WEIGHTED–EP ELEMENTS IN C^*-ALGEBRAS*

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Abstract. The weighted-EP elements in C*-algebras are defined and characterized.

Key words. EP elements, Moore–Penrose inverse, group inverse, C^*-algebra.

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1. Introduction. There are many equivalent characterizations of EP elements in a ring or C^*-algebra (see, for example, [10, 19, 21, 23, 24, 27]), many more still for Banach or Hilbert space operators and matrices (see [1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 18, 22, 26]). In [30], Tian and Wang defined weighted–EP matrices and presented characterizations of weighted–EP matrices using various rank formulas for matrices. In this paper, weighted–EP elements of C^*-algebras are studied using different methods, extending the results from [30] to more general settings.

Let $\mathcal{A}$ be a unital C^*–algebra with the unit 1. An element $a \in \mathcal{A}$ is *regular* if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. The set of all regular elements of $\mathcal{A}$ is denoted by $\mathcal{A}^-$. An element $a \in \mathcal{A}$ satisfying $a^* = a$ is called *symmetric* (or *Hermitian*). An element $x \in \mathcal{A}$ is positive if $x = y^*y$ for some $y \in \mathcal{A}$. Alternatively, $x \in \mathcal{A}$ is positive if x is Hermitian and $\sigma(x) \subseteq [0, +\infty)$, where the spectrum of element x is denoted by $\sigma(x)$.

An element $a \in \mathcal{A}$ is *group invertible* if there exists $a^\# \in \mathcal{A}$ such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad aa^\# = a^\#a.$$

Recall that $a^\#$ is uniquely determined by these equations. The group inverse $a^\#$ exists if and only if $a\mathcal{A} = a^2\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^2$ if and only if $a \in a^2\mathcal{A} \cap \mathcal{A}a^2$ (see [12, 28]). We use $\mathcal{A}^\#$ to denote the set of all group invertible elements of $\mathcal{A}$. The group inverse $a^\#$ double commutes with a, that is, $ax = xa$ implies $a^\#x = xa^\#$ [6, 11].

An element $a^\dagger \in \mathcal{A}$ is the *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{A}$, if

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the following hold [29]:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that above conditions hold (see [14, 17]). The set of all Moore–Penrose invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^\dagger$.

THEOREM 1.1. [14] *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

DEFINITION 1.2. Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. The element $a \in \mathcal{A}$ has *the weighted MP-inverse* with weights e, f if there exists $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^\dagger_{e,f}$ if it exists [6]. The set of all weighted MP-invertible elements of $\mathcal{A}$ with weights e, f, will be denoted by $\mathcal{A}^\dagger_{e,f}$.

THEOREM 1.3. [6] *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then the unique weighted MP-inverse $a^\dagger_{e,f}$ exists and*

$$a^\dagger_{e,f} = f^{-1/2}(e^{1/2}af^{-1/2})^\dagger e^{1/2}.$$

Define the mapping $x \mapsto x^{*e,f} = e^{-1}x^*f$, for all $x \in \mathcal{A}$. Notice that $(*, e, f) : \mathcal{A} \to \mathcal{A}$ is not an involution, because in general $(xy)^{*e,f} \neq y^{*e,f}x^{*e,f}$. Now, we formulate the following result which can be proved directly by the definition of the weighted MP-inverse.

THEOREM 1.4. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be positive invertible elements of $\mathcal{A}$. For any $a \in \mathcal{A}^-$, the following is satisfied:*

(a) $(a^\dagger_{e,f})^\dagger_{f,e} = a$;
(b) $(a^{*f,e})^\dagger_{f,e} = (a^\dagger_{e,f})^{*e,f}$;
(c) $a^{*f,e} = a^\dagger_{e,f}aa^{*f,e} = a^{*f,e}aa^\dagger_{e,f}$;
(d) $a^{*f,e}(a^\dagger_{e,f})^{*e,f} = a^\dagger_{e,f}a$;
(e) $(a^\dagger_{e,f})^{*e,f}a^{*f,e} = aa^\dagger_{e,f}$;
(f) $(a^{*f,e}a)^\dagger_{f,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}$;
(g) $(aa^{*f,e})^\dagger_{e,e} = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}$;
(h) $a^\dagger_{e,f} = (a^{*f,e}a)^\dagger_{f,f}a^{*f,e} = a^{*f,e}(aa^{*f,e})^\dagger_{e,e}$;
(i) $(a^{*e,f})^\dagger_{f,e} = a(a^{*f,e}a)^\dagger_{f,f} = (aa^{*f,e})^\dagger_{e,e}a$.

For $a \in \mathcal{A}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{A} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{A} : xa = 0\}.$$

Notice that,

$$(a^*)^\circ = a^\circ \Leftrightarrow {}^\circ(a^*) = {}^\circ a, \qquad\qquad a\mathcal{A} = a^*\mathcal{A} \Leftrightarrow \mathcal{A}a = \mathcal{A}a^*.$$

LEMMA 1.5. [10] *For $a \in \mathcal{A}$, $a \in \mathcal{A}^- \Leftrightarrow \mathcal{A} = (a^*\mathcal{A}) \oplus a^\circ$.*

The following result is very useful in the rest of paper and can be checked using properties of the weighted MP-inverse.

LEMMA 1.6. *Let $a \in \mathcal{A}^-$, and let e, f be invertible positive elements in $\mathcal{A}$. Then*

(a) $a^\dagger_{e,f}\mathcal{A} = a^\dagger_{e,f}a\mathcal{A} = f^{-1}a^*\mathcal{A}$;
(b) $(a^\dagger_{e,f})^*\mathcal{A} = (aa^\dagger_{e,f})^*\mathcal{A} = ea\mathcal{A}$;
(c) $a^\circ = (ea)^\circ$;
(d) $(a^*)^\circ = (f^{-1}a^*)^\circ$;
(e) $(a^\dagger_{e,f})^\circ = [(ea)^*]^\circ$;
(f) $[(a^\dagger_{e,f})^*]^\circ = (af^{-1})^\circ$;

Now, we state an important result related to the weighted Moore-Penrose inverse. In [19, Lemma 1.5], the following result is proved for the ordinary Moore-Penrose inverse. Observe that conditions (1.1) and (1.2) appear in the proof of [15, Theorem 10] also for the ordinary Moore-Penrose inverse.

LEMMA 1.7. *Let $a \in \mathcal{A}^-$, and let e, f be invertible positive elements in $\mathcal{A}$. Then*

$$a^\dagger_{e,f} = (a^{*f,e}a + 1 - a^\dagger_{e,f}a)^{-1}a^{*f,e} = a^{*f,e}(aa^{*f,e} + 1 - aa^\dagger_{e,f})^{-1}, \tag{1.1}$$

$$a^{*f,e}\mathcal{A}^{-1} = a^\dagger_{e,f}\mathcal{A}^{-1} \text{ and } \mathcal{A}^{-1}a^{*f,e} = \mathcal{A}^{-1}a^\dagger_{e,f}, \tag{1.2}$$

$$(a^{*f,e})^\circ = (a^\dagger_{e,f})^\circ \text{ and } {}^\circ(a^{*f,e}) = {}^\circ(a^\dagger_{e,f}). \tag{1.3}$$

Proof. By Theorem 1.4, we can verify

$$a^{*f,e} = (a^{*f,e}a + 1 - a^\dagger_{e,f}a)a^\dagger_{e,f} = a^\dagger_{e,f}(aa^{*f,e} + 1 - aa^\dagger_{e,f}),$$

$$(a^{*f,e}a + 1 - a^\dagger_{e,f}a)^{-1} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f} + 1 - a^\dagger_{e,f}a$$

and

$$(aa^{*f,e} + 1 - aa^{\dagger}_{e,f})^{-1} = (a^{\dagger}_{e,f})^{*e,f} a^{\dagger}_{e,f} + 1 - aa^{\dagger}_{e,f}.$$

Thus, the part (1.1) holds and it implies the equalities (1.2) and (1.3). □

We recall the definition of EP elements.

DEFINITION 1.8. An element $a \in \mathcal{A}^-$ is *EP* if $aa^\dagger = a^\dagger a$.

LEMMA 1.9. [19] *An element $a \in \mathcal{A}$ is EP, if $a \in \mathcal{A}^-$ and $a\mathcal{A} = a^*\mathcal{A}$ (or, equivalently, if $a \in \mathcal{A}^-$ and $a^\circ = (a^*)^\circ$).*

The condition $a\mathcal{A} = a^*\mathcal{A}$ gave the EP elements their name for *equal projections* onto the range of a and a^* in the case of matrices and closed range Hilbert space operators. These elements are important since they are characterized by commutativity with their Moore–Penrose inverse. Also notice that EP elements are those elements for which the group and the Moore–Penrose inverse exist and coincide.

In this paper, as an extension of EP elements, we are concerned with elements of a C^*-algebra which commute with their weighted Moore-Penrose inverse. These elements are called weighted–EP elements. In particular, we give several equivalent conditions for an element of C^*-algebra to be weighted–EP. The motivation for this paper is an interesting paper by Tian and Wang [30]. They studied such characterizations for weighted–EP complex square matrices.

We conclude this section with the following results on reverse order law for the Moore-Penrose inverse of a product, which be used later.

LEMMA 1.10. [25, Theorem 2.4] *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following conditions are equivalent:*

(a) $(ab)^\dagger = b^\dagger a^\dagger$;
(b) $a^*ab = bb^\dagger a^*ab$ *and* $abb^* = abb^*a^\dagger a$;

THEOREM 1.11. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then $(ab)^\dagger = b^\dagger a^\dagger$ if and only if $a^*ab\mathcal{A} \subseteq b\mathcal{A}$ and $bb^*a^*\mathcal{A} \subseteq a^*\mathcal{A}$.*

Proof. $\Longrightarrow$: If $(ab)^\dagger = b^\dagger a^\dagger$, then, by Lemma 1.10, $a^*ab = bb^\dagger a^*ab$ and $abb^* = abb^*a^\dagger a$ implying

$$bb^*a^* = (abb^*) = (abb^*a^\dagger a)^* = a^*(a^\dagger)^*bb^*a^*.$$

Hence, $a^*ab\mathcal{A} \subseteq b\mathcal{A}$ and $bb^*a^*\mathcal{A} \subseteq a^*\mathcal{A}$.

$\Longleftarrow$: Conversely, from $a^*ab\mathcal{A} \subseteq b\mathcal{A}$ and $bb^*a^*\mathcal{A} \subseteq a^*\mathcal{A}$, we conclude that $a^*ab = bx$, for some $x \in \mathcal{A}$, and $bb^*a^* = a^*y$, for some $y \in \mathcal{A}$. Then the equalities

$$bb^\dagger a^*ab = bb^\dagger bx = bx = a^*ab$$

and

$$abb^*a^\dagger a = (a^\dagger abb^*a^*)^* = (a^\dagger aa^*y)^* = (a^*y)^* = (bb^*a^*)^* = abb^*$$

imply $(ab)^\dagger = b^\dagger a^\dagger$, by Lemma 1.10. □

2. Weighted–EP elements in C^*-algebras. First, we state the definition of weighted–EP elements in C^*-algebras.

Definition 2.1. An element $a \in \mathcal{A}$ is said to be *weighted–EP* with respect to two invertible positive elements $e, f \in \mathcal{A}$ (or weighted–EP w.r.t. (e,f)) if both ea and af^{-1} are EP, that is $a \in \mathcal{A}^-$, $ea\mathcal{A} = (ea)^*\mathcal{A}$ and $af^{-1}\mathcal{A} = (af^{-1})^*\mathcal{A}$.

In the following theorem, a number of necessary and sufficient conditions for an element to be weighted–EP are presented.

Theorem 2.2. *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:*

(I) *a is weighted–EP w.r.t. (e,f);*
(II) *a is weighted–EP w.r.t. (f,e);*
(III) *a is both weighted–EP w.r.t. (e,e) and w.r.t. (f,f);*
(IV) *$ea\mathcal{A} = fa\mathcal{A} = a^*\mathcal{A}$;*
(V) *$e^{-1}a^*\mathcal{A} = f^{-1}a^*\mathcal{A} = a\mathcal{A}$;*
(VI) *$a^\dagger_{e,f}\mathcal{A} = a\mathcal{A}$ and $(a^\dagger_{e,f})^*\mathcal{A} = a^*\mathcal{A}$;*
(VII) *a^* is weighted–EP w.r.t. (e^{-1},f^{-1});*
(VIII) *$aa^\dagger_{e,f} = a^\dagger_{e,f}a$;*
(IX) *$a \in \mathcal{A}^\#$ and $a^k = a^\dagger_{e,f}aa^k = a^kaa^\dagger_{e,f}$, for any/some integer $k \geq 1$;*
(X) *$a^\dagger_{e,f} = a(a^\dagger_{e,f})^2 = (a^\dagger_{e,f})^2a$;*
(XI) *$a \in \mathcal{A}^\#$ and $a^\# = a^\dagger_{e,f}$;*
(XII) *$a \in \mathcal{A}^\#$ and both $eaa^\#$ and $faa^\#$ are Hermitian;*
(XIII) *$a \in \mathcal{A}^\#$ and $a^\#a^\dagger_{e,f} = a^\dagger_{e,f}a^\#$;*
(XIV) *$a \in \mathcal{A}^\#$ and $aa^\#a^\dagger_{e,f} = a^\dagger_{e,f}a^\#a$;*
(XV) *$a \in a^\dagger_{e,f}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}a^\dagger_{e,f}$;*
(XVI) *$a \in a^\dagger_{e,f}\mathcal{A} \cap \mathcal{A}a^\dagger_{e,f}$;*
(XVII) *$a\mathcal{A}^{-1} = f^{-1}a^*\mathcal{A}^{-1}$ and $\mathcal{A}^{-1}a = \mathcal{A}^{-1}a^*e$;*
(xviii) *$\mathcal{A}^{-1}a^* = \mathcal{A}^{-1}af^{-1}$ and $a^*\mathcal{A}^{-1} = ea\mathcal{A}^{-1}$;*
(XIX) *there exists $x \in \mathcal{A}$ such that $a = e^{-1}a^*xa^*f$;*
(XX) *$a = (ae^{-1})^\dagger ae^{-1}afa(fa)^\dagger$;*
(XXI) *$a \in \mathcal{A}^\#$ and a^k is weighted–EP w.r.t. (e,f), for any/some integer $k \geq 1$;*
(XXII) *aa^*a is weighted–EP w.r.t. (e,f);*
(XXIII) *$a^\circ = [(ea)^*]^\circ$ and $(a^*)^\circ = (af^{-1})^\circ$;*

(XXIV) $\mathcal{A} = e^{-1}a^*\mathcal{A} \oplus (a^*)^\circ = a^*\mathcal{A} \oplus (a^*f)^\circ$;

(XXV) $a^\dagger$ *is weighted–EP w.r.t.* (e^{-1}, f^{-1});

(XXVI) $a^\dagger_{e,f}$ *is weighted–EP w.r.t.* (e,f);

(XXVII) $a \in \mathcal{A}^\#$ *and* $a^{2k-1} = a^\dagger_{e,f}a^{2k+1}a^\dagger_{e,f}$, *for any/some integer* $k \geq 1$;

(XXVIII) $a \in \mathcal{A}^\#$ *and* $aa^\dagger_{e,f}a^\dagger_{e,f}a = a^\dagger_{e,f}aaa^\dagger_{e,f}$;

(XXIX) $a \in \mathcal{A}^\#$ *and* $a^\#$ *is weighted–EP w.r.t.* (e,f);

(XXX) $a \in \mathcal{A}^\#$ *and* $aa^\# = aa^\dagger_{e,e} = aa^\dagger_{f,f}$ (*or* $aa^\# = a^\dagger_{e,e}a = a^\dagger_{f,f}a$);

(XXXI) $a \in \mathcal{A}^\#$ *and* $aa^\# = aa^\dagger_{e,f} = aa^\dagger_{f,e}$ (*or* $aa^\# = a^\dagger_{f,e}a = a^\dagger_{e,f}a$);

(XXXII) $a \in \mathcal{A}^\#$, $aa^\dagger_{e,e}e^{-1}a^*a = e^{-1}a^*aaa^\dagger_{e,e}$ *and* $aa^\dagger_{f,f}f^{-1}a^*a = f^{-1}a^*aaa^\dagger_{f,f}$;

(XXXIII) $a \in \mathcal{A}^\#$, $aa^\dagger_{e,f}e^{-1}a^*a = e^{-1}a^*aaa^\dagger_{e,f}$ *and* $aa^\dagger_{f,e}f^{-1}a^*a = f^{-1}a^*aaa^\dagger_{f,e}$;

(XXXIV) $a \in \mathcal{A}^\#$, $a^\dagger_{e,e}aaa^*e = aa^*ea^\dagger_{e,e}a$ *and* $a^\dagger_{f,f}aaa^*f = aa^*fa^\dagger_{f,f}a$;

(XXXV) $a \in \mathcal{A}^\#$, $a^\dagger_{f,e}aaa^*e = aa^*ea^\dagger_{f,e}a$ *and* $a^\dagger_{e,f}aaa^*f = aa^*fa^\dagger_{e,f}a$;

(XXXVI) $a \in \mathcal{A}^\#$ *and* $a^kaa^\dagger_{e,f} + a^\dagger_{e,f}aa^k = 2a^k$, *for any/some integer* $k \geq 1$;

(XXXVII) $a \in \mathcal{A}^\#$ *and* $a^\dagger_{e,f}a^\#a + aa^\#a^\dagger_{e,f} = 2a^\dagger_{e,f}$;

(XXXVIII) $a \in \mathcal{A}^\#$ *and* $a^{*f,e} = a^{*f,e}aa^\# = a^\#aa^{*f,e}$;

(XXXIX) $a \in \mathcal{A}^\#$ *and* $a^{*f,e}aa^\# + a^\#aa^{*f,e} = 2a^{*f,e}$;

(XL) $a \in \mathcal{A}^\#$ *and* $a^kaa^\dagger_{e,f} + (a^kaa^\dagger_{e,f})^* = a^\dagger_{e,f}aa^k + (a^\dagger_{e,f}aa^k)^* = a^k + (a^k)^*$, *for any/some integer* $k \geq 1$;

(XLI) $aa^\dagger_{e,f}(a + \lambda a^\dagger_{e,f}) = (a + \lambda a^\dagger_{e,f})aa^\dagger_{e,f}$ *and* $a^\dagger_{e,f}a(a + \lambda a^\dagger_{e,f}) = (a + \lambda a^\dagger_{e,f})a^\dagger_{e,f}a$, *for any/some complex number* $\lambda \neq 0$;

(XLII) $ab = ba \ \Rightarrow \ a^\dagger_{e,f}b = ba^\dagger_{e,f}$;

(XLIII) $a^\dagger_{e,f} = f(a)$, *for some function* f *holomorphic in a neighbourhood of* $\sigma(a)$;

(XLIV) $(a + \lambda a^\dagger_{e,e})\mathcal{A} = (a + \lambda a^\dagger_{f,f})\mathcal{A} = (\lambda a + a^3)\mathcal{A}$ *and* $\mathcal{A}(a + \lambda a^\dagger_{e,e}) = \mathcal{A}(a + \lambda a^\dagger_{f,f}) = \mathcal{A}(\lambda a + a^3)$, *for any/some complex number* $\lambda \neq 0$;

(XLV) $(a + \lambda a^\dagger_{e,f})\mathcal{A} = (\lambda a + a^3)\mathcal{A}$ *and* $\mathcal{A}(a + \lambda a^\dagger_{e,f}) = \mathcal{A}(\lambda a + a^3)$, *for any/some complex number* $\lambda \neq 0$;

(XLVI) $(a + \lambda a^\dagger_{e,e})^\circ = (a + \lambda a^\dagger_{f,f})^\circ = (\lambda a + a^3)^\circ$ *and* ${}^\circ(a + \lambda a^\dagger_{e,e}) = {}^\circ(a + \lambda a^\dagger_{f,f}) = {}^\circ(\lambda a + a^3)$, *for any/some complex number* $\lambda \neq 0$;

(XLVII) $(a + \lambda a^\dagger_{e,f})^\circ = (\lambda a + a^3)^\circ$ *and* ${}^\circ(a + \lambda a^\dagger_{f,e}) = {}^\circ(\lambda a + a^3)$, *for any/some complex number* $\lambda \neq 0$;

(XLVIII) $a \in \mathcal{A}^\#$ *and* $(a^\dagger_{e,f})^2a^\# = a^\dagger_{e,f}a^\#a^\dagger_{e,f} = a^\#(a^\dagger_{e,f})^2$;

(XLIX) $a \in \mathcal{A}^\#$ *and* $a(a^\dagger_{e,f})^2 = a^\# = (a^\dagger_{e,f})^2a$;

(L) $a \in \mathcal{A}^\#$, $a^{*f,e}a^\dagger_{e,f} = a^{*f,e}a^\#$ *and* $a^\dagger_{e,f}a^{*f,e} = a^\#a^{*f,e}$;

(LI) $a \in \mathcal{A}^\#$ *and* $(a^\dagger_{e,f})^2 = (a^\#)^2$;

(LII) $a^{*e,f} = a^{*e,f}a^\dagger_{e,f}a = aa^\dagger_{e,f}a^{*e,f}$;

(LIII) $a \in \mathcal{A}^\#$ *and* $(a^\#)^{*e,f} = aa^\#(a^\#)^{*e,f} = (a^\#)^{*e,f}a^\#a$ (*or* $(a^\#)^{*f,e} = aa^\#(a^\#)^{*f,e} = (a^\#)^{*f,e}a^\#a$);

(LIV) $a \in \mathcal{A}^\#$ *and* $a^\dagger_{e,f}(a^\#)^2 = (a^\#)^2a^\dagger_{e,f}$;

(LV) $a \in \mathcal{A}^{\#}$ *and* $a^k a^{\dagger}_{e,f} = a^{\dagger}_{e,f} a^k$*, for any/some integer* $k \geq 1$*;*

(LVI) $aa^{\dagger}_{e,f}(a+\lambda a^{*e,f}) = (a+\lambda a^{*e,f})aa^{\dagger}_{e,f}$ *and* $a^{\dagger}_{e,f}a(a+\lambda a^{*e,f}) = (a+\lambda a^{*e,f})a^{\dagger}_{e,f}a$*, for any/some complex number* $\lambda \neq 0$*;*

(LVII) $a \in \mathcal{A}^{\#}$, $aa^{\dagger}_{e,e}(aa^*e - e^{-1}a^*a) = (aa^*e - e^{-1}a^*a)aa^{\dagger}_{e,e}$ *and* $aa^{\dagger}_{f,f}(aa^*f - f^{-1}a^*a) = (aa^*f - f^{-1}a^*a)aa^{\dagger}_{f,f}$*;*

(LVIII) $a \in \mathcal{A}^{\#}$, $aa^{\dagger}_{e,f}(aa^*e - e^{-1}a^*a) = (aa^*e - e^{-1}a^*a)aa^{\dagger}_{e,f}$ *and* $aa^{\dagger}_{f,e}(aa^*f - f^{-1}a^*a) = (aa^*f - f^{-1}a^*a)aa^{\dagger}_{f,e}$*;*

(LIX) $a \in \mathcal{A}^{\#}$, $a^{\dagger}_{e,e}a(aa^*e - e^{-1}a^*a) = (aa^*e - e^{-1}a^*a)a^{\dagger}_{e,e}a$ *and* $a^{\dagger}_{f,f}a(aa^*f - f^{-1}a^*a) = (aa^*f - f^{-1}a^*a)a^{\dagger}_{f,f}a$*;*

(LX) $a \in \mathcal{A}^{\#}$, $a^{\dagger}_{f,e}a(aa^*e - e^{-1}a^*a) = (aa^*e - e^{-1}a^*a)a^{\dagger}_{f,e}a$ *and* $a^{\dagger}_{e,f}a(aa^*f - f^{-1}a^*a) = (aa^*f - f^{-1}a^*a)a^{\dagger}_{e,f}a$*;*

(LXI) $a \in \mathcal{A}^{\#}$ *and* $(a^{s+t})^{\dagger} = (a^s)^{\dagger}_{e,1}(a^t)^{\dagger}_{1,e} = (a^s)^{\dagger}_{f,1}(a^t)^{\dagger}_{1,f}$*, for any/some integers* $s,t \geq 1$*;*

(LXII) $a \in \mathcal{A}^{\#}$ *and* $(a^{s+t})^{\dagger}_{e,f} = (a^s)^{\dagger}_{f,f}(a^t)^{\dagger}_{e,f} = (a^s)^{\dagger}_{e,f}(a^t)^{\dagger}_{e,e}$*, for any/some integers* $s,t \geq 1$*.*

Proof. (I) $\Rightarrow$ (II): Assume that a is weighted–EP w.r.t. (e,f), i.e. ea and af^{-1} are EP. From $ea\mathcal{A} = (ea)^*\mathcal{A}$, we obtain $a\mathcal{A} = e^{-1}a^*e\mathcal{A} = (ae^{-1})^*e\mathcal{A}$ implying $ae^{-1}e\mathcal{A} = (ae^{-1})^*e\mathcal{A}$, that is, $ae^{-1}\mathcal{A} = (ae^{-1})^*\mathcal{A}$. In the same way $af^{-1}\mathcal{A} = (af^{-1})^*\mathcal{A}$ implies $fa\mathcal{A} = (fa)^*\mathcal{A}$. Hence, fa and ae^{-1} are EP, i.e. a is weighted–EP w.r.t. (f,e).

(II) $\Rightarrow$ (I): This implication can be proved in the same way as (I) $\Rightarrow$ (II).

(III) $\Leftrightarrow$ (I): Obviously, because (I) $\Leftrightarrow$ (II).

(IV) $\Leftrightarrow$ (I): Notice that $(ea)^*\mathcal{A} = a^*\mathcal{A}$ and $af^{-1}\mathcal{A} = a\mathcal{A}$. Now, $ea\mathcal{A} = fa\mathcal{A} = a^*\mathcal{A}$ is equivalent to $ea\mathcal{A} = a^*e\mathcal{A}$ and $a\mathcal{A} = f^{-1}a^*\mathcal{A}$, that is, $ea\mathcal{A} = (ea)^*\mathcal{A}$ and $af^{-1}\mathcal{A} = (af^{-1})^*\mathcal{A}$. These equalities mean that ea and af^{-1} are EP, i.e. a is weighted–EP w.r.t. (e,f).

(V) $\Leftrightarrow$ (IV): This is easy to check.

(VI) $\Leftrightarrow$ (I): By Lemma 1.6, $a^{\dagger}_{e,f}\mathcal{A} = a\mathcal{A}$ and $(a^{\dagger}_{e,f})^*\mathcal{A} = a^*\mathcal{A}$ is equivalent to $f^{-1}a^*\mathcal{A} = a\mathcal{A}$ and $ea\mathcal{A} = a^*\mathcal{A}$ which is $(af^{-1})^*\mathcal{A} = af^{-1}\mathcal{A}$ and $ea\mathcal{A} = (ea)^*\mathcal{A}$.

(VII) $\Leftrightarrow$ (V): Using the equivalence (I) $\Leftrightarrow$ (V) for a^*, we have that a^* is weighted–EP w.r.t. (e^{-1},f^{-1}) if and only if $e^{-1}a^*\mathcal{A} = f^{-1}a^*\mathcal{A} = a\mathcal{A}$.

(VIII) $\Rightarrow$ (VI): The equality $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$ gives

$$a\mathcal{A} = a^{\dagger}_{e,f}aa\mathcal{A} \subset a^{\dagger}_{e,f}\mathcal{A} = aa^{\dagger}_{e,f}a^{\dagger}_{e,f}\mathcal{A} \subset a\mathcal{A},$$

and

$$a^*\mathcal{A} = (aaa^{\dagger}_{e,f})^*\mathcal{A} \subset (a^{\dagger}_{e,f})^*\mathcal{A} = (a^{\dagger}_{e,f}a^{\dagger}_{e,f}a)^*\mathcal{A} \subset a^*\mathcal{A},$$

i.e. $a^{\dagger}_{e,f}\mathcal{A} = a\mathcal{A}$ and $(a^{\dagger}_{e,f})^*\mathcal{A} = a^*\mathcal{A}$.

(VI) $\Rightarrow$ (IX): Since $a^{\dagger}_{e,f}\mathcal{A} = a\mathcal{A}$, then $a = a^{\dagger}_{e,f}y$ for some $y \in \mathcal{A}$. Now,

$$a = a^{\dagger}_{e,f}y = a^{\dagger}_{e,f}a(a^{\dagger}_{e,f}y) = a^{\dagger}_{e,f}aa,$$

and $a^k = a^{\dagger}_{e,f}aa^k$, for any/some integer $k \geq 1$. Analogously, the assumption $(a^{\dagger}_{e,f})^*\mathcal{A} = a^*\mathcal{A}$ implies $a^* = (a^{\dagger}_{e,f})^*x$ for some $x \in \mathcal{A}$ and

$$a^* = (a^{\dagger}_{e,f})^*x = (a^{\dagger}_{e,f}aa^{\dagger}_{e,f})^*x = (aa^{\dagger}_{e,f})^*(a^{\dagger}_{e,f})^*x = (aa^{\dagger}_{e,f})^*a^* = (aaa^{\dagger}_{e,f})^*.$$

Applying involution to this equality, we get $a = aaa^{\dagger}_{e,f}$ and, for any/some integer $k \geq 1$, $a^k = a^k aa^{\dagger}_{e,f}$. Notice that, from $a \in a^2\mathcal{A} \cap \mathcal{A}a^2$, it follows $a^{\#}$ exists.

(IX) $\Rightarrow$ (VIII): If $a \in \mathcal{A}^{\#}$ and $a^k = a^{\dagger}_{e,f}aa^k = a^k aa^{\dagger}_{e,f}$, for any/some integer $k \geq 1$, then

$$aa^{\dagger}_{e,f} = (a^{\#})^k(a^{k+1}a^{\dagger}_{e,f}) = (a^{\#})^k a^k = a^k(a^{\#})^k = a^{\dagger}_{e,f}a^{k+1}(a^{\#})^k = a^{\dagger}_{e,f}a.$$

(X) $\Rightarrow$ (VIII): Applying the equality $a^{\dagger}_{e,f} = a(a^{\dagger}_{e,f})^2 = (a^{\dagger}_{e,f})^2a$, we obtain $a^{\dagger}_{e,f}a = a((a^{\dagger}_{e,f})^2a) = aa^{\dagger}_{e,f}$.

(VIII) $\Rightarrow$ (X): Obviously.

(VIII) $\Leftrightarrow$ (XI) $\Leftrightarrow$ (XII): By the uniquely determined group and weighted-MP inverse.

(XIII) $\Rightarrow$ (VIII): The hypothesis $a^{\#}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\#}$ implies

$$aa^{\dagger}_{e,f} = a^2(a^{\#}a^{\dagger}_{e,f}) = a^2a^{\dagger}_{e,f}a^{\#} = a^2a^{\dagger}_{e,f}a(a^{\#})^2 = a^2(a^{\#})^2 = aa^{\#}$$

and

$$a^{\dagger}_{e,f}a = (a^{\dagger}_{e,f}a^{\#})a^2 = a^{\#}a^{\dagger}_{e,f}a^2 = (a^{\#})^2aa^{\dagger}_{e,f}a^2 = a^{\#}a.$$

Therefore, $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}a$.

(XI) $\Rightarrow$ (XIII): From the equality $a^{\#} = a^{\dagger}_{e,f}$, we have $a^{\#}a^{\dagger}_{e,f} = (a^{\dagger}_{e,f})^2 = a^{\dagger}_{e,f}a^{\#}$ So, the condition (XIII) holds.

(XIV) $\Rightarrow$ (VIII): Suppose that $a \in \mathcal{A}^{\#}$ and $aa^{\#}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\#}a$. Then we get the equality (VIII):

$$\begin{aligned} aa^{\dagger}_{e,f} &= a(aa^{\#}a^{\dagger}_{e,f}) = aa^{\dagger}_{e,f}a^{\#}a = aa^{\dagger}_{e,f}aa^{\#} = aa^{\#} \\ &= a^{\#}aa^{\dagger}_{e,f}a = (aa^{\#}a^{\dagger}_{e,f})a = a^{\dagger}_{e,f}a^{\#}aa = a^{\dagger}_{e,f}a. \end{aligned}$$

(XI) $\Rightarrow$ (XIV): By $a^\# = a^\dagger_{e,f}$, obviously, the condition (XIV) is satisfied.

(VIII) $\Rightarrow$ (XV): Using $aa^\dagger_{e,f} = a^\dagger_{e,f}a$, we can verify that $a = (a^2 + 1 - a^\dagger_{e,f}a)a^\dagger_{e,f}$ and $(a^2+1-a^\dagger_{e,f}a)^{-1} = (a^\dagger_{e,f})^2+1-a^\dagger_{e,f}a$. Thus, $a \in \mathcal{A}^{-1}a^\dagger_{e,f}$. Since $aa^\dagger_{e,f} = a^\dagger_{e,f}a$, by previous equalities, we conclude that $a = a^\dagger_{e,f}(a^2+1-aa^\dagger_{e,f})$ and $(a^2+1-aa^\dagger_{e,f})^{-1} = (a^\dagger_{e,f})^2 + 1 - aa^\dagger_{e,f}$ which yields $a \in a^\dagger_{e,f}\mathcal{A}^{-1}$. Hence, (XV) holds.

(XV) $\Rightarrow$ (XVI): Obviously.

(XVI) $\Rightarrow$ (IX): From the condition $a \in a^\dagger_{e,f}\mathcal{A}$, we have $a = a^\dagger_{e,f}x$, for some $x \in \mathcal{A}$, and $a^k - a^\dagger_{e,f}aa^k = (a^\dagger_{e,f} - a^\dagger_{e,f}aa^\dagger_{e,f})xa^{k-1} = 0$, for integer $k \geq 1$. In the similar way, $a \in \mathcal{A}a^\dagger_{e,f}$ gives $a^k = a^kaa^\dagger_{e,f}$, for integer $k \geq 1$. When $k = 1$, we observe that $a \in a^2\mathcal{A} \cap \mathcal{A}a^2$ and $a^\#$ exists. So, the condition (IX) holds.

(XV) $\Leftrightarrow$ (XVII): The assumption $a \in a^\dagger_{e,f}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}a^\dagger_{e,f}$ is equivalent to $a\mathcal{A}^{-1} = a^\dagger_{e,f}\mathcal{A}^{-1}$ and $\mathcal{A}^{-1}a = \mathcal{A}^{-1}a^\dagger_{e,f}$. By Lemma 1.7, we observe that these equalities hold if and only if $a\mathcal{A}^{-1} = a^{*f,e}\mathcal{A}^{-1} = f^{-1}a^*\mathcal{A}^{-1}$ and $\mathcal{A}^{-1}a = \mathcal{A}^{-1}a^{*f,e} = \mathcal{A}^{-1}a^*e$.

(XVII) $\Leftrightarrow$ (XVIII): Applying the involution, we check this equivalence.

(XIX) $\Rightarrow$ (II): Suppose that there exists $x \in \mathcal{A}$ such that $a = e^{-1}a^*xa^*f$. Then $a \in e^{-1}a^*\mathcal{A} \cap \mathcal{A}a^*f = a^{*e,f}\mathcal{A} \cap \mathcal{A}a^{*e,f} = a^\dagger_{f,e}\mathcal{A} \cap \mathcal{A}a^\dagger_{f,e}$, by Lemma 1.7. Now, by (xvi) $\Leftrightarrow$ (i), we deduce that a is weighted–EP w.r.t. (f, e).

(II) $\Rightarrow$ (XIX): If a is weighted–EP w.r.t. (f, e), by the equivalence (I) $\Leftrightarrow$ (XVI), $a \in a^\dagger_{f,e}\mathcal{A} \cap \mathcal{A}a^\dagger_{f,e} = e^{-1}a^*\mathcal{A} \cap \mathcal{A}a^*f$. Therefore, for some $y, z \in \mathcal{A}$, $a = e^{-1}a^*y = za^*f$ and $a = aa^\dagger_{f,e}a = e^{-1}a^*(ya^\dagger_{f,e}z)a^*f$. For $x = ya^\dagger_{f,e}z$, the statement (XIX) is satisfied.

(XX) $\Rightarrow$ (II): Since $a = (ae^{-1})^\dagger ae^{-1}afa(fa)^\dagger$, we conclude that $a \in (ae^{-1})^\dagger\mathcal{A} \cap \mathcal{A}(fa)^\dagger = (ae^{-1})^*\mathcal{A} \cap \mathcal{A}(fa)^* = a^{*e,f}\mathcal{A} \cap \mathcal{A}a^{*e,f} = a^\dagger_{f,e}\mathcal{A} \cap \mathcal{A}a^\dagger_{f,e}$. Using (XVI) $\Leftrightarrow$ (I), we observe that a is weighted–EP w.r.t. (f, e).

(II) $\Rightarrow$ (XX): The condition (II) implies that ae^{-1} and fa are EP and then

$$\begin{aligned}(ae^{-1})^\dagger ae^{-1}afa(fa)^\dagger &= ((ae^{-1})^\dagger ae^{-1}ae^{-1})efa(fa)^\dagger = ae^{-1}efa(fa)^\dagger \\ &= f^{-1}(fafa(fa)^\dagger) = f^{-1}fa = a.\end{aligned}$$

Thus, the condition (XX) holds.

(XXI) $\Leftrightarrow$ (XII): Applying the equivalence (I) $\Leftrightarrow$ (XII) for a^k, $k \geq 1$, we see that $a \in \mathcal{A}^\#$ and a^k is weighted–EP w.r.t. (e,f) if and only if $a \in \mathcal{A}^\#$ and $ea^k(a^k)^\#$, $fa^k(a^k)^\#$ are Hermitian which is equivalent to $a \in \mathcal{A}^\#$ and $eaa^\#$, $faa^\#$ are Hermitian, by $(a^k)^\# = (a^\#)^k$.

(XXII) $\Leftrightarrow$ (IV): Notice that, from $a \in \mathcal{A}^-$, it follows that $aa^*a \in \mathcal{A}^-$ and $(aa^*a)^\dagger = a^\dagger(a^*)^\dagger a^\dagger$. Using the equivalence (I) $\Leftrightarrow$ (IV) for aa^*a, we observe that

aa^*a is weighted–EP w.r.t. (e,f) is equivalent to $eaa^*a\mathcal{A} = faa^*a\mathcal{A} = a^*aa^*\mathcal{A}$. Since $a \in \mathcal{A}^-$, then $a^\dagger$ exists and $ea = eaa^\dagger a = eaa^*(a^\dagger)^* = eaa^*aa^\dagger(a^\dagger)^*$. Consequently, $eaa^*a\mathcal{A} = ea\mathcal{A}$ and in the same way $faa^*a\mathcal{A} = fa\mathcal{A}$. By $a^* = a^*aa^\dagger = a^*aa^\dagger aa^\dagger = a^*aa^*(a^\dagger)^*a^\dagger$, we conclude $a^*aa^*\mathcal{A} = a^*\mathcal{A}$. Hence, $eaa^*a\mathcal{A} = faa^*a\mathcal{A} = a^*aa^*\mathcal{A}$ is equivalent to (IV).

(XXIII) $\Leftrightarrow$ (I): Observe that a is weighted–EP w.r.t. (e,f) if and only if elements ea and af^{-1} are EP. By the definition of EP elements, this is equivalent to $(ea)^\circ = [(ea)^*]^\circ$ and $(af^{-1})^\circ = [(af^{-1})^*]^\circ = (f^{-1}a^*)^\circ$, which can be written as $a^\circ = [(ea)^*]^\circ$ and $(af^{-1})^\circ = (a^*)^\circ$, by Lemma 1.6.

(V) $\Rightarrow$ (XXIV): The condition (V) gives $e^{-1}a^*\mathcal{A} = a\mathcal{A}$ and its equivalent condition (IV) imply $fa\mathcal{A} = a^*\mathcal{A}$. For $a^*, a^*f \in \mathcal{A}^-$, by Lemma 1.5, it follows $\mathcal{A} = a\mathcal{A} \oplus (a^*)^\circ = fa\mathcal{A} \oplus (a^*f)^\circ$. Thus, $\mathcal{A} = e^{-1}a^*\mathcal{A} \oplus (a^*)^\circ = a^*\mathcal{A} \oplus (a^*f)^\circ$.

(XXIV) $\Rightarrow$ (VII): From $\mathcal{A} = e^{-1}a^*\mathcal{A} \oplus (a^*)^\circ = a^*\mathcal{A} \oplus (a^*f)^\circ$, we see that $\mathcal{A} = e^{-1}a^*\mathcal{A} \oplus (e^{-1}a^*)^\circ = a^*f\mathcal{A} \oplus (a^*f)^\circ$. Define the left regular representation $L_a : \mathcal{A} \to \mathcal{A}$ by $L_a(x) = ax$ for all $x \in \mathcal{A}$. Now, $\mathcal{A} = R(L_{e^{-1}a^*}) \oplus N(L_{e^{-1}a^*}) = R(L_{a^*f}) \oplus N(L_{a^*f})$ which implies that $L_{e^{-1}a^*}$ i L_{a^*f} are EP operators. According to [3, Remark 12], necessary and sufficient condition for $a \in \mathcal{A}$ to be EP is that $L_a \in \mathcal{L}(\mathcal{A})$ is EP. So, elements $e^{-1}a^*$, a^*f are EP and a^* is weighted–EP w.r.t. (e^{-1},f^{-1}).

(XXV) $\Leftrightarrow$ (V): By the equivalence (I) $\Leftrightarrow$ (V) for $a^\dagger$, we get that $a^\dagger$ is weighted–EP w.r.t. (e^{-1},f^{-1}) if and only if $e(a^\dagger)^*\mathcal{A} = f(a^\dagger)^*\mathcal{A} = a^\dagger\mathcal{A}$. Recall that $a^\dagger\mathcal{A} = a^*\mathcal{A}$ and $(a^\dagger)^*\mathcal{A} = a\mathcal{A}$. Now,

$$e(a^\dagger)^*\mathcal{A} = a^\dagger\mathcal{A} \ \Leftrightarrow \ (a^\dagger)^*\mathcal{A} = e^{-1}a^*\mathcal{A} \ \Leftrightarrow \ a\mathcal{A} = e^{-1}a^*\mathcal{A}$$

and, similarly, $f(a^\dagger)^*\mathcal{A} = a^\dagger\mathcal{A} \Leftrightarrow a\mathcal{A} = f^{-1}a^*\mathcal{A}$.

(XXVI) $\Leftrightarrow$ (VI): If we apply the equivalence (I) $\Leftrightarrow$ (iv) for $a^\dagger_{e,f}$, then $a^\dagger_{e,f}$ is weighted–EP w.r.t. $(e,f) \Leftrightarrow ea^\dagger_{e,f}\mathcal{A} = fa^\dagger_{e,f}\mathcal{A} = (a^\dagger_{e,f})^*\mathcal{A}$. By Lemma 1.6, we obtain

$$ea^\dagger_{e,f}\mathcal{A} = (a^\dagger_{e,f})^*\mathcal{A} \ \Leftrightarrow \ ea^\dagger_{e,f}\mathcal{A} = ea\mathcal{A} \ \Leftrightarrow \ a^\dagger_{e,f}\mathcal{A} = a\mathcal{A}$$

and

$$fa^\dagger_{e,f}\mathcal{A} = (a^\dagger_{e,f})^*\mathcal{A} \ \Leftrightarrow \ a^*\mathcal{A} = (a^\dagger_{e,f})^*\mathcal{A}.$$

(XXVII) $\Rightarrow$ (VIII): Assume that $a \in \mathcal{A}^\#$ and $a^{2k-1} = a^\dagger_{e,f}a^{2k+1}a^\dagger_{e,f}$, for any/some integer $k \geq 1$. Consequently, we have

$$aa^\dagger_{e,f} = (a^\#)^{2k}a^{2k+1}a^\dagger_{e,f} = (a^\#)^{2k}a(a^\dagger_{e,f}a^{2k+1}a^\dagger_{e,f}) = (a^\#)^{2k}aa^{2k-1} = a^\# a$$

and

$$a^\dagger_{e,f}a = (a^\dagger_{e,f}a^{2k+1}a^\dagger_{e,f})a(a^\#)^{2k} = a^{2k-1}a(a^\#)^{2k} = aa^\#$$

implying $aa^\dagger_{e,f} = a^\dagger_{e,f}a$.

The implication (VIII) $\Rightarrow$ (XXVII) is easy to check.

(XXVIII) $\Rightarrow$ (IX): Suppose that $a \in \mathcal{A}^\#$ and $aa^\dagger_{e,f}a^\dagger_{e,f}a = a^\dagger_{e,f}aaa^\dagger_{e,f}$. Now, observe that

$$\begin{aligned} a^\dagger_{e,f}aaa^\dagger_{e,f} &= aa^\dagger_{e,f}a^\dagger_{e,f}a = aa^\#(aa^\dagger_{e,f}a^\dagger_{e,f}a)a^\#a \\ &= a^\#aa^\dagger_{e,f}aaa^\dagger_{e,f}aa^\# = a^\#a. \end{aligned} \tag{2.1}$$

First, if k is a positive integer, then multiplying the equality (2.1) by a^k from the left side, we get $a^kaa^\dagger_{e,f} = a^k$ and then multiplying the equality (2.1) by a^k from the right side, we obtain $a^\dagger_{e,f}aa^k = a^k$. So, the condition (IX) is satisfied.

The implication (VIII) $\Rightarrow$ (XXVIII) is obvious.

(XXIX) $\Leftrightarrow$ (IV): Using (I) $\Leftrightarrow$ (IV) for $a^\#$, we deduce that $a^\#$ is weighted–EP w.r.t. (e,f) if and only if $ea^\#\mathcal{A} = fa^\#\mathcal{A} = (a^\#)^*\mathcal{A}$. This is equivalent to (iv), because $a^\#\mathcal{A} = a\mathcal{A}$, $a^\#\mathcal{A} = a\mathcal{A}$ and $(a^\#)^*\mathcal{A} = a^*\mathcal{A}$.

(XXX) $\Rightarrow$ (XII): From $aa^\# = aa^\dagger_{e,e} = aa^\dagger_{f,f}$ we conclude that elements $eaa^\# = eaa^\dagger_{e,e}$ and $faa^\# = faa^\dagger_{f,f}$ are Hermitian.

(III) $\Rightarrow$ (XXX): Since a is weighted–EP w.r.t. (e,f) implies $a^\# = a^\dagger_{e,f}$, then a is both weighted–EP w.r.t. (e,e) and w.r.t. (f,f) gives $a^\# = a^\dagger_{e,e} = a^\dagger_{f,f}$. Thus, $aa^\# = aa^\dagger_{e,e} = aa^\dagger_{f,f}$.

(I) $\Rightarrow$ (XXXI) $\Rightarrow$ (XII): This part follows similarly as (III) $\Rightarrow$ (XXX) $\Rightarrow$ (XII), using the equivalence (I) $\Leftrightarrow$ (II).

(XXXII) $\Rightarrow$ (XXX): By the equality $aa^\dagger_{e,e}e^{-1}a^*a = e^{-1}a^*aaa^\dagger_{e,e}$, we have

$$\begin{aligned} a^* = a^*aa^\dagger &= a^*aaa^\#a^\dagger = e(e^{-1}a^*aaa^\dagger_{e,e})aa^\#a^\dagger \\ &= eaa^\dagger_{e,e}e^{-1}a^*aaa^\#a^\dagger = eaa^\dagger_{e,e}e^{-1}a^*. \end{aligned} \tag{2.2}$$

Applying the involution to (2.2), we obtain $a = ae^{-1}eaa^\dagger_{e,e} = aaa^\dagger_{e,e}$ which yields $a^\#a = aa^\dagger_{e,e}$. In the same way, the assumption $aa^\dagger_{f,f}f^{-1}a^*a = f^{-1}a^*aaa^\dagger_{f,f}$ implies $a^\#a = aa^\dagger_{f,f}$. Therefore, the condition (XXX) holds.

(III) $\Rightarrow$ (XXXII): The condition (III) gives that $a^\# = a^\dagger_{e,e} = a^\dagger_{f,f}$. Then we get

$$\begin{aligned} aa^\dagger_{e,e}e^{-1}a^*a &= e^{-1}eaa^\dagger_{e,e}e^{-1}a^*a = e^{-1}(ae^{-1}eaa^\dagger_{e,e})^*a \\ &= e^{-1}(aaa^\#)^*a = e^{-1}a^*a = e^{-1}a^*aaa^\dagger_{e,e} \end{aligned}$$

and similarly $aa^\dagger_{f,f}f^{-1}a^*a = f^{-1}a^*aaa^\dagger_{f,f}$.

The implications (I) $\Rightarrow$ (XXXIII) $\Rightarrow$ (XXX), (III) $\Rightarrow$ (XXXIV) $\Rightarrow$ (XXX) and (I) $\Rightarrow$ (XXXV) $\Rightarrow$ (XXX) can be proved in the same way as (III) $\Rightarrow$ (XXXII) $\Rightarrow$ (XXX).

(XXXVI) $\Rightarrow$ (VIII): Multiplying $a^k aa^{\dagger}_{e,f} + a^{\dagger}_{e,f} aa^k = 2a^k$, $k \geq 1$, from the right side by $(a^{\#})^k$, we obtain

$$a^k aa^{\dagger}_{e,f}(a^{\#})^k + a^{\dagger}_{e,f} aaa^{\#} = 2aa^{\#}.$$

Further, the equality

$$a^k aa^{\dagger}_{e,f} a(a^{\#})^{k+1} + a^{\dagger}_{e,f} a = 2aa^{\#},$$

gives $aa^{\#} + a^{\dagger}_{e,f} a = 2aa^{\#}$, i.e. $a^{\dagger}_{e,f} a = aa^{\#}$. Similarly, multiplying $a^k aa^{\dagger}_{e,f} + a^{\dagger}_{e,f} aa^k = 2a^k$ from the left side by $(a^{\#})^k$, we show that $aa^{\dagger}_{e,f} = aa^{\#}$. So, $a^{\dagger}_{e,f} a = aa^{\dagger}_{e,f}$.

(VIII) $\Rightarrow$ (XXXVI) $\wedge$ (XXXVII): We can easily check this implication.

(XXXVII) $\Rightarrow$ (VIII): Multiplying the equality $a^{\dagger}_{e,f} a^{\#} a + aa^{\#} a^{\dagger}_{e,f} = 2a^{\dagger}_{e,f}$ by a first from the right side, we get $a^{\#} a = a^{\dagger}_{e,f} a$ and then from the left side, we obtain $aa^{\#} = a^{\dagger}_{e,f} a$. Hence, we deduce that $a^{\dagger}_{e,f} a = aa^{\dagger}_{e,f}$.

(XXXVIII) $\Rightarrow$ (XII): The condition $a^{*f,e} = a^{*f,e} aa^{\#} = a^{\#} aa^{*f,e}$ is equivalent to $a^* = a^* eaa^{\#} e^{-1} = fa^{\#} af^{-1} a^*$. Then, from

$$(eaa^{\#})^* = (a^{\#})^* a^* e = (a^{\#})^* a^* eaa^{\#} e^{-1} e = (aa^{\#})^* eaa^{\#}$$

and

$$(fa^{\#} a)^* = a^* (a^{\#})^* f = fa^{\#} af^{-1} a^* (a^{\#})^* f = fa^{\#} af^{-1} (a^{\#} a)^* f,$$

we conclude that elements $eaa^{\#}$ and $fa^{\#} a$ are Hermitian.

(XII) $\Rightarrow$ (XXXVIII): If $eaa^{\#}$ is Hermitian, then

$$a^{*f,e} aa^{\#} = f^{-1} a^* eaa^{\#} = f^{-1} (eaa^{\#} a)^* = f^{-1} (ea)^* = f^{-1} a^* e = a^{*f,e}.$$

In the same way, since $faa^{\#}$ is Hermitian, it follows $a^{*f,e} = a^{\#} aa^{*f,e}$.

(XXXIX) $\Rightarrow$ (XXXVIII): Multiplying the equality $a^{*f,e} aa^{\#} + a^{\#} aa^{*f,e} = 2a^{*f,e}$ by $aa^{\dagger}_{e,f}$ from the right side, we get

$$a^{*f,e} aa^{\dagger}_{e,f} + a^{\#} aa^{*f,e} aa^{\dagger}_{e,f} = 2a^{*f,e} aa^{\dagger}_{e,f}.$$

By Theorem 1.4, we have $a^{*f,e} + a^{\#} aa^{*f,e} = 2a^{*f,e}$, which implies $a^{\#} aa^{*f,e} = a^{*f,e}$. Similarly, multiplying the equality $a^{*f,e} aa^{\#} + a^{\#} aa^{*f,e} = 2a^{*f,e}$ from the left side by a and then by $a^{\dagger}_{e,f}$, we obtain $a^{*f,e} aa^{\#} = a^{*f,e}$. Hence, (XXIX) is satisfied.

The implication (XXXVIII) $\Rightarrow$ (XXXIX) is obvious.

(XL) $\Rightarrow$ (IX): Multiplying the condition $a^k a a^\dagger_{e,f} + (a^k a a^\dagger_{e,f})^* = a^k + (a^k)^*$, $k \geq 1$, by a from the right side, we see that

$$(a^k a a^\dagger_{e,f})^* a = (a^k)^* a. \tag{2.3}$$

Applying the involution to (2.3), we obtain

$$a^* a^k a a^\dagger_{e,f} = a^* a^k,$$

which gives

$$a^k a a^\dagger_{e,f} = a a^\dagger a^k a a^\dagger_{e,f} = (a^\dagger)^*(a^* a^k a a^\dagger_{e,f}) = (a^\dagger)^* a^* a^k = a^k.$$

In the same way, multiplying the hypothesis $a^\dagger_{e,f} a a^k + (a^\dagger_{e,f} a a^k)^* = a^k + (a^k)^*$, $k \geq 1$, by a from the left side, we show $a^\dagger_{e,f} a a^k = a^k$. Thus, (ix) holds.

The implication (VIII) $\Rightarrow$ (XL) $\wedge$ (XLI) is obvious.

(XLI) $\Rightarrow$ (VIII): The equality $a a^\dagger_{e,f}(a + \lambda a^\dagger_{e,f}) = (a + \lambda a^\dagger_{e,f}) a a^\dagger_{e,f}$ is equivalent to

$$a + \lambda a a^\dagger_{e,f} a^\dagger_{e,f} = a a a^\dagger_{e,f} + \lambda a^\dagger_{e,f}. \tag{2.4}$$

Multiplying (2.4) from the left side by $a^\dagger_{e,f}$, we get

$$a^\dagger_{e,f} a + \lambda a^\dagger_{e,f} a^\dagger_{e,f} = a^\dagger_{e,f} a a a^\dagger_{e,f} + \lambda a^\dagger_{e,f} a^\dagger_{e,f},$$

which yields $a^\dagger_{e,f} a = a^\dagger_{e,f} a a a^\dagger_{e,f}$. Analogously, $a^\dagger_{e,f} a(a + \lambda a^\dagger_{e,f}) = (a + \lambda a^\dagger_{e,f}) a^\dagger_{e,f} a$ implies $a a^\dagger_{e,f} = a^\dagger_{e,f} a a a^\dagger_{e,f}$. Therefore, $a^\dagger_{e,f} a = a a^\dagger_{e,f}$.

(XLII) $\Rightarrow$ (VIII): Assume that $ab = ba$ implies $a^\dagger_{e,f} b = b a^\dagger_{e,f}$. If $b = a$, then $a^\dagger_{e,f} a = a a^\dagger_{e,f}$.

(XI) $\Rightarrow$ (XLII): By $a^\# = a^\dagger_{e,f}$ and the double commutativity of $a^\#$, from $ab = ba$ we obtain $a^\# b = b a^\#$, i.e. $a^\dagger_{e,f} b = b a^\dagger_{e,f}$.

(XLIII) $\Rightarrow$ (VIII): Let $a^\dagger_{e,f} = f(a)$, for some function f holomorphic in a neighbourhood of $\sigma(a)$. By a property of the holomorphic calculus, $a^\dagger_{e,f}$ commutes with a.

(XI) $\Rightarrow$ (XLIII): From $a^\dagger_{e,f} = a^\#$ and, by [20, Theorem 4.4], $a^\# = f(a)$, where f is holomorphic in a neighbourhood of $\sigma(a)$, and $f(\lambda) = 0$ in a neighbourhood of 0, $f(\lambda) = \lambda^{-1}$ in a neighbourhood of $\sigma(a) \backslash \{0\}$, it follows (XLIII).

(XLIV) $\Rightarrow$ (III): Since $(a+\lambda a^\dagger_{e,e})\mathcal{A} = (\lambda a + a^3)\mathcal{A}$, $\lambda \neq 0$, then $a+\lambda a^\dagger_{e,e} = (\lambda a + a^3)x$, for some $x \in \mathcal{A}$. Now, from

$$\begin{aligned} a+\lambda aa^\dagger_{e,e}a^\dagger_{e,e} &= aa^\dagger_{e,e}(a+\lambda a^\dagger_{e,e}) = aa^\dagger_{e,e}(\lambda a + a^3)x \\ &= (\lambda a + a^3)x = a+\lambda a^\dagger_{e,e}, \end{aligned}$$

we conclude that $aa^\dagger_{e,e}a^\dagger_{e,e} = a^\dagger_{e,e}$. In the same way, $\mathcal{A}(a+\lambda a^\dagger_{e,e}) = \mathcal{A}(\lambda a + a^3)$ gives $a^\dagger_{e,e}a^\dagger_{e,e}a = a^\dagger_{e,e}$. So, $a^\dagger_{e,e} = aa^\dagger_{e,e}a^\dagger_{e,e} = a^\dagger_{e,e}a^\dagger_{e,e}a$, which implies that a is weighted–EP w.r.t. (e,e), by (I) $\Leftrightarrow$ (X).

Similarly, from the equalities $(a+\lambda a^\dagger_{f,f})\mathcal{A} = (\lambda a + a^3)\mathcal{A}$ and $\mathcal{A}(a+\lambda a^\dagger_{f,f}) = \mathcal{A}(\lambda a + a^3)$, for $\lambda \neq 0$, we can show that a is weighted–EP w.r.t. (f,f).

(III) $\Rightarrow$ (XLIV): The condition (III) implies $a^\# = a^\dagger_{e,e} = a^\dagger_{f,f}$. Then, for $\lambda \neq 0$, by

$$a+\lambda a^\dagger_{e,e} = a+\lambda a^\# = (a^3+\lambda a)(a^\#)^2 \in (a^3+\lambda a)\mathcal{A} \tag{2.5}$$

and

$$a^3+\lambda a = (a+\lambda a^\#)a^2 = (a+\lambda a^\dagger_{e,e})a^2 \in (a+\lambda a^\dagger_{e,e})\mathcal{A}, \tag{2.6}$$

we deduce $(a+\lambda a^\dagger_{e,e})\mathcal{A} = (\lambda a + a^3)\mathcal{A}$. In the same way, it follows the rest of condition (XLIV).

(XI) $\Rightarrow$ (XLV) $\Rightarrow$ (X): It follows in the same way as the part (III) $\Leftrightarrow$ (XLIV).

(XLVI) $\Rightarrow$ (XXIII): Assume that $(a+\lambda a^\dagger_{e,e})^\circ = (\lambda a + a^3)^\circ$, $\lambda \neq 0$. If $ax = 0$, for some $x \in \mathcal{A}$, then $(\lambda a + a^3)x = 0$ implies $(a+\lambda a^\dagger_{e,e})x = 0$. Now, we conclude $a^\dagger_{e,e}x = 0$ and $a^\circ \subset (a^\dagger_{e,e})^\circ$. Therefore, by Lemma 1.6, $a^\circ \subset [(ea)^*]^\circ$.

Let ${}^\circ(a+\lambda a^\dagger_{e,e}) = {}^\circ(\lambda a + a^3)$, $\lambda \neq 0$ and $(ea)^*x = 0$, for some $x \in \mathcal{A}$. Applying the involution, we see that $x^*ea = 0$ which gives $x^*e(\lambda a + a^3) = 0$. Then $x^*e(a+\lambda a^\dagger_{e,e}) = 0$ and, consequently, $x^*ea^\dagger_{e,e} = 0$, i.e. $(a^\dagger_{e,e})^*ex = 0$. By this equality, we have

$$ax = ae^{-1}(ea^\dagger_{e,e}a)^*x = ae^{-1}a^*(a^\dagger_{e,e})^*ex = 0.$$

Hence, $[(ea)^*]^\circ \subset a^\circ$ and $a^\circ = [(ea)^*]^\circ$.

The equalities $(a+\lambda a^\dagger_{f,f})^\circ = (\lambda a + a^3)^\circ$ and ${}^\circ(a+\lambda a^\dagger_{f,f}) = {}^\circ(\lambda a + a^3)$, for $\lambda \neq 0$, imply $(a^*)^\circ = (af^{-1})^\circ$ in the similar way.

(III) $\Rightarrow$ (XLVI): The assumption (III) gives $a^\# = a^\dagger_{e,e} = a^\dagger_{f,f}$, so by (2.5) and (2.6), we deduce ${}^\circ(a+\lambda a^\dagger_{e,e}) = {}^\circ(\lambda a + a^3)$, for $\lambda \neq 0$. Similarly, we can prove the rest of (XLVI).

(III) $\Rightarrow$ (XLVII) $\Rightarrow$ (XXIII): Similarly as (III) $\Rightarrow$ (XLVI) $\Rightarrow$ (XXIII).

(XLVIII) $\Rightarrow$ (VIII): Using the equality $(a^\dagger_{e,f})^2 a^\# = a^\dagger_{e,f} a^\# a^\dagger_{e,f}$, first we get

$$\begin{aligned}(a^\dagger_{e,f})^2 a^\# &= ((a^\dagger_{e,f})^2 a^\#) a a^\# = a^\dagger_{e,f} a^\# a^\dagger_{e,f} a a^\# \\ &= a^\dagger_{e,f} (a^\#)^2 a a^\dagger_{e,f} a a^\# = a^\dagger_{e,f} (a^\#)^2\end{aligned}$$

and then

$$\begin{aligned}a a^\dagger_{e,f} &= a^3 (a^\#)^2 a^\dagger_{e,f} = a^3 a^\dagger_{e,f} a (a^\#)^2 a^\dagger_{e,f} = a^3 (a^\dagger_{e,f} a^\# a^\dagger_{e,f}) \\ &= a^3 ((a^\dagger_{e,f})^2 a^\#) = a^3 a^\dagger_{e,f} (a^\#)^2 = a^3 a^\dagger_{e,f} a (a^\#)^3 = a a^\#.\end{aligned}$$

We can show that $a^\dagger_{e,f} a^\# a^\dagger_{e,f} = a^\# (a^\dagger_{e,f})^2$ implies $a^\dagger_{e,f} a = a a^\#$ in the same way. Thus, $a^\dagger_{e,f} a = a a^\dagger_{e,f}$.

The implication (XI) $\Rightarrow$ (XLVIII) $\wedge$ (XLIX) is obvious.

(XLIX) $\Rightarrow$ (VIII): From the hypothesis $a(a^\dagger_{e,f})^2 = a^\# = (a^\dagger_{e,f})^2 a$, we have

$$a a^\dagger_{e,f} = a a^\# a a^\dagger_{e,f} = a a (a^\dagger_{e,f})^2 a a^\dagger_{e,f} = a(a(a^\dagger_{e,f})^2) = a a^\#$$

and

$$a^\dagger_{e,f} a = a^\dagger_{e,f} a a^\# a = a^\dagger_{e,f} a (a^\dagger_{e,f})^2 a a = ((a^\dagger_{e,f})^2 a) a = a^\# a.$$

Therefore, we deduce that $a^\dagger_{e,f} a = a a^\dagger_{e,f}$.

(L) $\Rightarrow$ (XLIX): The equalities $a^{*f,e} a^\dagger_{e,f} = a^{*f,e} a^\#$ and $a^\dagger_{e,f} a^{*f,e} = a^\# a^{*f,e}$ are equivalent to $a^* e a^\dagger_{e,f} = a^* e a^\#$ and $a^\dagger_{e,f} f^{-1} a^* = a^\# f^{-1} a^*$. By $a^* e a^\dagger_{e,f} = a^* e a^\#$, we obtain

$$\begin{aligned}a(a^\dagger_{e,f})^2 &= e^{-1} (e a a^\dagger_{e,f})^* a^\dagger_{e,f} = e^{-1} (a^\dagger_{e,f})^* (a^* e a^\dagger_{e,f}) = e^{-1} (a^\dagger_{e,f})^* a^* e a^\# \\ &= e^{-1} e a a^\dagger_{e,f} a^\# = a a^\dagger_{e,f} a (a^\#)^2 = a^\#.\end{aligned}$$

Analogously, from $a^\dagger_{e,f} f^{-1} a^* = a^\# f^{-1} a^*$, we get $a^\# = (a^\dagger_{e,f})^2 a$. So, the condition (XLIX) holds.

The implication (XI) $\Rightarrow$ (L) $\wedge$ (LI) is obvious.

(LI) $\Rightarrow$ (XLIX): Multiplying $(a^\dagger_{e,f})^2 = (a^\#)^2$ by a first from the left side and then from the right side, we observe that (XLIX) is satisfied.

(LII) $\Leftrightarrow$ (VIII): The assumption $a^{*e,f} = a^{*e,f} a^\dagger_{e,f} a = a a^\dagger_{e,f} a^{*e,f}$ is equivalent to $a^* f = a^* f a^\dagger_{e,f} a$ and $e^{-1} a^* = a a^\dagger_{e,f} e^{-1} a^*$. Applying the involution to these equalities, we see that they are equivalent to $fa = f a^\dagger_{e,f} a a$ and $a e^{-1} = a e^{-1} e a a^\dagger_{e,f} e^{-1}$, i.e. $a = a^\dagger_{e,f} a a = a a a^\dagger_{e,f} \Leftrightarrow a^\dagger_{e,f} a = a a^\dagger_{e,f}$.

(LIII) $\Rightarrow$ (XII): Since $(a^\#)^{*e,f} = aa^\#(a^\#)^{*e,f} = (a^\#)^{*e,f}a^\#a$ can be written as $(a^\#)^* = eaa^\#e^{-1}(a^\#)^* = (a^\#)^*fa^\#af^{-1}$, we get

$$(eaa^\#)^* = (a^\#)^*a^*e = eaa^\#e^{-1}(a^\#)^*a^*e = eaa^\#e^{-1}(eaa^\#)^*$$

and

$$(fa^\#a)^* = a^*(a^\#)^*f = a^*(a^\#)^*fa^\#af^{-1}f = (a^\#a)^*fa^\#a.$$

So, we conclude that $eaa^\#$ and $fa^\#a$ are Hermitian.

(XI) $\Rightarrow$ (LIII): It is easy to check this part, by Theorem 1.4.

(LIV) $\Rightarrow$ (XII): The condition $a^\dagger_{e,f}(a^\#)^2 = (a^\#)^2a^\dagger_{e,f}$ gives

$$eaa^\# = ea^3a^\dagger_{e,f}a(a^\#)^3 = ea^3(a^\dagger_{e,f}(a^\#)^2) = ea^3(a^\#)^2a^\dagger_{e,f} = eaa^\dagger_{e,f}$$

and

$$fa^\#a = f(a^\#)^3aa^\dagger_{e,f}a^3 = f((a^\#)^2a^\dagger_{e,f})a^3 = fa^\dagger_{e,f}(a^\#)^2a^3 = fa^\dagger_{e,f}a.$$

Therefore, $eaa^\#$ and $fa^\#a$ are Hermitian elements.

The implication (XI) $\Rightarrow$ (LIV) $\wedge$ (LV)is obvious.

(LV) $\Rightarrow$ (VIII): Suppose that $a \in \mathcal{A}^\#$ and $a^ka^\dagger_{e,f} = a^\dagger_{e,f}a^k$, for any/some integer $k \geq 1$. Then

$$aa^\dagger_{e,f} = (a^\#)^{k-1}(a^ka^\dagger_{e,f}) = (a^\#)^{k-1}a^\dagger_{e,f}a^k = (a^\#)^kaa^\dagger_{e,f}a^k = a^\#a$$

and

$$a^\dagger_{e,f}a = (a^\dagger_{e,f}a^k)(a^\#)^{k-1} = a^ka^\dagger_{e,f}(a^\#)^{k-1} = a^ka^\dagger_{e,f}a(a^\#)^k = aa^\#.$$

Hence, $aa^\dagger_{e,f} = a^\dagger_{e,f}a$.

(LVI) $\Rightarrow$ (VIII): The equality $aa^\dagger_{e,f}(a + \lambda a^{*e,f}) = (a + \lambda a^{*e,f})aa^\dagger_{e,f}$, for $\lambda \neq 0$, is equivalent to

$$a + \lambda aa^\dagger_{e,f}a^{*e,f} = aaa^\dagger_{e,f} + \lambda a^{*e,f}aa^\dagger_{e,f}. \tag{2.7}$$

Multiplying (2.7) from the right side by a, we observe that

$$aa^\dagger_{e,f}a^{*e,f}a = e^{-1}a^{*e,f}. \tag{2.8}$$

Multiplying (2.8) from the right side by $a^\dagger_{f,e}$, we get $aa^\dagger_{e,f}a^{*e,f} = a^{*e,f}$.

Similarly, from $a^{\dagger}_{e,f}a(a+\lambda a^{*e,f}) = (a+\lambda a^{*e,f})a^{\dagger}_{e,f}a$, for $\lambda \neq 0$, we obtain $a^{*e,f}a^{\dagger}_{e,f}a = a^{*e,f}$. Thus, the condition (LII) is satisfied.

(VIII) $\Rightarrow$ (LVI): If $a^{\dagger}_{e,f}a = aa^{\dagger}_{e,f}$, then

$$\begin{aligned}(a+\lambda a^{*e,f})aa^{\dagger}_{e,f} &= a + \lambda e^{-1}a^*fa^{\dagger}_{e,f}a = a + e^{-1}(fa^{\dagger}_{e,f}aa)^* \\ &= a + \lambda e^{-1}a^*f = a + \lambda e^{-1}(ae^{-1}eaa^{\dagger}_{e,f})^*f \\ &= a + \lambda aa^{\dagger}_{e,f}e^{-1}a^*f = aa^{\dagger}_{e,f}(a+\lambda a^{*e,f}).\end{aligned}$$

The second equality follows similarly.

(LVII) $\Leftrightarrow$ (XXXII): Notice that the assumption $aa^{\dagger}_{e,e}(aa^*e - e^{-1}a^*a) = (aa^*e - e^{-1}a^*a)aa^{\dagger}_{e,e}$ is equivalent to $aa^*e - aa^{\dagger}_{e,e}e^{-1}a^*a = aa^*e - e^{-1}a^*aaa^{\dagger}_{e,e}$, i.e. $aa^{\dagger}_{e,e}e^{-1}a^*a = e^{-1}a^*aaa^{\dagger}_{e,e}$.

In the same way, the equality $aa^{\dagger}_{f,f}(aa^*f - f^{-1}a^*a) = (aa^*f - f^{-1}a^*a)aa^{\dagger}_{f,f}$ holds if and only if $aa^{\dagger}_{f,f}f^{-1}a^*a = f^{-1}a^*aaa^{\dagger}_{f,f}$.

The equivalences (LVIII) $\Leftrightarrow$ (XXXIII), (LIX) $\Leftrightarrow$ (XXXIV) and (L) $\Leftrightarrow$ (XXXV) follow similarly as (LVII) $\Leftrightarrow$ (XXXII).

(LXI) $\Leftrightarrow$ (IV): For $s,t \geq 1$ and $a \in \mathcal{A}^{\#}$, notice that $a^s, a^t, a^{s+t} \in \mathcal{A}^{\#}$ and then $a^s, a^t, a^{s+t} \in \mathcal{A}^-$. By Theorem 1.3, $(a^{s+t})^{\dagger} = (a^s)^{\dagger}_{e,1}(a^t)^{\dagger}_{1,e}$ is equivalent to $[(a^te^{-1/2})(e^{1/2}a^s)]^{\dagger} = (e^{1/2}a^s)^{\dagger}(a^te^{-1/2})^{\dagger}$ which holds, by Theorem 1.11, if and only if $e^{-1/2}(a^t)^*a^{s+t}\mathcal{A} \subseteq e^{1/2}a^s\mathcal{A}$ and $e^{1/2}a^s(a^{s+t})^*\mathcal{A} \subseteq e^{-1/2}(a^t)^*\mathcal{A}$, i.e. $(a^t)^*a^{s+t}\mathcal{A} \subseteq ea^s\mathcal{A}$ and $ea^s(a^{s+t})^*\mathcal{A} \subseteq (a^t)^*\mathcal{A}$. By elementary computations, this is equivalent to $a^*\mathcal{A} \subseteq ea\mathcal{A}$ and $ea\mathcal{A} \subseteq a^*\mathcal{A}$, that is $a^*\mathcal{A} = ea\mathcal{A}$. Analogy, $(a^{s+t})^{\dagger} = (a^s)^{\dagger}_{f,1}(a^t)^{\dagger}_{1,f}$, $s,t \geq 1 \Leftrightarrow a^*\mathcal{A} = fa\mathcal{A}$.

(LXII) $\Leftrightarrow$ (IV): Observe that, for $s,t \geq 1$, $(a^{s+t})^{\dagger}_{e,f} = (a^s)^{\dagger}_{f,f}(a^t)^{\dagger}_{e,f}$ is equivalent to $[(e^{1/2}a^tf^{-1/2})(f^{1/2}a^sf^{-1/2})]^{\dagger} = (f^{1/2}a^sf^{-1/2})^{\dagger}(e^{1/2}a^tf^{-1/2})^{\dagger}$, by Theorem 1.3. Using Theorem 1.11, the previous equality is equivalent to $f^{-1/2}(a^t)^*ea^{t+s}f^{-1/2}\mathcal{A} \subseteq f^{1/2}a^sf^{-1/2}\mathcal{A}$ and $f^{1/2}a^sf^{-1}(a^{t+s})^*e^{1/2}\mathcal{A} \subseteq f^{-1/2}(a^t)^*e^{1/2}\mathcal{A}$, that is, $(a^t)^*ea^{t+s}\mathcal{A} \subseteq fa^s\mathcal{A}$ and $fa^sf^{-1}(a^{t+s})^*\mathcal{A} \subseteq (a^t)^*\mathcal{A}$. It follows, by elementary computations, that this is equivalent to $a^*\mathcal{A} \subseteq fa\mathcal{A}$ and $fa\mathcal{A} \subseteq a^*\mathcal{A}$, i.e. $a^*\mathcal{A} = fa\mathcal{A}$. Similarly, $(a^{s+t})^{\dagger}_{e,f} = (a^s)^{\dagger}_{e,f}(a^t)^{\dagger}_{e,e}$, $s,t \geq 1$ if and only if $a^*\mathcal{A} = ea\mathcal{A}$.
□

From the previous theorem, we can get the following result.

Corollary 2.3. *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:*

(a) *a is weighted–EP w.r.t. (e,f);*

(b) $a^{*f,e}\mathcal{A} = a^{*e,f}\mathcal{A} = a\mathcal{A}$ (*or* $a^{*f,e}\mathcal{A} = a\mathcal{A}$ *and* $\mathcal{A}a^{*f,e} = \mathcal{A}a$);

(c) $(a^{*f,e})^\circ = a^\circ$ *and* $^\circ(a^{*f,e}) = {}^\circ a$;

(d) $a\mathcal{A}^{-1} = a^{*f,e}\mathcal{A}^{-1}$ *and* $\mathcal{A}^{-1}a = \mathcal{A}^{-1}a^{*f,e}$;

(e) $\mathcal{A}^{-1}a^* = \mathcal{A}^{-1}(a^{*f,e})^*$ *and* $a^*\mathcal{A}^{-1} = (a^{*f,e})^*\mathcal{A}^{-1}$;

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