MIXED-TYPE REVERSE ORDER LAW FOR PRODUCTS OF THREE OPERATORS

Nebojša Č. Dinčić[1] and Dragan S. Djordjević [1]

Abstract

We present new results related to the mixed-type reverse order law for the Moore-Penrose inverse of various products of three operators on Hilbert spaces. Some finite dimensional results are extended to infinite dimensional settings.

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1 Introduction

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X,Y)$ denote the set of all linear bounded operators from X to Y. If $A \in \mathcal{L}(X,Y)$, then we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the range and the null-space of A.

The Moore-Penrose inverse of given $A \in \mathcal{L}(X,Y)$ is the (unique when it exists) operator $A^\dagger \in \mathcal{L}(Y,X)$ satisfying the following Penrose equations:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed.

If $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,Z)$ are invertible, then the well-known rule $(BA)^{-1} = A^{-1}B^{-1}$ is the reverse order rule of the ordinary inverse. However, if we suppose that operators A, B, BA have closed ranges, then the analogous rule for the Moore-Penrose inverse $(BA)^\dagger = A^\dagger B^\dagger$ does not hold in general. There exists the well-known result giving equivalent conditions such that the reverse order rule holds for the Moore-Penrose inverse, and that is: AA^* commutes with $B^\dagger B$, and $AA^\dagger$ commutes with B^*B (see the following references: [11] for matrices, [2], [3] and [14] for closed-range linear bounded operators on Hilbert spaces, [15] for Moore-Penrose invertible elements in rings with involution). There are also equivalent conditions such that the rule of the form $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ holds ([13]). Besides these facts, there are several papers investigating various forms of the reverse order rule, mostly for complex matrices, but some of them dealing with operators

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on Hilbert spaces, or elements in rings with involutions. This is implied by at least two facts: first, the Moore-Penrose inverse is important in solving various types of equations; and second, the reverse order rule is a very useful computational tool.

This research is motivated by the results obtained in [17], where the reverse order rule is investigated for a triple matrix product. Notice that results in [17] are obtained using finite dimensional methods, mostly rank of a complex matrix. In this paper we extend results from [17] to infinite dimensional Hilbert spaces, using operator matrices.

For this purpose, we list some properties of the Moore-Penrose inverse.

Lemma 1.1. *$A \in \mathcal{L}(X,Y)$ has a closed range if and only if AA^* has a closed range. Moreover, $A^\dagger = A^*(AA^*)^\dagger$.*

Lemma 1.2. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof is straightforward.

Lemma 1.3. ([8]) *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^* A_1 + A_2^* A_2$ *maps* $\mathcal{R}(A^*)$ *into itself and* $D > 0$ (*meaning* $D \geq 0$ *invertible*). *Also,*

$$A^\dagger = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i *denotes different operators in any of these two cases.*

The following result is well-known, and it can be found in [4], p.127.

Lemma 1.4. *Let* $A \in \mathcal{L}(Y,Z)$ *and* $B \in \mathcal{L}(X,Y)$ *have closed ranges. Then* AB *has a closed range if and only if* $A^\dagger ABB^\dagger$ *has a closed range.*

Notice the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A,B] = AB - BA$ denotes the commutator of A and B. On the other hand, if $U \in \mathcal{L}(X,Z)$ and $V \in \mathcal{L}(Y,Z)$, then $[U \quad V] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Z$ denote the matrix form of the corresponding operator.

The following result is proved in [8], Lemma 2.1.

Lemma 1.5. *Let* X, Y *be Hilbert spaces, let* $C \in \mathcal{L}(X,Y)$ *have a closed range, and let* $D \in \mathcal{L}(Y)$ *be Hermitian and invertible. Then* $\mathcal{R}(DC) = \mathcal{R}(C)$ *if and only if* $[D, CC^\dagger] = 0$.

We also use the following result from [5], which can easily be extended from matrices to linear bounded Hilbert space operators.

Lemma 1.6. *Let* $H_i, i = \overline{1,4}$ *be Hilbert spaces, let* $C \in \mathcal{L}(H_1,H_2)$, $X \in \mathcal{L}(H_2,H_3)$ *and* $B \in \mathcal{L}(H_3,H_4)$ *be closed range operators. Then:*

$$C(BXC)^\dagger B = X^\dagger$$

if and only if:

$$\mathcal{R}(B^*BX) = \mathcal{R}(X) \quad \text{and} \quad \mathcal{N}(XCC^*) = \mathcal{N}(X).$$

We shall frequently use the following fact: if $T, S \in \mathcal{L}(H)$ are selfadjoint, then TS is selfadjoint if and only if $TS = ST$.

2 Main results

In this section we prove the results concerning the reverse order law for the Moore-Penrose inverse of various products of linear bounded Hilbert space operators. Throughout this paper X_1, X_2, X_3, X_4 denote arbitrary Hilbert spaces, and $A_k \in \mathcal{L}(X_{k+1}, X_k)$, $k = 1, 2, 3$, denote linear bounded operators. Also, let $M = A_1A_2A_3$.

Theorem 2.1. *Let A_1, A_3, M, $A_1^\dagger M A_3^\dagger$ have closed ranges. Then the following statements are equivalent:*
(a) $M^\dagger = A_3^\dagger (A_1^\dagger M A_3^\dagger)^\dagger A_1^\dagger$;
(b) $\mathcal{R}(A_1A_1^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}(A_3^*A_3M^*) = \mathcal{R}(M^*)$.

Proof. Using Lemma 1.2, we conclude that the operator A_1 has the following matrix form:

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_1^*) \\ \mathcal{N}(A_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_1) \\ \mathcal{N}(A_1^*) \end{bmatrix},$$

where A_{11} is invertible. Then

$$A_1^\dagger = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_1) \\ \mathcal{N}(A_1^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_1^*) \\ \mathcal{N}(A_1) \end{bmatrix}.$$

From Lemma 1.3 it also follows that operators $A_k, k = 2, 3$, have the following matrix forms:

$$A_k = \begin{bmatrix} A_{k1} & 0 \\ A_{k2} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_k^*) \\ \mathcal{N}(A_k) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_{k-1}^*) \\ \mathcal{N}(A_{k-1}) \end{bmatrix},$$

where $D_k = A_{k1}^*A_{k1} + A_{k2}^*A_{k2}$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A_k^*))$. Then

$$A_k^\dagger = \begin{bmatrix} D_k^{-1}A_{k1}^* & D_k^{-1}A_{k2}^* \\ 0 & 0 \end{bmatrix}.$$

We use the notation $M_1 = A_{11}A_{21}A_{31}$, so the matrix form of M is the following:

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

First we find an equivalent form for the statement (a). We have

$$W = A_1^\dagger M A_3^\dagger = \begin{pmatrix} A_{11}^{-1}M_1D_3^{-1}A_{31}^* & A_{11}^{-1}M_1D_3^{-1}A_{32}^* \\ 0 & 0 \end{pmatrix},$$

and consequently

$$W^\dagger = W^*(WW^*)^\dagger = \begin{pmatrix} A_{31}D_3^{-1/2}(A_{11}^{-1}M_1D_3^{-1/2})^\dagger & 0 \\ A_{32}D_3^{-1/2}(A_{11}^{-1}M_1D_3^{-1/2})^\dagger & 0 \end{pmatrix}.$$

It follows that

$$A_3^\dagger W^\dagger A_1^\dagger = \begin{pmatrix} D_3^{-1/2}(A_{11}^{-1}M_1D_3^{-1/2})^\dagger A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$M^\dagger = A_3^\dagger(A_1^\dagger M A_3^\dagger)^\dagger A_1^\dagger$$

is equivalent to:

$$M_1^\dagger = D_3^{-1/2}(A_{11}^{-1}M_1D_3^{-1/2})^\dagger A_{11}^{-1},$$

or, in other words:

$$(A_{11}^{-1}M_1D_3^{-1/2})^\dagger = D_3^{1/2}M_1^\dagger A_{11}.$$

By checking the Penrose equations, the last formula holds if and only if

$$[A_{11}A_{11}^*, M_1M_1^\dagger] = 0 \quad \text{and} \quad [D_3, M_1^\dagger M_1] = 0. \tag{1}$$

Hence, the statement (a) is equivalent to (1).

Now, we find the equivalent statement to (b). The conditions

$$\mathcal{R}(A_1A_1^*M) = \mathcal{R}(M) \quad \text{and} \quad \mathcal{R}(A_3^*A_3M^*) = \mathcal{R}(M^*)$$

are equivalent to

$$\mathcal{R}(A_{11}A_{11}^*M_1) = \mathcal{R}(M_1) \quad \text{and} \quad \mathcal{R}(D_3M_1^*) = \mathcal{R}(M_1^*).$$

By Lemma 1.5, the last statement is equivalent to (1).

Hence, (b) is equivalent to (a). □

Theorem 2.2. *Let $A_1, A_3, M, A_1^*MA_3^*$ have closed ranges. Then the following statements are equivalent:*

(a) $M^\dagger = A_3^*(A_1^*MA_3^*)^\dagger A_1^*$;

(b) $\mathcal{R}(A_1A_1^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}(A_3^*A_3M^*) = \mathcal{R}(M^*)$.

Proof. The statement (b) is equivalent to (1) (see a previous theorem).

In order to prove that (a) is also equivalent to (1), we use the notations from a previous theorem. Let $V = A_1^* M A_3^*$. We see that

$$V^\dagger = V^*(VV^*)^\dagger = \begin{bmatrix} A_{31} D_3^{-1/2} (A_{11}^* M_1 D_3^{1/2})^\dagger & 0 \\ A_{32} D_3^{-1/2} (A_{11}^* M_1 D_3^{1/2})^\dagger & 0 \end{bmatrix}.$$

Now,

$$M^\dagger = A_3^* (A_1^* M A_3^*)^\dagger A_1^*$$

if and only if

$$M_1^\dagger = D_3^{1/2} (A_{11}^* M_1 D_3^{1/2})^\dagger A_{11}^*,$$

or, equivalently:

$$(A_{11}^* M D_3^{1/2})^\dagger = D_3^{-1/2} M_1^\dagger (A_{11}^*)^{-1}.$$

By checking the Penrose equations we obtain that the last statement is equivalent to (1).

Thus, (a) is equivalent to (b). □

Theorem 2.3. *Let $A_1, A_3, M, (A_1A_1^*)^\dagger M (A_3^* A_3)^\dagger$ have closed ranges. Then the following statements are equivalent:*

(a) $M^\dagger = (A_3^* A_3)^\dagger [(A_1 A_1^*)^\dagger M (A_3^* A_3)^\dagger]^\dagger (A_1 A_1^*)^\dagger$*;*

(b) $\mathcal{R}((A_1 A_1^*)^2 M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^* A_3)^2 M^*) = \mathcal{R}(M^*)$.

Proof. Using the notation and the method described in Theorem 2.1, we conclude that the statement (a) is equivalent to the following:

$$M_1^\dagger = D_3^{-1} ((A_{11} A_{11}^*)^{-1} M_1 D_3^{-1})^\dagger (A_{11} A_{11}^*)^{-1},$$

or, equivalently $((A_{11} A_{11}^*)^{-1} M_1 D_3^{-1})^\dagger = D_3 M_1^\dagger A_{11} A_{11}^*$. Using the Penrose equations, it follows that the last equality holds if and only if

$$[M_1 M_1^\dagger, (A_{11} A_{11}^*)^2] = 0 \quad \text{and} \quad [D_3^2, M_1^\dagger M_1] = 0.$$

Using Lemma 1.5 it follows that the last conditions hold if and only if

$$\mathcal{R}((A_{11} A_{11}^*)^2 M_1) = \mathcal{R}(M_1) \quad \text{and} \quad \mathcal{R}(D_3^2 M_1^*) = \mathcal{R}(M_1^*).$$

Now it is easy to see that (a) is equivalent to (b). □

Remark 2.1. *The equality*

$$M^\dagger = (A_3^*A_3)^\dagger[(A_1A_1^*)^\dagger M(A_3^*A_3)^\dagger]^\dagger(A_1A_1^*)^\dagger$$

is often written in the equivalent form:

$$M^\dagger = (A_3^*A_3)^\dagger[(A_3^\dagger A_2^* A_1^\dagger)^\dagger]^*(A_1A_1^*)^\dagger.$$

Previous forms are equal, since

$$\begin{aligned}(A_1A_1^*)^\dagger M(A_3^*A_3)^\dagger &= (A_1A_1^*)^\dagger A_1A_2A_3(A_3^*A_3)^\dagger\\ &= (A_1^*)^\dagger A_2(A_3^*)^\dagger = (A_3^\dagger A_2^* A_1^\dagger)^*.\end{aligned}$$

In the rest of the paper we shall use the following fact. If $S \in \mathcal{L}(H)$ is selfadjoint, then $\mathcal{R}(S)$ is closed if and only if 0 is not a point of accumulation of the spectrum of S. From the spectral mapping theorem it follows that $\mathcal{R}(S^n)$ is closed for every non-negative integer n.

Now, we have a generalization of a previous theorem.

Proposition 2.1. *Under the assumptions of the Theorem 2.3, the following statements are equivalent (k and l are non-negative integers):*
(a) $M^\dagger = ((A_3^*A_3)^\dagger)^l[((A_1A_1^*)^\dagger)^k M((A_3^*A_3)^\dagger)^l]^\dagger((A_1A_1^*)^\dagger)^k$;
(b) $\mathcal{R}((A_1A_1^*)^{2k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{2l}M^*) = \mathcal{R}(M^*)$.

Proof. Using the method described in Theorem 2.1, we conclude that the statement of this theorem is equivalent to the following:

$$\begin{aligned}M_1^\dagger &= D_3^{-l}((A_{11}A_{11}^*)^{-k}M_1D_3^{-l})^\dagger(A_{11}A_{11}^*)^{-k}\\ &\Leftrightarrow \mathcal{R}((A_{11}A_{11}^*)^{2k}M_1) = \mathcal{R}(M_1) \text{ and } \mathcal{R}(D_3^{2l}M_1^*) = \mathcal{R}(M_1^*).\end{aligned}$$

It is not difficult to see that both members from previous equivalence are actually equivalent to:

$$[(A_{11}A_{11}^*)^{2k}, M_1M_1^\dagger] = 0 \text{ and } [D_3^{2l}, M_1^\dagger M_1] = 0,$$

so we have the proof completed. □

Theorem 2.4. *Let A_1, A_3, M, $A_1A_1^*MA_3^*A_3$ have closed ranges. Then the following statements are equivalent:*
(a) $M^\dagger = A_3^*A_3(A_1A_1^*MA_3^*A_3)^\dagger A_1A_1^*$;
(b) $\mathcal{R}((A_1A_1^*)^2M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^2M^*) = \mathcal{R}(M^*)$.

Proof. We conclude that the statement (a) is equivalent to the following:

$$M_1^\dagger = D_3(A_{11}A_{11}^*M_1D_3)^\dagger A_{11}A_{11}^*,$$

which is equivalent to $(A_{11}A_{11}^*M_1D_3)^\dagger = D_3^{-1}M_1^\dagger(A_{11}A_{11}^*)^{-1}$. Using the Penrose equations, we conclude that the last statement is equivalent to

$$[(A_{11}A_{11}^*)^2, M_1M_1^\dagger] = 0 \quad \text{and} \quad [D_3^2, M_1^\dagger M_1] = 0.$$

From Lemma 1.5 we conclude that the last statement is equivalent to

$$\mathcal{R}((A_{11}A_{11}^*)^2M) = \mathcal{R}(M) \quad \text{and} \quad \mathcal{R}(D_3^2M^*) = \mathcal{R}(M^*).$$

We conclude that (b) is equivalent to (a). □

Previous theorem can be generalized in the following way.

Proposition 2.2. *Under the assumptions of the Theorem 2.4, the following statements are equivalent (k and l are non-negative integers):*
(a) $M^\dagger = (A_3^*A_3)^l[(A_1A_1^*)^kM(A_3^*A_3)^l]^\dagger(A_1A_1^*)^k$,
(b) $\mathcal{R}((A_1A_1^*)^{2k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{2l}M^*) = \mathcal{R}(M^*)$.

Proof. The statement of this theorem is equivalent to the following:

$$\begin{aligned} M_1^\dagger &= D_3^l((A_{11}A_{11}^*)^kMD_3^l)^\dagger(A_{11}A_{11}^*)^k \\ &\Leftrightarrow \mathcal{R}((A_{11}A_{11}^*)^{2k}M_1) = \mathcal{R}(M_1) \text{ and } \mathcal{R}(D_3^{2l}M_1^*) = \mathcal{R}(M_1^*). \end{aligned}$$

It is not dificiult to see that both members from previous equivalence are actually equivalent to:

$$[(A_{11}A_{11}^*)^{2k}, M_1M_1^\dagger] = 0 \text{ and } [D_3^{2l}, M_1^\dagger M_1] = 0,$$

so we have the proof completed. □

Theorem 2.5. *Let A_1, A_3, M, $(A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger$ have closed ranges. Then the following statements are equivalent.*
(a) $M^\dagger = (A_3A_3^*A_3)^\dagger((A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger)^\dagger(A_1A_1^*A_1)^\dagger$;
(b) $\mathcal{R}((A_1A_1^*)^3M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proof. We denote:

$$\begin{aligned} T &= (A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger \\ &= \begin{pmatrix} (A_{11}A_{11}^*A_{11})^{-1}M_1D_3^{-2}A_{31}^* & (A_{11}A_{11}^*A_{11})^{-1}M_1D_3^{-2}A_{32}^* \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and consequently:

$$T^\dagger = T^*(TT^*)^\dagger = \begin{pmatrix} A_{31}D_3^{-1/2}((A_{11}A_{11}^*A_{11})^{-1}M_1D_3^{-3/2})^\dagger & 0 \\ A_{32}D_3^{-1/2}((A_{11}A_{11}^*A_{11})^{-1}M_1D_3^{-3/2})^\dagger & 0 \end{pmatrix}.$$

Further, we compute

$$S = A_3A_3^*A_3 = \begin{pmatrix} A_{31}D_3 & 0 \\ A_{32}D_3 & 0 \end{pmatrix},$$

and

$$S^\dagger = (S^*S)^\dagger S^* = \begin{pmatrix} D_3^{-2}A_{31}^* & D_3^{-2}A_{32}^* \\ 0 & 0 \end{pmatrix}.$$

The statement (a) is equivalent to the following

$$M^\dagger = (A_3A_3^*A_3)^\dagger((A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger)^\dagger(A_1A_1^*A_1)^\dagger,$$

which is the same as:

$$M_1^\dagger = D_3^{-3/2}((A_{11}A_{11}^*A_{11})^{-1}M_1D_3^{-3/2})^\dagger(A_{11}A_{11}^*A_{11})^{-1},$$

or, in other words:

$$((A_{11}A_{11}^*A_{11})^{-1}M_1D_3^{-3/2})^\dagger = D_3^{3/2}M_1^\dagger A_{11}A_{11}^*A_{11}.$$

From the Penrose equations, it follows that the last statement holds if and only if:

$$[(A_{11}A_{11}^*A_{11})^2, M_1M_1^\dagger] = 0 \quad \text{and} \quad [D_3^3, M_1^\dagger M_1] = 0.$$

By Lemma 1.5, this is equivalent to

$$\mathcal{R}((A_1A_1^*A_1)^2M) = \mathcal{R}(M) \quad \text{and} \quad \mathcal{R}(D_3^3M_1^*) = \mathcal{R}(M_1^*),$$

which is equivalent to (b). □

Remark 2.2. *The equation*

$$M^\dagger = (A_3A_3^*A_3)^\dagger((A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger)^\dagger(A_1A_1^*A_1)^\dagger$$

is often written in the equivalent form:

$$M^\dagger = (A_3A_3^*A_3)^\dagger((A_1^*A_1)^\dagger A_2(A_3A_3^*)^\dagger)^\dagger(A_1A_1^*A_1)^\dagger.$$

The equivalentness follows from:

$$\begin{aligned} (A_1A_1^*A_1)^\dagger A_1 &= A_1^\dagger(A_1^*)^\dagger A_1^\dagger A_1 = A_1^\dagger(A_1^*)^\dagger = (A_1^*A_1)^\dagger, \\ A_3(A_3A_3^*A_3)^\dagger &= A_3A_3^\dagger(A_3^*)^\dagger A_3^\dagger = (A_3A_3^*)^\dagger. \end{aligned}$$

Previous theorem can be generalized in the following way.

Proposition 2.3. *Under the assumptions of Theorem 2.5, the following statements are equivalent (k is a non-negative integer):*
(a) $M^\dagger = (A_3A_3^*A_3)^\dagger[((A_1A_1^*A_1)^\dagger)^k M(A_3A_3^*A_3)^\dagger]^\dagger((A_1A_1^*A_1)^\dagger)^k$,
(a) $\mathcal{R}((A_1A_1^*)^{3k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proof. We use the method described in Theorem 2.1, with some necessary changes. We start with the following:

$$S = A_3A_3^*A_3 = \begin{pmatrix} A_{31}D_3 & 0 \\ A_{32}D_3 & 0 \end{pmatrix},$$

which means that:

$$S^\dagger = (S^*S)^\dagger S^* = \begin{pmatrix} D_3^{-2}A_{31}^* & D_3^{-2}A_{32}^* \\ 0 & 0 \end{pmatrix}.$$

Now we denote:

$$\begin{aligned} W &= ((A_1A_1^*A_1)^\dagger)^k M(A_3A_3^*A_3)^\dagger \\ &= \begin{pmatrix} (A_{11}A_{11}^*A_{11})^{-k}M_1D_3^{-2}A_{31}^* & (A_{11}A_{11}^*A_{11})^{-k}M_1D_3^{-2}A_{32}^* \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and find that:

$$W^\dagger = W^*(WW^*)^\dagger = \begin{pmatrix} A_{31}D_3^{-2}((A_{11}A_{11}^*A_{11})^{-k}M_1D_3^{-3/2})^\dagger & 0 \\ A_{32}D_3^{-2}((A_{11}A_{11}^*A_{11})^{-k}M_1D_3^{-3/2})^\dagger & 0 \end{pmatrix}.$$

Therefore,

$$M^\dagger = A_3A_3^*A_3W^\dagger(A_1A_1^*A_1)^k$$

is equivalent to:

$$M_1^\dagger = D_3^{-3/2}((A_{11}A_{11}^*A_{11})^{-k}M_1D_3^{-3/2})^\dagger(A_{11}A_{11}^*A_{11})^{-k},$$

or further:

$$((A_{11}A_{11}^*A_{11})^{-k}M_1D_3^{-3/2})^\dagger = D_3^{3/2}M_1^\dagger(A_{11}A_{11}^*A_{11})^k.$$

By the Penrose equations, the last formula holds if and only if the following is satisfied:

$$[(A_{11}A_{11}^*)^{3k}, M_1M_1^\dagger] = 0 \text{ and } [D_3^3, M_1^\dagger M_1] = 0.$$

On the other side, by Lemma 1.5, the last conditions are equivalent to

$$\mathcal{R}((A_1A_1^*)^{3k}M) = \mathcal{R}(M) \text{ and } \mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*).$$

Thus, the proof is completed. □

Previous proposition can be proved in a slightly different form:

Proposition 2.4. *Under the conditions of Theorem 2.5, the following statements are equivalent (l is a non-negative integer):*
(a) $M^\dagger = ((A_3A_3^*A_3)^\dagger)^l[(A_1A_1^*A_1)^\dagger M((A_3A_3^*A_3)^\dagger)^l]^\dagger(A_1A_1^*A_1)^\dagger$,
(b) $\mathcal{R}((A_1A_1^*)^3M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{3l}M^*) = \mathcal{R}(M^*)$.

Proof. This proof is very similar to the previous one. Important differences are the following decompositions of spaces and operator matrix forms according to those decompositions:

$$A_3 = \begin{bmatrix} A_{31} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_3^*) \\ \mathcal{N}(A_3) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_3) \\ \mathcal{N}(A_3^*) \end{bmatrix},$$

where A_{31} is invertible, and

$$A_k = \begin{bmatrix} A_{k1} & A_{k2} \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_{k+1}) \\ \mathcal{N}(A_{k+1}^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_k) \\ \mathcal{N}(A_k^*) \end{bmatrix}, k = \overline{1,2},$$

where $D_k = A_{k1}A_{k1}^* + A_{k2}A_{k2}^*$ is invertible and positive on $\mathcal{L}(\mathcal{R}(A_{k+1}^*))$. Analogously, to the previous proof, we have:

$$M^\dagger = ((A_3A_3^*A_3)^\dagger)^l((A_1A_1^*A_1)^\dagger M((A_3A_3^*A_3)^\dagger)^l)^\dagger(A_1A_1^*A_1)^\dagger,$$

which is equivalent to:

$$M_1^\dagger = (A_{31}A_{31}^*A_{31})^{-l}(D_1^{\frac{3}{2}}M_1(A_{31}A_{31}^*A_{31})^{-l})^\dagger D_1^{\frac{3}{2}}.$$

The rest of the proof is clear. □

Theorem 2.6. *Let A_1, A_3, M, $(A_1A_1^*A_1)^*M(A_3A_3^*A_3)^*$ have closed ranges. Then the following statements are equivalent:*
(a) $M^\dagger = (A_3A_3^*A_3)^*((A_1A_1^*A_1)^*M(A_3A_3^*A_3)^*)^\dagger(A_1A_1^*A_1)^*$*;*
(b) $\mathcal{R}((A_1^*A_1A_1^*)^2M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proof. An immediate computation, analogous to the one from Theorem 2.1, yields:

$$M_1^\dagger = D_3^{3/2}(A_{11}^*A_{11}A_{11}^*M_1D_3^{3/2})^\dagger(A_{11}^*A_{11}A_{11}^*),$$

or, equivalently,

$$(A_{11}^*A_{11}A_{11}^*M_1D_3^{3/2})^\dagger = D_3^{-3/2}M_1^\dagger(A_{11}^*A_{11}A_{11}^*)^{-1},$$

which, by Lemma 1.5, holds if and only if:

$$[(A_{11}^*A_{11}A_{11}^*)^2, M_1M_1^\dagger] = 0 \quad \text{and} \quad [D_3^3, M_1^\dagger M_1] = 0.$$

This shows that (a) is equivalent to (b). □

Previous theorem can be generalized in the following way.

Proposition 2.5. *Under the assumptions of Theorem 2.6, the following statements are equivalent (k is a non-negative integer):*

(a) $M^\dagger = (A_3A_3^*A_3)^*[((A_1A_1^*A_1)^*)^k M(A_3A_3^*A_3)^*]^\dagger((A_1A_1^*A_1)^*)^k$,

(a) $\mathcal{R}((A_1A_1^*)^{3k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proof. We start with the following:

$$S = A_3A_3^*A_3 = \begin{pmatrix} A_{31}D_3 & 0 \\ A_{32}D_3 & 0 \end{pmatrix},$$

which means that

$$S^* = \begin{pmatrix} D_3A_{31}^* & D_3A_{32}^* \\ 0 & 0 \end{pmatrix},$$

and also

$$S^\dagger = (S^*S)^\dagger S^* = \begin{pmatrix} (D_3)^{-2}A_{31}^* & D_3^{-2}A_{32}^* \\ 0 & 0 \end{pmatrix}.$$

Now we denote:

$$\begin{aligned} W &= ((A_1A_1^*A_1)^*)^k M(A_3A_3^*A_3)^* \\ &= \begin{pmatrix} ((A_{11}A_{11}^*A_{11})^*)^k M_1D_3A_{31}^* & ((A_{11}A_{11}^*A_{11})^*)^k M_1D_3A_{32}^* \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and find $W^\dagger$ by using $W^\dagger = W^*(WW^*)^\dagger$ that:

$$W^\dagger = W^*(WW^*)^\dagger = \begin{pmatrix} A_{31}D_3^{-1/2}(((A_{11}A_{11}^*A_{11})^*)^k M_1D_3^{3/2})^\dagger & 0 \\ A_{32}D_3^{-1/2}(((A_{11}A_{11}^*A_{11})^*)^k M_1D_3^{3/2})^\dagger & 0 \end{pmatrix}.$$

Therefore,

$$M^\dagger = ((A_3A_3^*A_3)^*)^l W^\dagger((A_1A_1^*A_1)^*)^k$$

is equivalent to:

$$M_1^\dagger = D_3^{3/2}(((A_{11}A_{11}^*A_{11})^*)^k M_1D_3^{3/2})^\dagger((A_{11}A_{11}^*A_{11})^*)^k,$$

or in other words:

$$(((A_{11}A_{11}^*A_{11})^*)^k M_1D_3^{3/2})^\dagger = D_3^{-3/2}M_1^\dagger((A_{11}A_{11}^*A_{11})^*)^{-k}.$$

By the Penrose equations the last condition holds if and only if the following is satisfied:

$$[(A_{11}A_{11}^*)^{3k}, M_1M_1^\dagger] = 0 \text{ and } [D_3^3, M_1^\dagger M_1] = 0.$$

On the other hand, by Lemma 1.5, the last conditions are equivalent to

$$\mathcal{R}((A_1A_1^*)^{3k}M) = \mathcal{R}(M) \text{ and } \mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*).$$

This completes the proof. $\square$

Previous proposition is valid also in a slightly different form:

Proposition 2.6. *Under the conditions of Theorem 2.6, the following statements are equivalent (l is a non-negative integer):*
(a) $M^\dagger = ((A_3A_3^*A_3)^*)^l[(A_1A_1^*A_1)^*M((A_3A_3^*A_3)^*)^l]^\dagger(A_1A_1^*A_1)^*$,
(b) $\mathcal{R}((A_1A_1^*)^3M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{3l}M^*) = \mathcal{R}(M^*)$.

Proof. We use the approach similar to the one used in previous proposition, but space decompositions are different here:

$$A_3 = \begin{bmatrix} A_{31} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_3^*) \\ \mathcal{N}(A_3) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_3) \\ \mathcal{N}(A_3^*) \end{bmatrix},$$

where A_{31} is invertible, and

$$A_k = \begin{bmatrix} A_{k1} & A_{k2} \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_{k+1}) \\ \mathcal{N}(A_{k+1}^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_k) \\ \mathcal{N}(A_k^*) \end{bmatrix}, \ k = \overline{1,2},$$

where $D_k = A_{k1}A_{k1}^* + A_{k2}A_{k2}^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A_{k+1}^*))$. Now we obtain:

$$M^\dagger = ((A_3A_3^*A_3)^*)^lW^\dagger(A_1A_1^*A_1)^*,$$

which is equivalent to:

$$M_1^\dagger = ((A_{31}A_{31}^*A_{31})^*)^l(D_1^{\frac{3}{2}}M_1(((A_{31}A_{31}^*A_{31})^*)^l))^\dagger D_1^{\frac{3}{2}}.$$

The remains of the proof is clear. $\square$

Theorem 2.7. *Let A_1, A_3, M, $((A_1A_1^*)^2)^\dagger M((A_3^*A_3)^2)^\dagger$ have closed ranges. Then the following statements are equivalent:*
(a) $M^\dagger = ((A_3^*A_3)^\dagger)^2[((A_1A_1^*)^2)^\dagger M(A_3^*A_3)^2)^\dagger]^\dagger((A_1A_1^*)^\dagger)^2$;
(b) $\mathcal{R}((A_1A_1^*)^4M) = \mathcal{R}(M)$ *and* $\mathcal{R}((D_3)^4M^*) = \mathcal{R}(M^*)$.

Proof. An immediate computation, analogous to the one from Theorem 2.1, yields that (a) is equivalent to:

$$M_1^\dagger = D_3^{-2}((A_{11}A_{11}^*)^2M_1D_3^{-2})^\dagger(A_{11}A_{11}^*)^{-2},$$

or

$$((A_{11}A_{11}^*)^{-2}M_1D_3^{-2})^\dagger = D_3^2M_1^\dagger(A_{11}A_{11}^*)^2$$

which, by Penrose equations, holds if and only if:

$$[(A_{11}A_{11}^*)^4, M_1M_1^\dagger] = 0 \quad \text{and} \quad [D_3^4, M_1^\dagger M_1] = 0.$$

By Lemma 1.5, the last statement is equivalent to (b). □

Remark 2.3. *In the same way as in previous remarks, the equation*

$$M^\dagger = ((A_3^*A_3)^\dagger)^2[((A_1A_1^*)^2)^\dagger M(A_3^*A_3)^2)^\dagger]^\dagger((A_1A_1^*)^\dagger)^2$$

is often replaced by its equivalent:

$$M^\dagger = ((A_3^*A_3)^\dagger)^2((A_1^*A_1A_1^*)^\dagger A_2(A_3^*A_3A_3^*)^\dagger)^\dagger((A_1A_1^*)^\dagger)^2.$$

Previous theorem can be generalized in the following way.

Proposition 2.7. *Under the assumptions of Theorem 2.7, the following statements are equivalent (k and l are non-negative integers):*
(a) $M^\dagger = ((A_3^*A_3)^\dagger)^{2l}[((A_1A_1^*)^{2k})^\dagger M(A_3^*A_3)^{2l})^\dagger]^\dagger((A_1A_1^*)^\dagger)^{2k}$,
(b) $\mathcal{R}((A_1A_1^*)^{4k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{4l}M^*) = \mathcal{R}(M^*)$.

Proof. An immediate computation, analogous to the one from Theorem 2.1, yields:

$$M_1^\dagger = D_3^{-2l}((A_{11}^*A_{11}A_{11}^*)^{-2k}A_{11}^{-1}M_1D_3^{-2l})^\dagger(A_{11}A_{11}^*)^{-2k},$$

or

$$((A_{11}A_{11}^*)^{-2k}M_1D_3^{-2l})^\dagger = D_3^{2l}M_1^\dagger(A_{11}A_{11}^*)^{2k}$$

which, by Lemma 1.5, holds if and only if:

$$[(A_{11}A_{11}^*)^{4k}, M_1M_1^\dagger] = 0 \text{ and } [D_3^{4l}, M_1^\dagger M_1] = 0.$$

□

Theorem 2.8. *Let A_1, A_3, M, $(A_1A_1^*)^2M(A_3^*A_3)^2$ have closed ranges. Then the following statements are equivalent:*
(a) $M^\dagger = (A_3^*A_3)^2((A_1A_1^*)^2M(A_3^*A_3)^2)^\dagger(A_1A_1^*)^2$;
(b) $\mathcal{R}((A_1A_1^*)^4M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^4M^*) = \mathcal{R}(M^*)$.

Proof. An immediate computation, analogous to the one from Theorem 2.1, shows that (a) is equivalent to the following:

$$M_1^\dagger = D_3^2((A_{11}A_{11}^*)^2 M_1 D_3^2)^\dagger (A_{11}A_{11}^*)^2,$$

or, equivalently,

$$((A_{11}A_{11}^*)^2 M_1 D_3^2)^\dagger = D_3^{-2} M_1^\dagger (A_{11}A_{11}^*)^{-2},$$

which, by Penrose equations, is equivalent to

$$[(A_{11}A_{11}^*)^4, M_1 M_1^\dagger] = 0 \quad \text{and} \quad [D_3^4, M_1^\dagger M_1] = 0.$$

By Lemma 1.5, the last statements is equivalent to (b). □

Previous theorem can be generalized in the following way.

Proposition 2.8. *Under the assumptions of the Theorem 2.8, the following statements are equivalent (k and l are non-negative integers):*
(a) $M^\dagger = ((A_3^*A_3)^*)^{2l}(((A_1A_1^*)^*)^{2k} M((A_3^*A_3)^*)^{2l})^\dagger((A_1A_1^*)^*)^{2k}$,
(b) $\mathcal{R}((A_1A_1^*)^{4k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{4l}M^*) = \mathcal{R}(M^*)$.

Proof. An immediate computation, analogous to the one from Theorem 2.1, yields:

$$M_1^\dagger = D_3^{2l}((A_{11}A_{11}^*)^{2k} M_1 D_3^{2l})^\dagger (A_{11}A_{11}^*)^{2k},$$

or, equivalently,

$$((A_{11}A_{11}^*)^{2k} M_1 D_3^{2l})^\dagger = D_3^{-2l} M_1^\dagger (A_{11}A_{11}^*)^{-2k},$$

which, by Lemma 1.5, holds if and only if:

$$[(A_{11}A_{11}^*)^{4k}, M_1 M_1^\dagger] = 0 \text{ and } [D_3^{4l}, M_1^\dagger M_1] = 0.$$

□

3 Some equivalencies

The results presented in previous section are connected as follows.

Theorem 3.1. *The following statements are equivalent (provided that we apply the Moore-Penrose inverse to closed range operators):*

(a) $M^\dagger = A_3^\dagger(A_1^\dagger M A_3^\dagger)^\dagger A_1^\dagger$;

(b) $M^\dagger = A_3^*(A_1^* M A_3^*)^\dagger A_1^*$;

(c) $A_3^\dagger(A_1^\dagger M A_3^\dagger)^\dagger A_1^\dagger = (A_3^* A_3)^\dagger((A_1A_1^*)^\dagger M(A_3^*A_3)^\dagger)^\dagger(A_1A_1^*)^\dagger$;

(d) $A_3^*(A_1^\dagger M A_3^\dagger)^\dagger A_1^* = A_3^*A_3 M^\dagger A_1 A_1^*$;

(e) $A_3^*(A_1^* M A_3^*)^\dagger A_1^* = A_3^* A_3(A_1A_1^*MA_3^*A_3)^\dagger A_1A_1^*$;

(f) $\mathcal{R}(A_1A_1^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}(A_3^*A_3M^*) = \mathcal{R}(M^*)$.

Proof. From Theorem 2.1 and Theorem 2.2 it follows that $(a) \Leftrightarrow (b) \Leftrightarrow (f)$. Using the method described in those two theorems, we easily conclude that:

$$(c) \Leftrightarrow D_3^{-1/2}(A_{11}^{-1}M_1D_3^{-1/2})^\dagger A_{11}^{-1} = D_3^{-1}((A_{11}A_{11}^*)^{-1}M_1D_3^{-1})^\dagger(A_{11}A_{11}^*)^{-1};$$

while:

$$(e) \Leftrightarrow D_3^{1/2}(A_{11}^*M_1D_3^{1/2})^\dagger A_{11}^* = D_3(A_{11}A_{11}^*M_1D_3)^\dagger A_{11}A_{11}^*.$$

Let us now prove $(e) \Leftrightarrow (f)$. We have the following:

$$(e) \Leftrightarrow (A_{11}^*M_1D_3^{1/2})^\dagger = D_3^{1/2}(A_{11}A_{11}^*M_1D_3^{1/2})^\dagger A_{11}.$$

Let us now denote $X = A_{11}^*M_1D_3^{1/2}$, $B = A_{11}$ and $C = D_3^{1/2}$. By Lemma 1.6, we have the following chain of the equivalencies:

$$\begin{aligned}
& \mathcal{R}(A_{11}^*A_{11}A_{11}^*M_1D_3^{1/2}) = \mathcal{R}(A_{11}^*M_1D_3^{1/2}) \text{ and } \mathcal{N}(A_{11}^*M_1D_3^{3/2}) = \mathcal{N}(A_{11}^*M_1D_3^{1/2}) \\
\Leftrightarrow\ & \mathcal{R}(A_{11}^*A_{11}A_{11}^*M_1D_3^{1/2}) = \mathcal{R}(A_{11}^*M_1D_3^{1/2}) \text{ and } \mathcal{R}(D_3^{3/2}M_1^*A_{11}) = \mathcal{R}(D_3^{1/2}M_1^*A_{11}) \\
\Leftrightarrow\ & \mathcal{R}(A_{11}A_{11}^*M_1D_3^{1/2}) = \mathcal{R}(M_1D_3^{1/2}) \text{ and } \mathcal{R}(D_3M_1^*A_{11}) = \mathcal{R}(M_1^*A_{11}) \\
\Leftrightarrow\ & \mathcal{R}(A_{11}A_{11}^*M_1) = \mathcal{R}(M_1) \text{ and } \mathcal{R}(D_3M_1^*) = \mathcal{R}(M_1^*).
\end{aligned}$$

The last expression is, by Lemma 1.5, equivalent to (f), so we have just proved $(e) \Leftrightarrow (f)$. (During the proof, an obvious fact: $\mathcal{R}(PQ) = \mathcal{R}(SQ) \Rightarrow \mathcal{R}(P) = \mathcal{R}(Q)$ if Q is invertible, is used.)

Analogously, $(c) \Leftrightarrow (f)$ can be proved.

On the other hand,

$$(d) \Leftrightarrow (A_{11}^{-1}M_1D_3^{-1/2})^\dagger = D_3^{1/2}M_1^\dagger A_{11},$$

which is obviously equivalent to the (f).

□

Theorem 3.2. *The following statements are equivalent (provided that we apply the Moore-Penrose inverse to closed range operators):*

(a) $M^\dagger = (A_3^* A_3)^\dagger ((A_1 A_1^*)^\dagger M (A_3^* A_3)^\dagger)^\dagger (A_1 A_1^*)^\dagger$;

(b) $M^\dagger = A_3^* A_3 (A_1 A_1^* M A_3^* A_3)^\dagger A_1 A_1^*$;

(c) $A_3^\dagger (A_1^\dagger M A_3^\dagger)^\dagger A_1^\dagger = A_3^* (A_1^* M A_3^*)^\dagger A_1^*$;

(d) $\mathcal{R}((A_1 A_1^*)^2 M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^* A_3)^2 M^*) = \mathcal{R}(M^*)$.

Proof. From Theorem 2.3 and Theorem 2.4 it follows that $(a) \Leftrightarrow (b) \Leftrightarrow (d)$. Using the method described in those two theorems, we easily conclude that:

$$(c) \Leftrightarrow D_3^{-1/2} (A_{11}^{-1} M_1 D_3^{-1/2})^\dagger A_{11}^{-1} = D_3^{1/2} (A_{11}^* M_1 D_3^{1/2})^\dagger A_{11}^*.$$

Using the method described in the proof of Theorem 3.1 (phase $(e) \Leftrightarrow (f)$), it is easy to conclude $(c) \Leftrightarrow (e)$. □

Theorem 3.3. *The following statements are equivalent (provided that we apply the Moore-Penrose inverse to closed range operators):*

(a) $M^\dagger = (A_3 A_3^* A_3)^\dagger ((A_1 A_1^* A_1)^\dagger M (A_3 A_3^* A_3)^\dagger)^\dagger (A_1 A_1^* A_1)^\dagger$;

(b) $M^\dagger = (A_3 A_3^* A_3)^* ((A_1 A_1^* A_1)^* M (A_3 A_3^* A_3)^*)^\dagger (A_1 A_1^* A_1)^*$;

(c) $A_3^\dagger (A_1^\dagger M A_3^\dagger)^\dagger A_1^\dagger = A_3^* A_3 (A_1 A_1^* M A_3^* A_3)^\dagger A_1 A_1^*$;

(d) $(A_1^\dagger M A_3^\dagger)^\dagger = A_3 A_3^* A_3 (A_1 A_1^* M A_3^* A_3)^\dagger A_1 A_1^* A_1$;

(e) $\mathcal{R}((A_1 A_1^*)^3 M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^* A_3)^3 M^*) = \mathcal{R}(M^*)$.

Proof. From Theorem 2.5 and Theorem 2.6 it follows that $(a) \Leftrightarrow (b) \Leftrightarrow (e)$. Using the method described in those two theorems, we easily conclude that:

$$(c) \Leftrightarrow D_3^{-1/2} (A_{11}^{-1} M_1 D_3^{-1/2})^\dagger A_{11}^{-1} = D_3 (A_{11} A_{11}^* M_1 D_3)^\dagger A_{11} A_{11}^*;$$

also

$$(d) \Leftrightarrow A_{3i} D_3 (A_{11} A_{11}^* M_1 D_3)^\dagger A_{11} A_{11}^* A_{11} = A_{3i} D_3^{-1/2} (A_{11}^{-1} M_1 D_3^{-1/2})^\dagger, \quad i = 1, 2.$$

Using the method described in the proof of Theorem 3.1(phase $(e) \Leftrightarrow (f)$), it is easy to conclude $(c) \Leftrightarrow (e)$ and $(d) \Leftrightarrow (e)$. □

Theorem 3.4. *The following statements are equivalent (provided that we apply the Moore-Penrose inverse to closed range operators):*

(a) $M^\dagger = ((A_3^*A_3)^\dagger)^2(((A_1A_1^*)^2)^\dagger M((A_3^*A_3)^2)^\dagger)^\dagger((A_1A_1^*)^\dagger)^2$;

(b) $M^\dagger = (A_3^*A_3)^2((A_1A_1^*)^2M(A_3^*A_3)^2)^\dagger(A_1A_1^*)^2$;

(c) $\mathcal{R}((A_1A_1^*)^4M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^4M^*) = \mathcal{R}(M^*)$.

Proof. From Theorem 2.7 and Theorem 2.8 it follows that $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. $\square$

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Dragan S. Djordjević
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
E-mail: `dragan@pmf.ni.ac.rs` `dragandjordjevic70@gmail.com`

Nebojša Č. Dinčić
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
E-mail: `ndincic@hotmail.com`