatrix} V^*\right)_g \\
&= U\begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} V^*V\begin{bmatrix} G_1^{-1}(A_1+E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\
&= U\begin{bmatrix} (A_1+E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\
&= U\begin{bmatrix} (I+A_1^{-1}E_1)^{-1}A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^* \\
&= (I+A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S}.
\end{aligned}$$

□

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