Further results on the reverse order law for the Moore-Penrose inverse in rings with involution

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Abstract

We present some equivalent conditions of the reverse order law for the Moore–Penrose inverse in rings with involution, extending some well-known results to more general settings. Then we apply this result to obtain a set of equivalent conditions to the reverse order rule for the weighted Moore-Penrose inverse in C^*-algebras.

Key words and phrases: Moore-Penrose inverse, reverse order law, ring with involution.

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1 Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. If $a, b \in \mathcal{R}$ are invertible, then ab is invertible too and the inverse of the product ab satisfied the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. This formula cannot trivially be extended to the Moore–Penrose inverse of the product ab. In this paper we study necessary and sufficient conditions for the reverse order law for the Moore–Penrose inverse in the setting of rings with involution. The equivalent conditions for the reverse order law for the weighted Moore–Penrose inverse in C^*-algebras follows as a corollary.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ is self-adjoint (or Hermitian) if $a^* = a$. An element $a \in \mathcal{R}$ is regular if there exists some $b \in \mathcal{R}$ satisfying $aba = a$.

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The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $b \in \mathcal{R}$, if the following equations hold [13]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one b such that above conditions hold and such b is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

If a is a linear bounded operator between two Hilbert spaces, then $a^\dagger$ exists if and only if the range space of a is closed.

If $\delta \subset \{1,2,3,4\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1,2,3,4\} = \{a^\dagger\}$.

Definition 1.1. Let $\mathcal{R}$ be a ring with involution and let e, f two invertible Hermitian elements in $\mathcal{R}$. We say that the element $a \in \mathcal{R}$ has the weighted MP-inverse with weights e, f if there exists $b \in \mathcal{R}$ such that

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3')\ (eba)^* = eba, \quad (4')\ (fab)^* = fab.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^\dagger_{e,f}$ if it exists [10]. The set of all weighted MP-invertible elements of $\mathcal{R}$ with weights e, f will be denoted by $\mathcal{R}^\dagger_{e,f}$.

The reverse order law for the Moore-Penrose inverse is a useful computational tool in applications (solving linear equations in linear algebra or numerical analysis), and it is also interesting from the theoretical point of view.

Greville [6] proved that $(ab)^\dagger = b^\dagger a^\dagger$ holds for complex matrices, if and only if: $a^\dagger a$ commutes with bb^*, and $bb^\dagger$ commutes with a^*a. In the case of linear bounded operators on Hilbert spaces, the analogous result was proved by Bouldin [1], [2] and Izumino [9]. The corresponding result in rings with involution (using an extra assumption: $a^*a = 0$ implies $a = 0$) was proved in [10]. Detailed analysis of the reverse order law can be found in [16], and for multiple products in [8]. More results related to linear bounded operators on Hilbert spaces can be found in [3] and [4]. Recently, many results concerning the reverse order law for complex matrices appeared in Tian's papers [14] and [15], and Tian used finite dimensional methods (mostly properties of the rank of a complex matrices). In [10], [11] and [12] it is shown that the existence of the involution as a powerful tool in rings.

In this paper we present new results for the reverse order law for the Moore-Penrose inverse in rings with involution. Thus, we extend the results from [15] to more general settings.

The paper is organized as follows. We finish Section 1 by listing the most important properties of the MP-inverse. These properties will be used later, in proving our main results. Section 2 contains various equivalent conditions such that the reverse order law holds for the Moore-Penrose inverse. Although these results are known for complex rectangular matrices ([15]), we present new methods, depending on algebraic properties of rings with involution. As a corollary we obtain necessary and sufficient conditions to the reverse order law for the weighted Moore–Penrose inverse.

Now, we state some auxiliary results on the MP-inverse.

Theorem 1.1. [5, 12] *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;

(e) $a^* = a^\dagger aa^* = a^*aa^\dagger$;

(f) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$;

(g) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

Notice that if $a = a^* \in \mathcal{R}^\dagger$, then $aa^\dagger = a^\dagger a$ (meaning that a is an EP element of $\mathcal{R}$).

Lemma 1.1. [12] *If $a \in \mathcal{R}^\dagger$, then $aa^*a \in \mathcal{R}^\dagger$ and $(aa^*a)^\dagger = a^\dagger(a^*)^\dagger a^\dagger$.*

The following result is a consequence of a direct computation.

Lemma 1.2. *If $a \in \mathcal{R}^\dagger$, then $aa^*aa^*, a^*aa^*a \in \mathcal{R}^\dagger$, $[(aa^*)^2]^\dagger = [(aa^*)^\dagger]^2 = [(a^*)^\dagger a^\dagger]^2$ and $[(a^*a)^2]^\dagger = [(a^*a)^\dagger]^2 = [a^\dagger(a^*)^\dagger]^2$.*

Let $\mathcal{A}$ be a unital C^*–algebra. We state the following theorem.

Theorem 1.2. [7] *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

It is useful to express the weighted MP-inverse in terms of the ordinary MP-inverse.

Theorem 1.3. [10] *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then the unique weighted MP-inverse $a^\dagger_{e,f}$ exists and*

$$a^\dagger_{e,f} = e^{-1/2}(f^{1/2}ae^{-1/2})^\dagger f^{1/2}.$$

2 Reverse order law for the MP-inverse

In this section we present necessary and sufficient conditions such that the reverse order law for the Moore–Penrose inverse holds.

Theorem 2.1. *Let $\mathcal{R}$ be a ring with involution, and let $a, b \in \mathcal{R}^\dagger$. Then the following conditions are equivalent:*

(a) $ab \in \mathcal{R}^\dagger$ *and* $(ab)^\dagger = b^\dagger a^\dagger$;

(b) $ab, a^\dagger ab \in \mathcal{R}^\dagger$, $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$ *and* $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$;

(c) $ab, abb^\dagger \in \mathcal{R}^\dagger$, $(ab)^\dagger = b^\dagger (abb^\dagger)^\dagger$ *and* $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$;

(d) $ab, a^*ab \in \mathcal{R}^\dagger$, $(ab)^\dagger = (a^*ab)^\dagger a^*$ *and* $(a^*ab)^\dagger = b^\dagger (a^*a)^\dagger$;

(e) $ab, abb^* \in \mathcal{R}^\dagger$, $(ab)^\dagger = b^*(abb^*)^\dagger$ *and* $(abb^*)^\dagger = (bb^*)^\dagger a^\dagger$;

(f) $ab, a^*abb^* \in \mathcal{R}^\dagger$, $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$ *and* $(a^*abb^*)^\dagger = (bb^*)^\dagger (a^*a)^\dagger$;

(g) $ab, aa^*abb^*b \in \mathcal{R}^\dagger$, $(ab)^\dagger = b^*b(aa^*abb^*b)^\dagger aa^*$ *and* $(aa^*abb^*b)^\dagger = (bb^*b)^\dagger (aa^*a)^\dagger$;

(h) $ab, (a^*a)^2(bb^*)^2 \in \mathcal{R}^\dagger$, $(ab)^\dagger = b^*bb^*[(a^*a)^2(bb^*)^2]^\dagger a^*aa^*$ *and* $[(a^*a)^2(bb^*)^2]^\dagger = [(bb^*)^2]^\dagger [(a^*a)^2]^\dagger$;

(i) $(a^\dagger)^*b \in \mathcal{R}^\dagger$ *and* $[(a^\dagger)^*b]^\dagger = b^\dagger a^*$;

(j) $a(b^\dagger)^* \in \mathcal{R}^\dagger$ *and* $[a(b^\dagger)^*]^\dagger = b^*a^\dagger$.

Proof. (a) $\Rightarrow$ (b): Since $(ab)^\dagger = b^\dagger a^\dagger$, it follows that

$$a^\dagger ab\Big(b^\dagger a^\dagger a\Big)a^\dagger ab = a^\dagger(abb^\dagger a^\dagger ab) = a^\dagger ab,$$

$$b^\dagger a^\dagger a\Big(a^\dagger ab\Big)b^\dagger a^\dagger a = (b^\dagger a^\dagger abb^\dagger a^\dagger)a = b^\dagger a^\dagger a,$$

$$(a^\dagger abb^\dagger a^\dagger a)^* = a^\dagger abb^\dagger a^\dagger a,$$

$$(b^\dagger a^\dagger aa^\dagger ab)^* = (b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab = b^\dagger a^\dagger aa^\dagger ab.$$

Hence, $a^\dagger ab \in \mathcal{R}^\dagger$ and $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$. Then we have $(ab)^\dagger = b^\dagger a^\dagger = (b^\dagger a^\dagger a)a^\dagger = (a^\dagger ab)^\dagger a^\dagger$.

(b) $\Rightarrow$ (c): From $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$ and $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$, we get

$$(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger.$$

Further,

$$abb^\dagger\Big(bb^\dagger a^\dagger\Big)abb^\dagger = (abb^\dagger a^\dagger ab)b^\dagger = abb^\dagger,$$

$$bb^\dagger a^\dagger\Big(abb^\dagger\Big)bb^\dagger a^\dagger = b(b^\dagger a^\dagger abb^\dagger a^\dagger) = bb^\dagger a^\dagger,$$

$$(abb^\dagger bb^\dagger a^\dagger)^* = (abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger = abb^\dagger bb^\dagger a^\dagger,$$

$$(bb^\dagger a^\dagger abb^\dagger)^* = bb^\dagger a^\dagger abb^\dagger.$$

So, $abb^\dagger \in \mathcal{R}^\dagger$ and $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$. Now $(ab)^\dagger = b^\dagger a^\dagger = b^\dagger(bb^\dagger a^\dagger) = b^\dagger(abb^\dagger)^\dagger$.

(c) $\Rightarrow$ (d): If $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger$ and $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$, we deduce

$$(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger = b^\dagger bb^\dagger a^\dagger = b^\dagger a^\dagger.$$

By Theorem 1.1, we obtain

$$a^*ab\Big(b^\dagger a^\dagger(a^\dagger)^*\Big)a^*ab = a^*abb^\dagger a^\dagger aa^\dagger ab = a^*(abb^\dagger a^\dagger ab) = a^*ab,$$

$$b^\dagger a^\dagger(a^\dagger)^*\Big(a^*ab\Big)b^\dagger a^\dagger(a^\dagger)^* = (b^\dagger a^\dagger abb^\dagger a^\dagger)(a^\dagger)^* = b^\dagger a^\dagger(a^\dagger)^*,$$

$$(a^*abb^\dagger a^\dagger(a^\dagger)^*)^* = a^\dagger abb^\dagger a^\dagger a = (a^\dagger abb^\dagger a^\dagger a)^* = a^*abb^\dagger a^\dagger(a^\dagger)^*,$$

$$(b^\dagger a^\dagger(a^\dagger)^*a^*ab)^* = (b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab = b^\dagger a^\dagger(a^\dagger)^*a^*ab.$$

Thus, $a^*ab \in \mathcal{R}^\dagger$ and $(a^*ab)^\dagger = b^\dagger a^\dagger(a^\dagger)^* = b^\dagger(a^*a)^\dagger$, by Theorem 1.1. Then $(ab)^\dagger = b^\dagger a^\dagger = b^\dagger a^\dagger aa^\dagger = (b^\dagger a^\dagger(a^\dagger)^*)a^* = (a^*ab)^\dagger a^*$.

(d) $\Rightarrow$ (e): Suppose that (d) holds. By Theorem 1.1, we get

$$(ab)^\dagger = (a^*ab)^\dagger a^* = b^\dagger(a^*a)^\dagger a^* = b^\dagger a^\dagger$$

which implies

$$abb^*\Big((b^\dagger)^*b^\dagger a^\dagger\Big)abb^* = abb^\dagger bb^\dagger a^\dagger abb^* = (abb^\dagger a^\dagger ab)b^* = abb^*,$$

$$(b^\dagger)^*b^\dagger a^\dagger\Big(abb^*\Big)(b^\dagger)^*b^\dagger a^\dagger = (b^\dagger)^*(b^\dagger a^\dagger abb^\dagger a^\dagger) = (b^\dagger)^*b^\dagger a^\dagger,$$

$$(abb^*(b^\dagger)^*b^\dagger a^\dagger)^* = (abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger = abb^*(b^\dagger)^*b^\dagger a^\dagger,$$

$$((b^\dagger)^*b^\dagger a^\dagger abb^*)^* = bb^\dagger a^\dagger abb^\dagger = (bb^\dagger a^\dagger abb^\dagger)^* = (b^\dagger)^*b^\dagger a^\dagger abb^*,$$

i.e. $abb^* \in \mathcal{R}^\dagger$ and $(abb^*)^\dagger = (b^\dagger)^*b^\dagger a^\dagger = (bb^*)^\dagger a^\dagger$. Therefore, $(ab)^\dagger = b^\dagger a^\dagger = b^\dagger bb^\dagger a^\dagger = b^*((b^\dagger)^*b^\dagger a^\dagger) = b^*(abb^*)^\dagger$.

(e) $\Rightarrow$ (f): From the conditions in (e) and Theorem 1.1, we obtain

$$(ab)^\dagger = b^*(abb^*)^\dagger = b^*(bb^*)^\dagger a^\dagger = b^\dagger a^\dagger.$$

Using this equality and Theorem 1.1, we get

$$\begin{aligned} a^*abb^*\Big((b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*\Big)a^*abb^* &= a^*abb^\dagger bb^\dagger a^\dagger aa^\dagger abb^* \\ &= a^*(abb^\dagger a^\dagger ab)b^* = a^*abb^*, \end{aligned}$$

$$\begin{aligned} (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*\Big(a^*abb^*\Big)(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^* &= (b^\dagger)^*(b^\dagger a^\dagger abb^\dagger a^\dagger)(a^\dagger)^* \\ &= (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*, \end{aligned}$$

i.e. $(bb^*)^\dagger(a^*a)^\dagger = (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^* \in (a^*abb^*)\{1,2\}$. Since the elements $a^\dagger abb^\dagger a^\dagger a$, $bb^\dagger a^\dagger abb^\dagger$ are self-adjoint and

$$(a^*abb^*(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*)^* = (a^*abb^\dagger a^\dagger(a^\dagger)^*)^* = a^\dagger abb^\dagger a^\dagger a,$$

$$((b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^*abb^*)^* = ((b^\dagger)^*b^\dagger a^\dagger abb^*)^* = bb^\dagger a^\dagger abb^\dagger,$$

we conclude that the elements $a^*abb^*(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*$, $(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^*abb^*$ are self-adjoint, too. Hence, $a^*abb^* \in \mathcal{R}^\dagger$ and $(a^*abb^*)^\dagger = (b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^* = (bb^*)^\dagger(a^*a)^\dagger$. Now,

$$(ab)^\dagger = b^\dagger a^\dagger = b^\dagger bb^\dagger a^\dagger aa^\dagger = b^*((b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*)a^* = b^*(a^*abb^*)^\dagger a^*.$$

(f) $\Rightarrow$ (g): By the hypothesis (f) and Theorem 1.1, we have

$$(ab)^\dagger = b^*(a^*abb^*)^\dagger a^* = b^*(bb^*)^\dagger(a^*a)^\dagger a^* = b^\dagger a^\dagger.$$

Set $x = aa^*abb^*b$ and $y = b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger$.

We compute as follows:

$$\begin{aligned} xyx &= aa^*abb^*b\Big(b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger\Big)aa^*abb^*b \\ &= aa^*(abb^\dagger a^\dagger ab)b^*b \\ &= aa^*abb^*b = x, \end{aligned}$$

and

$$\begin{aligned} yxy &= b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger\Big(aa^*abb^*b\Big)b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger \\ &= b^\dagger(b^\dagger)^*(b^\dagger a^\dagger abb^\dagger a^\dagger)(a^\dagger)^*a^\dagger \\ &= b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger = y. \end{aligned}$$

Hence, $y \in x\{1,2\}$. From the equalities

$$\begin{aligned}(aa^*abb^*bb^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger)^* &= (aa^*abb^\dagger a^\dagger(a^\dagger)^*a^\dagger)^* = (a^\dagger)^*a^\dagger abb^\dagger a^\dagger aa^* \\ &= (a^\dagger)^*bb^\dagger a^* = (abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger,\end{aligned}$$

$$\begin{aligned}(b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger aa^*abb^*b)^* &= (b^\dagger(b^\dagger)^*b^\dagger a^\dagger abb^*b)^* = b^*bb^\dagger a^\dagger abb^\dagger(b^\dagger)^* \\ &= b^*a^\dagger a(b^\dagger)^* = (b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab,\end{aligned}$$

we have $y \in x\{3,4\}$. Hence, $x \in \mathcal{R}^\dagger$ and $x^\dagger = y$. By Lemma 1.1, it follows $(aa^*abb^*b)^\dagger = (bb^*b)^\dagger(aa^*a)^\dagger$. Then,

$$(ab)^\dagger = b^\dagger a^\dagger = b^\dagger bb^\dagger a^\dagger aa^\dagger = b^*b(b^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger)aa^* = b^*b(aa^*abb^*b)^\dagger aa^*.$$

(g) $\Rightarrow$ (h): From the conditions in (g) and Lemma 1.1, we obtain

$$(ab)^\dagger = b^*b(bb^*b)^\dagger(aa^*a)^\dagger aa^* = b^*bb^\dagger(b^\dagger)^*b^\dagger a^\dagger(a^\dagger)^*a^\dagger aa^* = b^\dagger a^\dagger.$$

Let $x = (a^*a)^2(bb^*)^2$ and $y = [(b^*)^\dagger b^\dagger]^2[a^\dagger(a^*)^\dagger]^2$. Then:

$$\begin{aligned}xyx &= a^*aa^*abb^*bb^*(b^*)^\dagger b^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger(a^*)^\dagger a^*aa^*abb^*bb^* \\ &= a^*aa^*abb^*bb^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger aa^*abb^*bb^* \\ &= a^*aa^*(abb^\dagger a^\dagger ab)b^*bb^* \\ &= a^*aa^*abb^*bb^* = x,\end{aligned}$$

and

$$\begin{aligned}yxy &= (b^*)^\dagger b^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger(a^*)^\dagger a^*aa^*abb^*bb^*(b^*)^\dagger b^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger(a^*)^\dagger \\ &= (b^*)^\dagger b^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger aa^*abb^*bb^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger(a^*)^\dagger \\ &= (b^*)^\dagger b^\dagger(b^*)^\dagger(b^\dagger a^\dagger abb^\dagger a^\dagger)(a^*)^\dagger a^\dagger(a^*)^\dagger \\ &= (b^*)^\dagger b^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger(a^*)^\dagger = y.\end{aligned}$$

Thus, $y \in x\{1,2\}$. From the hypothesis the elements $abb^\dagger a^\dagger$, $b^\dagger a^\dagger ab$ are self-adjoint and then

$$\begin{aligned}(xy)^* &= (a^*aa^*abb^*bb^*(b^*)^\dagger b^\dagger(b^*)^\dagger b^\dagger a^\dagger(a^*)^\dagger a^\dagger(a^*)^\dagger)^* \\ &= (a^*aa^*(abb^\dagger a^\dagger)(a^*)^\dagger a^\dagger(a^*)^\dagger)^* = a^\dagger(a^*)^\dagger a^\dagger abb^\dagger a^\dagger aa^*a \\ &= a^\dagger(a^*)^\dagger bb^\dagger a^*a = a^\dagger(abb^\dagger a^\dagger)^*a \\ &= a^\dagger abb^\dagger a^\dagger a\end{aligned}$$

and

$$\begin{aligned}
(yx)^* &= ((b^*)^\dagger b^\dagger (b^*)^\dagger b^\dagger a^\dagger (a^*)^\dagger a^\dagger (a^*)^\dagger a^* a a^* a b b^* b b^*)^* \\
&= ((b^*)^\dagger b^\dagger (b^*)^\dagger (b^\dagger a^\dagger ab) b^* b b^*)^* = b b^* b b^\dagger a^\dagger a b b^\dagger (b^*)^\dagger b^\dagger \\
&= b b^* a^\dagger a (b^*)^\dagger b^\dagger = b(b^\dagger a^\dagger ab)^* b^\dagger \\
&= b b^\dagger a^\dagger a b b^\dagger.
\end{aligned}$$

By these equalities, since the elements $a^\dagger abb^\dagger a^\dagger a$, $bb^\dagger a^\dagger abb^\dagger$ are self-adjoint, we conclude that $y \in x\{3,4\}$. Thus, we have $x \in \mathcal{R}^\dagger$, $x^\dagger = y$ and, by Lemma 1.2, $[(a^*a)^2(bb^*)^2]^\dagger = [(bb^*)^2]^\dagger[(a^*a)^2]^\dagger$. Now,

$$\begin{aligned}
(ab)^\dagger &= b^\dagger a^\dagger = b^* b b^* ((b^\dagger)^* b^\dagger (b^\dagger)^* b^\dagger a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^*) a^* a a^* \\
&= b^* b b^* [(a^*a)^2(bb^*)^2]^\dagger a^* a a^*.
\end{aligned}$$

(h) $\Rightarrow$ (i): By (h) and Lemma 1.2, we get

$$(ab)^\dagger = b^*bb^*[(bb^*)^2]^\dagger[(a^*a)^2]^\dagger a^*aa^* = b^*bb^*[(bb^*)^\dagger]^2[(a^*a)^\dagger]^2 a^*aa^* = b^\dagger a^\dagger.$$

We obtain

$$(a^\dagger)^* bb^\dagger a^* (a^\dagger)^* b = (a^\dagger)^* a^\dagger (abb^\dagger a^\dagger ab) = (a^\dagger)^* a^\dagger ab = (a^\dagger)^* b,$$

$$b^\dagger a^* (a^\dagger)^* bb^\dagger a^* = (b^\dagger a^\dagger abb^\dagger a^\dagger) aa^* = b^\dagger a^\dagger aa^* = b^\dagger a^\dagger,$$

$$((a^\dagger)^* bb^\dagger a^*)^* = abb^\dagger a^\dagger \text{ is self}-\text{adjoint},$$

$$b^\dagger a^* (a^\dagger)^* b = b^\dagger a^\dagger ab \text{ is self}-\text{adjoint}.$$

Hence, $(a^\dagger)^* b \in \mathcal{R}^\dagger$ and $[(a^\dagger)^* b]^\dagger = b^\dagger a^*$.

(i) $\Rightarrow$ (a): Using (i), we get the following:

$$abb^\dagger a^\dagger ab = aa^\dagger abb^\dagger a^* (a^\dagger)^* b = aa^*((a^\dagger)^* bb^\dagger a^* (a^\dagger)^* b) = aa^* (a^\dagger)^* b = ab,$$

$$b^\dagger a^\dagger abb^\dagger a^\dagger = (b^\dagger a^* (a^\dagger)^* bb^\dagger a^*)(a^\dagger)^* a^\dagger = b^\dagger a^* (a^\dagger)^* a^\dagger = b^\dagger a^\dagger,$$

$$(abb^\dagger a^\dagger)^* = (a^\dagger)^* bb^\dagger a^* \text{ is self}-\text{adjoint},$$

$$b^\dagger a^\dagger ab = b^\dagger a^* (a^\dagger)^* b \text{ is self}-\text{adjoint}.$$

Thus, $ab \in \mathcal{R}^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger$.

(h) $\Rightarrow$ (j) $\Rightarrow$ (a): This part can be proved in the same way as (h) $\Rightarrow$ (i) $\Rightarrow$ (a).

$\square$

Finally we formulate the following conjecture which is known to hold for complex matrices [15].

Conjecture. Let $\mathcal{R}$ be a ring with involution, and let $a, b, ab \in \mathcal{R}^\dagger$. Then the following statements are equivalent:

(a) $(ab)^\dagger = b^\dagger a^\dagger$;

(b) $(ab)^\dagger = b^\dagger a^\dagger abb^\dagger a^\dagger$.

Applying Theorem 2.1 for $a = b$, necessary and sufficient conditions for a to be *bi-dagger*, that is $(a^2)^\dagger = (a^\dagger)^2$, follow as a corollary.

Corollary 2.1. *Let $\mathcal{R}$ be a ring with involution, and let $a \in \mathcal{R}^\dagger$. Then the following conditions are equivalent:*

(a) $a^2 \in \mathcal{R}^\dagger$ *and* $(a^2)^\dagger = (a^\dagger)^2$*;*

(b) $a^2, a^\dagger a^2 \in \mathcal{R}^\dagger$, $(a^2)^\dagger = (a^\dagger a^2)^\dagger a^\dagger$ *and* $(a^\dagger a^2)^\dagger = (a^\dagger)^2 a$*;*

(c) $a^2, a^2 a^\dagger \in \mathcal{R}^\dagger$, $(a^2)^\dagger = a^\dagger (a^2 a^\dagger)^\dagger$ *and* $(a^2 a^\dagger)^\dagger = a(a^\dagger)^2$*;*

(d) $a^2, a^* a^2 \in \mathcal{R}^\dagger$, $(a^2)^\dagger = (a^* a^2)^\dagger a^*$ *and* $(a^* a^2)^\dagger = a^\dagger (a^* a)^\dagger$*;*

(e) $a^2, a^2 a^* \in \mathcal{R}^\dagger$, $(a^2)^\dagger = a^* (a^2 a^*)^\dagger$ *and* $(a^2 a^*)^\dagger = (aa^*)^\dagger a^\dagger$*;*

(f) $a^2, a^* a^2 a^* \in \mathcal{R}^\dagger$, $(a^2)^\dagger = a^* (a^* a^2 a^*)^\dagger a^*$ *and* $(a^* a^2 a^*)^\dagger = (aa^*)^\dagger (a^* a)^\dagger$*;*

(g) $a^2, (aa^* a)^2 \in \mathcal{R}^\dagger$, $(a^2)^\dagger = a^* a[(aa^* a)^2]^\dagger aa^*$ *and* $[(aa^* a)^2]^\dagger = [(aa^* a)^\dagger]^2$*;*

(h) $a^2, (a^* a)^2 (aa^*)^2 \in \mathcal{R}^\dagger$, $(a^2)^\dagger = a^* aa^* [(a^* a)^2 (aa^*)^2]^\dagger a^* aa^*$ *and* $[(a^* a)^2 (aa^*)^2]^\dagger = [(aa^*)^2]^\dagger [(a^* a)^2]^\dagger$*;*

(i) $(a^\dagger)^* a \in \mathcal{R}^\dagger$ *and* $[(a^\dagger)^* a]^\dagger = a^\dagger a^*$*;*

(j) $a(a^\dagger)^* \in \mathcal{R}^\dagger$ *and* $[a(a^\dagger)^*]^\dagger = a^* a^\dagger$*.*

We can also consider the reverse order law for the weighted Moore-Penrose inverse. Using the results from Theorem 2.1, we now can establish various equivalent conditions related to the weighted MP-inverse of a product of elements in C^*–algebra.

Let e, f be positive invertible elements of a unital C^*–algebra $\mathcal{A}$ and define $x^{*e,f} = e^{-1} x^* f$.

Corollary 2.2. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b \in \mathcal{A}$ are regular, then the following conditions are equivalent:*

(a) *ab is regular and $(ab)^\dagger_{e,h} = b^\dagger_{e,f} a^\dagger_{f,h}$;*

(b) *$ab, a^\dagger_{f,h}ab$ are regular, $(ab)^\dagger_{e,h} = (a^\dagger_{f,h}ab)^\dagger_{e,f} a^\dagger_{f,h}$ and $(a^\dagger_{f,h}ab)^\dagger_{e,f} = b^\dagger_{e,f} a^\dagger_{f,h} a$;*

(c) *$ab, abb^\dagger_{e,f}$ are regular, $(ab)^\dagger_{e,h} = b^\dagger_{e,f}(abb^\dagger_{e,f})^\dagger_{f,h}$ and $(abb^\dagger_{e,f})^\dagger_{f,h} = bb^\dagger_{e,f} a^\dagger_{f,h}$;*

(d) *$ab, a^{*f,h}ab$ are regular, $(ab)^\dagger_{e,h} = (a^{*f,h}ab)^\dagger_{e,f} a^{*f,h}$ and $(a^{*f,h}ab)^\dagger_{e,f} = b^\dagger_{e,f}(a^{*f,h}a)^\dagger_{f,f}$;*

(e) *$ab, abb^{*e,f}$ are regular, $(ab)^\dagger_{e,h} = b^{*e,f}(abb^{*e,f})^\dagger_{f,h}$ and $(abb^{*e,f})^\dagger_{f,h} = (bb^{*e,f})^\dagger_{f,f} a^\dagger_{f,h}$;*

(f) *$ab, a^{*f,h}abb^{*e,f}$ are regular, $(ab)^\dagger_{e,h} = b^{*e,f}(a^{*f,h}abb^{*e,f})^\dagger_{f,f} a^{*f,h}$ and $(a^{*f,h}abb^{*e,f})^\dagger = (bb^{*e,f})^\dagger_{f,f}(a^{*f,h}a)^\dagger_{f,f}$;*

(g) *$ab, aa^{*f,h}abb^{*e,f}b$ are regular, $(ab)^\dagger_{e,h} = b^{*e,f}b(aa^{*f,h}abb^{*e,f}b)^\dagger_{e,h} aa^{*f,h}$ and $(aa^{*f,h}abb^{*e,f}b)^\dagger_{e,h} = (bb^{*e,f}b)^\dagger_{e,f}(aa^{*f,h}a)^\dagger_{f,h}$;*

(h) *$ab, (a^{*f,h}a)^2(bb^{*e,f})^2$ are regular, $(ab)^\dagger_{e,h} = b^{*e,f}bb^{*e,f}[(a^{*f,h}a)^2(bb^{*e,f})^2]^\dagger_{f,f} \times a^{*f,h}aa^{*f,h}$ and $[(a^{*f,h}a)^2(bb^{*e,f})^2]^\dagger_{f,f} = [(bb^{*e,f})^2]^\dagger_{f,f}[(a^{*f,h}a)^2]^\dagger_{f,f}$;*

(i) *$(a^\dagger_{f,h})^{*h,f}b$ is regular and $[(a^\dagger_{f,h})^{*h,f}b]^\dagger_{e,h} = b^\dagger_{e,f} a^{*f,h}$;*

(j) *$a(b^\dagger_{e,f})^{*f,e}$ is regular and $[a(b^\dagger_{e,f})^{*f,e}]^\dagger_{e,h} = b^{*e,f} a^\dagger_{f,h}$.*

Proof. (a) $\Leftrightarrow$ (b): Suppose that $a_1 = h^{1/2}af^{-1/2}$ and $b_1 = f^{1/2}be^{-1/2}$. Then $a_1b_1 = h^{1/2}abe^{-1/2}$ and a_1, b_1, a_1b_1 are regular if and only if a, b, ab are regular, respectively. From Theorem 1.3, we have $a^\dagger_{f,h} = f^{-1/2}(h^{1/2}af^{-1/2})^\dagger h^{1/2} = f^{-1/2}a_1^\dagger h^{1/2}$, $b^\dagger_{e,f} = e^{-1/2}(f^{1/2}be^{-1/2})^\dagger f^{1/2} = e^{-1/2}b_1^\dagger f^{1/2}$ and $(ab)^\dagger_{e,h} = e^{-1/2}(a_1b_1)^\dagger h^{1/2}$.

It is easy to verify that ab is regular and $(ab)^\dagger_{e,h} = b^\dagger_{e,f}a^\dagger_{f,h}$ if and only if a_1b_1 is regular and $(a_1b_1)^\dagger = b_1^\dagger a_1^\dagger$. By Theorem 2.1, this is necessary and sufficient condition for $a_1b_1, a_1^\dagger a_1b_1$ are regular, $(a_1b_1)^\dagger = (a_1^\dagger a_1b_1)^\dagger a_1^\dagger$ and

$(a_1^\dagger a_1 b_1)^\dagger = b_1^\dagger a_1^\dagger a_1$ which is equivalent to $ab, a_{f,h}^\dagger ab$ are regular, $(ab)_{e,h}^\dagger = (a_{f,h}^\dagger ab)_{e,f}^\dagger a_{f,h}^\dagger$ and $(a_{f,h}^\dagger ab)_{e,f}^\dagger = b_{e,f}^\dagger a_{f,h}^\dagger a$.

The rest of the proof follows analogously. □

3 Conclusions

In this paper we consider some necessary and sufficient conditions for the reverse order law for the Moore–Penrose inverse in rings with involution. Applying this result we obtain the equivalent conditions for the reverse order rule for the weighted Moore-Penrose inverse of elements in C^*-algebras. All of these results are already known for complex matrices. In this case the authors used finite dimensional methods and in particular the matrix rank to prove equivalent conditions for the reverse order law. However, we applied purely algebraic techniques in proving the results which then apply to much more general setting of rings with involution. It would be interesting to extend this work to the Moore–Penrose inverse and the weighted Moore-Penrose inverse of a triple product.

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