ON a-WEYL'S THEOREM

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ABSTRACT. Let X be a complex infinite dimensional Banach space. We use $\sigma_a(T)$ and $\sigma_{ea}(T)$ respectively, to denote the approximate point spectrum and the essential approximate point spectrum of a bounded operator T on X. Also, $\pi_{a0}(T)$ denotes the set of all isolated eigenvalues of $\sigma_a(T)$ of finite geometric multiplicity. We give sufficient conditions such that the spectral mapping theorem holds for the set $\sigma_a(T)\backslash\pi_{a0}(T)$ and for all analytic functions which are not constant on the connected components of their domains. Using the Rakočević's concept from [5] and [7], we say that an operator T on X obeys a-Weyl's theorem provided that $\sigma_{ea}(T) = \sigma_a(T)\backslash\pi_{a0}(T)$. We investigate connections between a-Weyl's theorem and the spectral mapping theorem. We also prove some perturbation results concerning a-Weyl's theorem and various essential spectra.

Key words: Approximate point spectrum, essential approximate point spectrum, a-Weyl's theorem, spectral mapping theorems, perturbations of spectra.

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1. INTRODUCTION

Let X be a complex infinite dimensional Banach space and let $B(X)$ and $K(X)$, respectively, denote the Banach algebra of all bounded operators on X and the ideal of all compact operators on X. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of T and $\rho(T)$ denotes the resolvent set of T. The next sets are well-known semigroups of semi–Fredholm operators on X: $\Phi_+(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim\mathcal{N}(T) < \infty\}$ and $\Phi_-(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim X/\mathcal{R}(T) < \infty\}$. The semigroup of Fredholm operators is $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. If T is semi–Fredholm, define the index of T by $i(T) = \dim\mathcal{N}(T) - \dim X/\mathcal{R}(T)$. We also consider sets $\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}$ (Weyl operators) and $\Phi_+^-(X) = \{T \in \Phi_+(X) : i(T) \leq 0\}$ (see [**5**]). The Weyl spectrum of $T \in B(X)$ is $\sigma_w(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_0(X)\}$. We use $\sigma_p(T)$ and $\sigma_a(T)$, respectively, to denote the point spectrum and approximate point spectrum of $T \in B(X)$. Also, we use $\mathbf{C}$ to denote the complex plane. We use $\pi_{00}(T)$ to denote the set of all $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma(T)$ and $0 < \dim\mathcal{N}(T-\lambda) < \infty$. We say that T obeys Weyl's theorem, if

$$\sigma_w(T) = \sigma(T)\backslash\pi_{00}(T) \qquad \text{(see [}\mathbf{4}\text{]).}$$

Let π_{a0} denotes the set of all $\lambda \in \mathbf{C}$ such that λ is isolated in $\sigma_a(T)$ and $0 < \dim\mathcal{N}(T-\lambda) < \infty$. If $\lambda \in \pi_{00}(T)$, or $\lambda \in \pi_{a0}(T)$, then λ has finite geometric multiplicity. Also, $\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(X)\}$ is the essential approximate point spectrum. It is well-known that $\sigma_{ea}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_+^-(X)\}$ [**5**]. We take $\sigma_{ab}(T) = \cap\{\sigma_a(T+K) : K \in K(X), TK = KT\}$ to denote the Browder essential approximate point spectrum. It is well-known that $\lambda \notin \sigma_{ab}(T)$ if and only if $T - \lambda \in \Phi_+^-(X)$ and the ascent $a(T-\lambda) < \infty$ [**6**]. We say that T obeys a-Weyl's theorem, if

$$\sigma_{ea}(T) = \sigma_a(T)\backslash\pi_{a0}(T), \qquad \text{(see [}\mathbf{6}\text{], [}\mathbf{7}\text{]).}$$

It is well-known that if $T \in B(X)$ obeys a-Weyl's theorem, then it also obeys Weyl's theorem, but the converse is not true [**7**]. Let $\mathrm{Hol}\,(T)$ denotes the set of all complex-valued functions f, defined and regular in some neighbourhood of $\sigma(T)$, such that f is not constant on the connected components of its domain of definition. Recall the definition of the reduced minimum modulus of T (see [**7**] and references cited there):

$$\gamma(T) = \inf\left\{\frac{\|Ax\|}{\mathrm{dist}(x,\mathcal{N}(T))} : x \notin \mathcal{N}(T)\right\}.$$

It is well–known that $\gamma(T) > 0$ if and only if $\mathcal{R}(T)$ is closed. An operator $T \in B(X)$ is called a Riesz operator, provided that $T - \lambda \in \Phi(X)$ for all $\lambda \in \mathbf{C}\backslash\{0\}$. All quasinilpotent operators are Riesz operators. Finally, we use $a(T)$ to denote the ascent of $T \in B(X)$ and ∂S to denote the boundary of the set $S \subseteq \mathbf{C}$. Recall the definitions of ascent and descent in [**6**] and references cited there.

2. Results

We begin with one generalization of Oberai's result [**4**].

Theorem 2.1. *Let $T \in B(X)$ and $f \in Hol(T)$. Then*

$$\sigma_a(f(T))\backslash\pi_{a0}(f(T)) \subset f(\sigma_a(T)\backslash\pi_{a0}(T)).$$

Proof. It is well–known that $f(\sigma_a((T)) = \sigma_a(f(T))$ [**3**]. Suppose that $\lambda \in \sigma_a(f(T))\backslash\pi_{a0}(f(T)) \subset f(\sigma_a(T))$. We consider three cases.

Case I. Suppose that λ is not an isolated point of $f(\sigma_a(T))$. Then there is a convergent sequence (μ_n) in $\sigma_a(T)$ such that $f(\mu_n) \to \lambda$ and $\mu_n \to \mu_0$. Now, $\lambda = f(\mu_0) \in f(\sigma_a(T)\backslash\pi_{a0}(T))$.

Case II. Now, let λ be an isolated point of $f(\sigma_a(T))$ and λ is not an eigenvalue of $f(T)$. We can write

$$f(T) - \lambda = (T - \mu_1)\cdots(T - \mu_n)g(T), \tag{1}$$

such that $\mu_1 \in \sigma_a(T)$, $\mu_2, \dots, \mu_n \in \sigma(T)$ and $g(T)$ is invertible. Also, operators on the right side of (1) mutually commute. Since λ is not an eigenvalue of $f(T)$, non of $\mu_1, \dots, \mu_n$ can be an eigenvalue of T. Therefore

$$\lambda = f(\mu_1) \in f(\sigma_a(T)\backslash\pi_{a0}(T)).$$

Case III. Let λ be an eigenvalue of $f(T)$ of infinite geometric multiplicity. We also have

$$f(T) - \lambda = (T - \mu_1)\cdots(T - \mu_n)g(T).$$

Since λ is an eigenvalue of $f(T)$ of infinite multiplicity, we get that there is some μ_i, such that μ_i is an eigenvalue of T of infinite multiplicity. Then $\mu_i \in \sigma_a(T)\backslash\pi_{a0}(T)$ and $\lambda \in f(\sigma_a(T)\backslash\pi_{a0}(T))$. $\square$

Definition. We say that $T \in B(X)$ is a-isolated provided that all isolated points of $\sigma_a(T)$ are eigenvalues of T.

It is well-known that $\partial\sigma(T) \subset \sigma_a(T)$, so all isolated points of $\sigma(T)$ are also isolated points of $\sigma_a(T)$. Recall that $T \in B(X)$ is isolated, provided that all isolated points of $\sigma(T)$ are eigenvalues of T [**4**]. Now it is obvious that if T is a-isolated, then it is also isolated.

THEOREM 2.2. *Let $T \in B(X)$ be a-isolated and $f \in Hol(T)$. Then*

$$f(\sigma_a(T)\backslash\pi_{a0}(T)) = \sigma_a(f(T))\backslash\pi_{a0}(f(T)).$$

Proof. According to Theorem 2.1 it is enough to prove the inclusion $\subset$. Let $\lambda \in f(\sigma_a(T)\backslash\pi_{a0}(T)) \subset \sigma_a(f(T))$. Suppose that $\lambda \in \pi_{a0}(f(T))$. Then λ is isolated in $\sigma_a(f(T))$ and

$$f(T) - \lambda = (T - \mu_1)\cdots(T - \mu_n)g(T),$$

for some $\mu_1, \dots, \mu_n \in \sigma(T)$, and $g(T)$ is invertible. If some μ_i belongs to $\sigma_a(T)$, then μ_i is isolated in $\sigma_a(T)$ and it must be an eigenvalue of T. Since λ is an eigenvalue of finite multiplicity, then all $\mu_i \in \sigma_a(T)$ are eigenvalues of T of finite multiplicities. We get that all $\mu_i \in \sigma_a(T)$ are in $\pi_{a0}(T)$. Therefore we get the contradiction, since $\lambda \in f(\sigma_a(T)\backslash\pi_{a0}(T))$. $\square$

Let $T \in B(X)$ and let f be an analytic function defined in a neighbourhood of $\sigma(T)$. It is well-known that the next inclusion holds [**6**]:

$$\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T)). \tag{2}$$

It is also known that this inclusion may be proper [**5**]. The next theorem gives some sufficient conditions such that the equality holds in (2).

THEOREM 2.3. *Let $T \in B(X)$ be a-isolated, T obeys a-Weyl's theorem and let $f \in Hol(T)$. Then $f(T)$ obeys a-Weyl's theorem if and only if $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$.*

Proof. By Theorem 2.2 we have

$$f(\sigma_{ea}(T)) = f(\sigma_a(T) \setminus \pi_{a0}(T)) = \sigma_a(f(T)) \setminus \pi_{a0}(f(T)).$$

The right side is equal to $\sigma_{ea}(f(T))$ if and only if $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. $\square$

If λ is an isolated point of $\sigma(T)$ and $E(\lambda, T)$ denotes the corresponding spectral projection, then $\lambda \in \pi_0(T)$ if and only if $E(\lambda, T)$ is a finite-rank operator. We say that $\pi_0(T)$ consists of isolated eigenvalues of T of finite algebraic multiplicity. Notice that $\pi_0(T) \subseteq \pi_{a0}(T)$. We shall use the follwing Erovenko's result [**2**, Теорема 1]:

PROPOSITION 2.4. *Let $T \in B(X)$ and f is defined and analytic on some neighbourhood of $\sigma(T)$. If $\lambda_0 \in \sigma(T)$ and $f(\lambda_0) = \mu \in \pi_0(f(T))$, then $\lambda_0 \in \pi_0(T)$.*

The next results are also related to Theorems 2.2 and 2.3.

PROPOSITION 2.5. *Let $T \in B(X)$, $f \in Hol(T)$ and $\pi_{a0}(f(T)) = \pi_0(f(T))$. Then*

$$f(\sigma_a(T) \setminus \pi_{a0}(T)) = \sigma_a(f(T)) \setminus \pi_{a0}(f(T)).$$

Proof. By Theorem 2.1, it is enough to prove the inclusion $\subset$. Let

$$\lambda \in f(\sigma_a(T) \setminus \pi_{a0}(T)) \subset \sigma_a(f(T)).$$

Suppose that $\lambda \in \pi_{a0}(f(T)) = \pi_0(f(T))$. If $\mu \in \sigma(T)$ and $f(\mu) = \lambda$, by Proposition 2.4 it follows that $\mu \in \pi_0(T) \subset \pi_{a0}(T)$. So for all $\mu \in \sigma(T)$, if $f(\mu) = \lambda$ then $\mu \in \pi_{a0}(T) \subseteq \sigma_a(T)$. We get the contradiction, since $\lambda \in f(\sigma_a(T) \setminus \pi_{a0}(T))$. $\square$

Corollary 2.6. *Let $T \in B(X)$, $f \in Hol(T)$, $\pi_{a0}(f(T)) = \pi_0(f(T))$ and a-Weyl's theorem holds for T. Then a-Weyl's theorem holds for $f(T)$ if and only if*

$$f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)).$$

Proof. This proof is the same as the proof of Theorem 2.3. $\square$

Now, note that we can easily modify a result from [**5**, Theorem 5.6].

Theorem 2.7. *The operator $T \in B(X)$ obeys a-Weyl's theorem if and only if the next two conditions hold:*

(i) *if $\lambda \in \pi_{a0}(T)$, then $\mathcal{R}(T-\lambda)$ is closed and*
(ii) *if $T-\lambda \in \Phi_+^-(X)$, then λ is an isolated point of $\sigma_a(T)$.*

The main difference is that in [**5**] the second condition is: if $T-\lambda \in \Phi_+^-(X)$, then λ is not an interior point of $\sigma_a(T)$.

To verify the equivalence of our Theorem 2.7 and the statement from [**5**, Theorem 5.6], suppose that $T-\lambda \in \Phi_+^-(X)$ and λ is not an interior point of $\sigma_a(T)$. There is some $\epsilon > 0$, such that if $0 < |\mu-\lambda| < \epsilon$ then $T-\mu \in \Phi_+^-(X)$ and $\dim\mathcal{N}(T-\mu)$ is a constant not greater then $\dim\mathcal{N}(T-\lambda)$. Some of those μ belongs to the set $\mathbf{C}\backslash\sigma_a(T)$, so $\dim\mathcal{N}(T-\mu) = 0$ if $0 < |\mu-\lambda| < \epsilon$. We get that λ is an isolated point of $\sigma_a(T)$.

The next results are connected with the Browder essential approximate point spectrum.

Proposition 2.8. *If $T \in B(X)$ obeys a-Weyl's theorem, then $\sigma_{ea}(T) = \sigma_{ab}(T)$.*

Proof. Notice that $\sigma_{ea}(T) \subset \sigma_{ab}(T)$. If T obeys a-Weyl's theorem, then conditions (i) and (ii) of Theorem 2.7 are valid. Suppose that $\lambda \in \sigma_{ab}(T)\backslash \sigma_{ea}(T)$. Then $T-\lambda \in \Phi_+^-(X)$ and by Theorem 2.7, we get that λ is an isolated point of $\sigma_a(T)$. Also, it is well-known that $\sigma_{ab}(T) = \sigma_{ea}(T) \bigcup \{\lambda \in$

$\sigma_a(T) : \lambda$ is not an isolated point of $\sigma_a(T)\}$ [**6**, Corollary 2.4.]. It follows that λ must not be an isolated point of $\sigma_a(T)$ and we get the contradiction. $\square$

Corollary 2.9. *Suppose that $T \in B(X)$ obeys a-Weyl's theorem. If $T - \lambda \in \Phi_+^-(X)$, then the ascent of $T - \lambda$ is finite.*

Proof. Follows by Proposition 2.8 and [**6**, Theorem 2.1]. $\square$

Theorem 2.10. *Let $T \in B(X)$. Then T obeys a-Weyl's theorem if and only if the next conditions hold:*

(i) *if $\lambda \in \sigma_a(T) \backslash \sigma_{ea}(T)$, then $a(T-\lambda) < \infty$, and*
(ii) *if $\lambda \in \pi_{a0}(T)$, then $\lambda \notin \sigma_{ab}(T)$.*

Proof. Suppose that (i) and (ii) hold. If $\lambda \in \pi_{a0}(T)$, then $\lambda \notin \sigma_{ea}(T)$. Suppose that $\lambda \in \sigma_a(T) \backslash \pi_{a0}(T)$ and $\lambda \notin \sigma_{ea}(T)$. By (i) we get that $a(T-\lambda) < \infty$ and by [**6**] $\lambda \notin \sigma_{ab}(T)$. Since $\lambda \in \sigma_a(T) \backslash \sigma_{ab}(T)$, we have that λ is an isolated point of $\sigma_a(T)$ and $\dim \mathcal{N}(T-\lambda) < \infty$, so $\lambda \in \pi_{a0}(T)$. This is in contradiction with $\lambda \in \sigma_a(T) \backslash \pi_{a0}(T)$. The opposite implication follows by Proposition 2.8 and [**6**, Theorem 2.1]. $\square$

Theorem 2.11. *Let $T \in B(X)$ and let f be an arbitrary analytic function, defined on a neighbourhood of $\sigma(T)$. If the next three conditions hold for f and T:*

(i) *if $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$, then $a(T-\lambda) < \infty$,*
(ii) *if $\lambda \in \pi_{a0}(t)$, then $f(\lambda) \notin \sigma_{ea}(f(t))$,*
(iii) *$f(T)$ obeys a-Weyl's theorem,*

then T obeys a-Weyl's theorem.

Proof. By Theorem 2.10, it is enough to prove that if $\lambda \in \pi_{a0}(T)$, then $\lambda \notin \sigma_{ab}(T)$. Suppose that $\lambda \in \pi_{a0}(T) \cap \sigma_{ab}(T)$. Then $f(\lambda) \in f(\sigma_{ab}(T)) = \sigma_{ab}(f(T))$ [**6**]. Since $f(T)$ obeys a-Weyl's theorem, by Proposition 2.8

we get $\sigma_{ab}(f(T)) = \sigma_{ea}(f(T))$, or $f(\lambda) \in \sigma_{ea}(f(T))$. By (ii), we get the contradiction. $\square$

THEOREM 2.12. *Let $T \in B(X)$ be such that $\sigma_{ea}(T) = \sigma_{ab}(T)$. Then T obeys a-Weyl's theorem if and only if one of the following three conditions holds:*

(i) *if $\lambda \in \pi_{a0}(T)$, then $\mathcal{R}(T-\lambda)$ is closed.*

(ii) *if $\lambda \in \pi_{a0}(T)$, then γ is discontinuous at $T-\lambda$.*

(iii$_1$) *if $\lambda \in \pi_{00}(T)$, then the descent of $T-\lambda$ is finite, and*
(iii$_2$) *if $\lambda \in \pi_{a0}(T)\backslash\pi_{00}(T)$, then $\mathcal{R}(T-\lambda)$ is closed.*

Proof. (i) Follows by Theorem 2.7 and [**6**, Corollary 2.4].

(ii) If the condition (ii) holds, by [**7**, Theorem 2.4.] it follows that T obeys a-Weyl's theorem. Now suppose that the condition (i) holds, i.e. a-Weyl's theorem holds for T. Let $\lambda \in \Delta_a^s(T) = \{\mu : T-\mu \in \Phi_+^-(X), 0 < \dim\mathcal{N}(T-\mu)\}$. Then $\lambda \notin \sigma_{ea}(T) = \sigma_{ab}(T)$, λ is an isolated point of $\sigma_a(T)$ and $\lambda \in \pi_{a0}(T)$. The rest of the proof follows again from [**7**, Theorem 2.4].

(iii) If the condition (iii) holds, by [**7**, Theorem 2.9] it follows that T obeys a-Weyl's theorem. We now prove the opposite implication. We use the next sets: $\Delta_4^s(T) = \{\lambda \in \mathbf{C} : A-\lambda \in \Phi(X), i(A-\lambda) = 0\}$, $\Delta_-^s(T) = \{\lambda \in \Delta_a^s(T) : \dim\mathcal{N}(T-\lambda) < \operatorname{codim}\mathcal{R}(T-\lambda) < \infty\}$ and $\Delta_{-\infty}^s(T) = \{\lambda \in \Delta_a^s(T) : \operatorname{codim}\mathcal{R}(T-\lambda) = \infty\}$, introduced in [**7**]. Suppose that $\lambda \in \Delta_4^s(T) \cup \Delta_-^s(T)$. Then $\lambda - T \in \Phi_-^+(X)$ and $\lambda \notin \sigma_{ea}(T) = \sigma_{ab}(T)$. Now by [**6**], it follows that the ascent of $T-\lambda$ is finite. Suppose that $\lambda \in \Delta_{-\infty}^s(T)$. Then $T-\lambda \in \Phi_+^-(X)$, so $\lambda \notin \sigma_{ea}(T) = \sigma_{ab}(T)$. By [**6**] we get that λ is an isolated point of $\sigma_a(T)$. There exists a neighbourhood $B(\lambda)$ of λ, such that for all $\mu \in B(\lambda)\backslash\{\lambda\}$ it is satisfied $\dim\mathcal{N}(T) = 0$. We get that λ satisfies the condition (λ) of [**7**]. By [**7**, Theorem 2.9] it follows that T obeys a-Weyl's theorem. $\square$

We now prove some perturbation theorems.

THEOREM 2.13. *Let $T \in B(X)$ and let N be a quasinilpotent operator commuting with T. Then $\sigma_{ea}(T) = \sigma_{ea}(T+N)$.*

Proof. We shall use the following well-known fact [**8**, 30. Theorem]: if $T \in \Phi_+(X)$, K is a Riesz operator and $KT = TK$, then $T + \lambda K \in \Phi_+(X)$ for all $\lambda \in \mathbf{C}$. It is enough to prove the implication: if $0 \notin \sigma_{ea}(T)$, then $0 \notin \sigma_{ea}(T+N)$. Suppose that $0 \notin \sigma_{ea}(T)$. Then $T \in \Phi_+^-(X)$ and $T + \lambda N \in \Phi_+(X)$ for all $\lambda \in \mathbf{C}$. Now it is obvious that T and $T+N$ are in the same component of $\Phi_+(X)$, so $i(T+N) = i(T) \leq 0$ and $T + N \in \Phi_+^-(X)$. We get that $0 \notin \sigma_{ea}(T+N)$. $\square$

One can easily verify the next well-known fact:

$$\text{(3)} \qquad \begin{aligned} &\text{if } T, K \in B(X),\ K \text{ is nilpotent and } TK = KT, \\ &\text{then } \sigma_a(T) = \sigma_a(T+K). \end{aligned}$$

THEOREM 2.14. *Let $T \in B(X)$ and let N be a nilpotent operator commuting with T. If a-Weyl's theorem holds for T then it also holds for $T+N$.*

Proof. Firstly we prove that $\pi_{a0}(T+N) = \pi_{a0}(T)$. It is enough to prove that if $0 \in \pi_{a0}(T)$, then $0 \in \pi_{a0}(T+N)$. Suppose that $0 \in \pi_{a0}(T)$, so $0 < \dim\mathcal{N}(T) < \infty$.

We prove that $\dim\mathcal{N}(T+N) < \infty$. If $(T+N)x = 0$ for some $x \neq 0$, then $Tx = -Nx$. Since N commutes with T, it follows that for every positive integer m: $T^m x = (-1)^m N^m x$. Let n be the smallest positive integer such that $N^n = 0$. We get that there is some positive integer r, $r \leq n$, such that $T^r x = 0$. Thus $\mathcal{N}(T+N) \subseteq \mathcal{N}(T^r)$ and $\mathcal{N}(T+N)$ is finite dimensional.

We prove that $\dim\mathcal{N}(T+N) > 0$. There is some $x \neq 0$ such that $Tx = 0$. Then $(T+N)^n x = 0$, $0 \in \sigma_p(T+N) \subseteq \sigma_a(T+N)$ and $\dim\mathcal{N}(T+N) > 0$. By (3) we know that $\sigma_a(T) = \sigma_a(T+N)$, so it follows that $0 \in \pi_{a0}(T+N)$.

Thus, using Theorem 2.13 we get

$$\sigma_{ea}(T+N) = \sigma_{ea}(T) = \sigma_a(T)\backslash\pi_{a0}(T) = \sigma_a(T+N)\backslash\pi_{a0}(T+N).$$

Thus a-Weyl's theorem holds for $T+N$. $\square$

Now we give one simple connection with the work of Buoni [**1**]. Let H be a complex, infinite dimensional Hilbert space, $T \in B(H)$ and $\alpha \in \rho(T)$. For arbitrary $\mu \in \mathbf{C}\backslash\{0\}$, there is some $\lambda \in \mathbf{C}$ such that $(\lambda - \alpha)\mu = 1$. Let $A = (T-\alpha)^{-1}$. It is well-known that $\lambda \in \sigma(T)$ if and only if $\mu \in \sigma(A)$ [**1**]. We prove some analogous results concerning the sets $\sigma_a(T)$ and $\sigma_{ea}(T)$.

Lemma 2.15. *Let T, A, α, λ and μ be as above.*

(a) *$\lambda \in \sigma_a(T)$ if and only if $\mu \in \sigma_a(A)$.*
(b) *$\lambda \in \sigma_{ea}(T)$ if and only if $\mu \in \sigma_{ea}(A)$.*
(c) *$\lambda \in \pi_{a0}(T)$ if and only if $\mu \in \pi_{a0}(A)$.*

Proof. (a) If $\lambda \notin \sigma_a(T)$, then $T-\lambda$ is one–to–one and $\mathcal{R}(T-\lambda)$ is closed. Now, by [**1**, Lema 2.2. and Lema 2.3.], $A-\mu$ is 1–1 and $\mathcal{R}(A-\mu)$ is closed, so $\mu \notin \sigma_a(A)$. The opposite implication is analogous.

(b) If $\lambda \notin \sigma_{ea}(T)$, then $T-\lambda \in \Phi_+^-(X)$. By [**1**], $A-\mu \in \Phi_+(X)$ and $i(A-\mu) = i(T-\lambda) \leq 0$, so $\mu \notin \sigma_{ea}(A)$. The opposite implication is analogous.

(c) Suppose that $\lambda \in \sigma_a(T)$ and λ is not isolated in $\sigma_a(T)$. Then there is a sequence λ_i of $\sigma_a(T)$, such that $\lambda_i \to \lambda$ and $\lambda_i \neq \alpha$. We may take $\mu_i = 1/(\lambda_i - \alpha) \in \sigma_a(A)$ (by (a)) and it is obvious that $\mu_i \to \mu$, so $\mu \in \sigma_a(A)$ and μ is not isolated in $\sigma_a(A)$.We get that λ is an isolated point of $\sigma_a(T)$ if and only if μ is an isolated point of $\sigma_a(A)$. Now, if $\lambda \in \pi_{a0}(T)$, since $\dim\mathcal{N}(A-\mu) = \dim\mathcal{N}(T-\lambda)$, we get that $\mu \in \pi_{a0}(A)$. The opposite implication is obvious. $\square$

Using Lemma 2.15, we get the next

Corollary 2.16. *If $0 \notin \sigma_a(T)$, then a-Weyl's theorem holds for T if and only if it holds for A.*

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