Reverse order law in C^*-algebras

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Abstract

We study equivalent conditions for the reverse order law $(a_1a_2\dots a_n)^\dagger = a_n^\dagger(a_1^\dagger a_1a_2\dots a_na_n^\dagger)^\dagger a_1^\dagger$ in C^*-algebras. As corollaries, we obtain some recent and special results.

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1 Introduction

Let S be a semigroup with the unit 1. If $a, b \in S$ are invertible, then ab is invertible too and the inverse of the product ab satisfies the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. This formula cannot trivially be extended to the Moore–Penrose inverse of the product ab. Many authors studied this problem and proved some equivalent conditions for $(ab)^\dagger = b^\dagger a^\dagger$, as well as $(abc)^\dagger = c^\dagger b^\dagger a^\dagger$ in setting of matrices, operators, or elements of rings [1, 2, 3, 5, 7, 9, 10, 12, 13, 15, 16, 18, 21, 22, 23, 24, 25, 26]. Since the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ does not always hold, it is not easy to simplify various expressions that involve the Moore-Penrose inverse of a product. In addition to $(ab)^\dagger = b^\dagger a^\dagger$, $(ab)^\dagger$ may be expressed as $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger$ etc. These equalities are called mixed-type reverse order laws for the Moore-Penrose inverse of a product. When investigating various reverse order laws for $(ab)^\dagger$ or $(abc)^\dagger$, we notice that some of them are in fact equivalent (see [18, 20, 21, 24]). In this paper we consider necessary and sufficient conditions for mixed-type reverse order laws to hold for the Moore-Penrose inverse of a product of n elements in C^*-algebras.

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The Moore-Penrose inverse have lots of applications in numerical linear algebra, as well as in approximation methods in general Hilbert spaces. Particularly, the reverse order law is an useful computational tool. Hence, we investigate the reverse order rule in C^*-algebras.

Let $\mathcal{A}$ be a unital C^*–algebra with the unit 1. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. The set of all regular elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^-$. An element $a \in \mathcal{A}$ is selfadjoint if $a^* = a$.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{A}$ is the element $b \in \mathcal{A}$, if the following equations hold [19]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [19]), and such b is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

If $\delta \subset \{1, 2, 3, 4\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1, 2, 3, 4\} = \{a^\dagger\}$.

The following essential result on the existence of the Moore-Penrose inverse is proved in [11]. An alternative proof can be found in [14], an its generalization is proved in [8].

Theorem 1.1. *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

The following result is well-known and frequently used in the rest of the paper.

Theorem 1.2. [6, 17] *For any $a \in \mathcal{A}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;

(e) $a^* = a^\dagger aa^* = a^*aa^\dagger$;

(f) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$;

(g) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

From the last theorem we see that the following chain of equivalences hold:

$$a \in \mathcal{A}^\dagger \Leftrightarrow a^* \in \mathcal{A}^\dagger \Leftrightarrow aa^* \in \mathcal{A}^\dagger \Leftrightarrow a^*a \in \mathcal{A}^\dagger.$$

One of the basic topics in the theory of generalized inverses is to investigate various reverse order laws related to generalized inverses products. The reverse order law for the Moore-Penrose inverse is an useful computational tool in applications (solving linear equations in linear algebra or numerical analysis), and it is also interesting from the theoretical point of view.

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [9]. A Hilbert space version of their result was given by Isumino [13]. They proved that $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ holds if and only if $\mathcal{R}((a^*)^\dagger b) = \mathcal{R}(ab)$ and $\mathcal{R}(b^\dagger a^*) = \mathcal{R}((ab)^*)$, where for linear operators a and b, $\mathcal{R}(\cdot)$ denotes the range of an operator. Many results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's papers [20] and [21], where the author used finite dimensional methods (mostly properties of the rank of a complex matrices). Moreover, the operator analogues of these results are proved in [4] for linear operators on Hilbert spaces, using the operator matrices. In [18], a set of equivalent conditions for this reverse order rule for the Moore-Penrose inverse in the setting of C^*–algebra is presented, extending the results for complex matrices from [21]. Using some rank formulas, Tian [24] investigated necessary and sufficient conditions for a group of mixed-type reverse order laws for the Moore-Penrose inverse of a triple matrix product. Some related results can also be found in [12, 25, 26].

In this paper we present new results for the reverse order law $(a_1a_2\dots a_n)^\dagger = a_n^\dagger(a_1^\dagger a_1a_2\dots a_na_n^\dagger)^\dagger a_1^\dagger$ in C^*–algebras. As a consequence we get equivalent conditions for the reverse order law the Moore-Penrose inverse for a product of two or tree elements. Some of these results are known for reverse order laws $(ab)^\dagger$ or $(abc)^\dagger$ of complex rectangular matrices [20, 25] and Hilbert space operators [4], but we present new methods, depending on C^*–algebra properties. Thus, we extend the known results for matrices [20, 25] and operators [4] to more general settings.

2 Reverse order law for the MP-inverse

In this section we present necessary and sufficient conditions such that the reverse order law $(a_1a_2\dots a_n)^\dagger = a_n^\dagger(a_1^\dagger a_1a_2\dots a_na_n^\dagger)^\dagger a_1^\dagger$ holds. Before the main theorem, using Theorem 1.2, we can easily get the following result.

Lemma 2.1. *Let $\mathcal{A}$ be a unital C^*–algebra, let $a_1, a_2, \dots, a_n \in \mathcal{A}$ and let $a = a_1 a_2 \cdots a_n$. If $a_1, a_n \in \mathcal{A}^-$, then the following statements are equivalent:*

(1) $a \in \mathcal{A}^-$;

(2) $a_1^\dagger a a_n^\dagger \in \mathcal{A}^-$;

(3) $a_1^* a a_n^* \in \mathcal{A}^-$;

(4) $a_1^\dagger a a_n^* \in \mathcal{A}^-$;

(5) $a_1^* a a_n^\dagger \in \mathcal{A}^-$;

(6) $(a_1^\dagger)^* a_2 \dots a_n \in \mathcal{A}^-$;

(7) $(a_1^* a_1)^\dagger a_2 \dots a_n a_n^* \in \mathcal{A}^-$;

(8) $a_1 \dots a_{n-1} (a_n^\dagger)^* \in \mathcal{A}^-$;

(9) $a_1^* a_1 \dots a_{n-1} (a_n a_n^*)^\dagger \in \mathcal{A}^-$;

(10) $a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger \in \mathcal{A}^-$;

(11) $a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1 \in \mathcal{A}^-$;

(12) $(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^* (a_1^* a_1)^\dagger \in \mathcal{A}^-$;

(13) $a_1^\dagger a \in \mathcal{A}^-$;

(14) $a a_n^\dagger \in \mathcal{A}^-$;

(15) $(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger \in \mathcal{A}^-$;

(16) $a_1^\dagger a_1 \dots a_{n-1} (a_n^\dagger)^* \in \mathcal{A}^-$;

(17) $a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger \in \mathcal{A}^-$;

(18) $a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1 \in \mathcal{A}^-$;

(19) $a_1^* a \in \mathcal{A}^-$;

(20) $a a_n^* \in \mathcal{A}^-$;

(21) $(a_1^* a_1)^\dagger a_2 \dots a_n \in \mathcal{A}^-$;

(22) $(a_1^\dagger)^* a_2 \dots a_n a_n^* \in \mathcal{A}^-$;

(23) $a_1^* a_1 \dots a_{n-1}(a_n^\dagger)^* \in \mathcal{A}^-$;

(24) $a_1 \dots a_{n-1}(a_n a_n^*)^\dagger \in \mathcal{A}^-$;

(25) $(a_1^\dagger)^* a_2 \dots a_{n-1}(a_n a_n^*)^\dagger \in \mathcal{A}^-$;

(24) $(a_1^* a_1)^\dagger a_2 \dots a_{n-1}(a_n^\dagger)^* \in \mathcal{A}^-$;

(26) $a_1 a_1^* a a_n^* a_n \in \mathcal{A}^-$;

(27) $(a_1 a_1^*)^\dagger a (a_n^* a_n)^\dagger \in \mathcal{A}^-$.

Now, we formulate the main results which consists a number of equivalent conditions for mixed-type reverse order laws of a product of n elements in C^*-algebra.

Theorem 2.1. *Let $\mathcal{A}$ be a unital C^*–algebra, let $a_1, a_2, \dots, a_n \in \mathcal{A}$ and let $a = a_1 a_2 \dots a_n$. If $a_1, a_n, a \in \mathcal{A}^-$, then the following statements are equivalent:*

(a1) $a^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$;

(a2) $a^\dagger = a_n^* (a_1^* a a_n^*)^\dagger a_1^*$;

(a3) $a^\dagger = a_n^* (a_1^\dagger a a_n^*)^\dagger a_1^\dagger$;

(a4) $a^\dagger = a_n^\dagger (a_1^* a a_n^\dagger)^\dagger a_1^*$;

(b1) $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^*$;

(b2) $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger = a_n^* [(a_1^* a_1)^\dagger a_2 \dots a_n a_n^*]^\dagger a_1^\dagger$;

(b3) $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger = a_n^* (a_1^\dagger a a_n^*)^\dagger a_1^*$;

(c1) $[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = a_n^* (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$;

(c2) $[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = a_n^\dagger [a_1^* a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger a_1^*$;

(c4) $[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = a_n^* (a_1^* a a_n^\dagger)^\dagger a_1^*$;

(d1) $(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger = a_1 (a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger a_n$;

(d2) $(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger = (a_1^\dagger)^* [(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^* (a_1^* a_1)^\dagger]^\dagger (a_n^\dagger)^*$;

(e1) $(a_1^\dagger a)^\dagger a_1^\dagger = a_n^\dagger (a a_n^\dagger)^\dagger$;

(e2) $(a_1^\dagger a)^\dagger a_1^* = a_n^\dagger[(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger]^\dagger$;

(e3) $[a_1^\dagger a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger a_1^\dagger = a_n^*(aa_n^\dagger)^\dagger$;

(e4) $(a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n = a_1(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger$;

(e5) $(a_1^* a)^\dagger a_1^* = a_n^*(aa_n^*)^\dagger$;

(e6) $[(a_1^* a_1)^\dagger a_2 \dots a_n]^\dagger a_1^\dagger = a_n^*[(a_1^\dagger)^* a_2 \dots a_n a_n^*]^\dagger$;

(e7) $[a_1^* a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger a_1^* = a_n^\dagger[a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger$;

(e8) $a_n^\dagger[(a_1^\dagger)^* a_2 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger = [(a_1^* a_1)^\dagger a_2 \dots a_{n-1}(a_n^\dagger)^*]^\dagger a_1^\dagger$;

(e9) $(a_1^\dagger a)^\dagger a_1^\dagger = a_n^*(aa_n^*)^\dagger$;

(e10) $(a_1^* a)^\dagger a_1^* = a_n^\dagger(aa_n^\dagger)^\dagger$;

(e11) $(a_1 a_1^* a a_n^* a_n)^\dagger = a_n^\dagger(a_1^* a a_n^*)^\dagger a_1^\dagger$;

(e12) $[(a_1 a_1^*)^\dagger a (a_n^* a_n)^\dagger]^\dagger = a_n^*(a_1^\dagger a a_n^\dagger)^\dagger a_1^*$;

(f1) $(a_1^\dagger a)^\dagger = a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger$ *and* $(aa_n^\dagger)^\dagger = (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$;

(f2) $(a_1^\dagger a)^\dagger = a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger$ *and* $[(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger]^\dagger = (a_1^\dagger a a_n^\dagger)^\dagger a_1^*$;

(f3) $[a_1^\dagger a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = a_n^*(a_1^\dagger a a_n^\dagger)^\dagger$ *and* $(aa_n^\dagger)^\dagger = (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$;

(f4) $(a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger = a_1(a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger$ *and* $(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger = (a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger a_n$;

(f5) $(a_1^* a)^\dagger = a_n^*(a_1^* a a_n^*)^\dagger$ *and* $(aa_n^*)^\dagger = (a_1^* a a_n^*)^\dagger a_1^*$;

(f6) $[(a_1^* a_1)^\dagger a_2 \dots a_n]^\dagger = a_n^*[(a_1^* a_1)^\dagger a_2 \dots a_n a_n^*]^\dagger$ *and* $[(a_1^\dagger)^* a_2 \dots a_n a_n^*]^\dagger = [(a_1^* a_1)^\dagger a_2 \dots a_n a_n^*]^\dagger a_1^\dagger$;

(f7) $[a_1^* a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = a_n^\dagger[a_1^* a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger$ *and* $[a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger = [a_1^* a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger a_1^*$;

(f8) $[(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger = (a_1^\dagger)^*[(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^* a_1)^\dagger]^\dagger$ *and* $[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^* a_1)^\dagger]^\dagger = [(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^* a_1)^\dagger]^\dagger (a_n^\dagger)^*$;

(f9) $(a_1^\dagger a)^\dagger = a_n^*(a_1^\dagger a a_n^*)^\dagger$ *and* $(aa_n^*)^\dagger = (a_1^\dagger a a_n^*)^\dagger a_1^\dagger$;

(f10) $(a_1^* a)^\dagger = a_n^\dagger(a_1^* a a_n^\dagger)^\dagger$ *and* $(aa_n^\dagger)^\dagger = (a_1^* a a_n^\dagger)^\dagger a_1^*$;

(f11) $(a_1^\dagger a)^\dagger = a_n^*(a_1^\dagger a a_n^*)^\dagger$ *and* $(aa_n^\dagger)^\dagger = (a_1^* a a_n^\dagger)^\dagger a_1^*$;

(g1) $(a_1^\dagger a)^\dagger = a^\dagger a_1$ *and* $(aa_n^\dagger)^\dagger = a_n a^\dagger$;

(g2) $(a_1^\dagger a)^\dagger = [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^*$ *and* $[(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger]^\dagger = a_n[(a_1^\dagger)^* a_2 \dots a_n]^\dagger$;

(g3) $[a_1^\dagger a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = [a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger a_1$ *and* $(aa_n^\dagger)^\dagger = (a_n^\dagger)^*[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger$;

(g4) $(a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger = (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n^\dagger$ *and* $(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger = a_1^\dagger (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger$;

(g5) $(a_1^* a)^\dagger = a^\dagger (a_1^\dagger)^*$ *and* $(aa_n^*)^\dagger = (a_n^\dagger)^* a^\dagger$;

(g6) $[(a_1^* a_1)^\dagger a_2 \dots a_n]^\dagger = [(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1$ *and* $[(a_1^\dagger)^* a_2 \dots a_n a_n^*]^\dagger = (a_n^\dagger)^*[(a_1^\dagger)^* a_2 \dots a_n]^\dagger$;

(g7) $[a_1^* a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger = [a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger (a_1^\dagger)^*$ *and* $[a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger = a_n[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger$;

(g8) $[(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger = (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n^*$ *and* $[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^* a_1)^\dagger]^\dagger = a_1^*(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger$;

(g9) $(a_1^\dagger a)^\dagger = a^\dagger a_1$ *and* $(aa_n^*)^\dagger = (a_n^\dagger)^* a^\dagger$;

(g10) $(a_1^* a)^\dagger = a^\dagger (a_1^\dagger)^*$ *and* $(aa_n^\dagger)^\dagger = a_n a^\dagger$;

(h1) $(a_1^* a a_n^*)^\dagger = (a_n^\dagger)^* a^\dagger (a_1^\dagger)^*$;

(h2) $[(a_1^* a_1)^\dagger a_2 \dots a_n a_n^*]^\dagger = (a_n^\dagger)^*[(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1$;

(h3) $[a_1^* a_1 \dots a_{n-1}(a_n a_n^*)^\dagger]^\dagger = a_n[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger (a_1^\dagger)^*$;

(h4) $[(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^* a_1)^\dagger]^\dagger = a_1^*(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n^*$;

(h5) $(a_1^\dagger a a_n^\dagger)^\dagger = a_n a^\dagger a_1$;

(h6) $(a_1^\dagger a a_n^\dagger)^\dagger = a_n[(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^*$;

(h7) $(a_1^\dagger a a_n^\dagger)^\dagger = (a_n^\dagger)^*[a_1 \dots a_{n-1}(a_n^\dagger)^*]^\dagger a_1$;

(h8) $(a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger = a_1^\dagger (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n^\dagger$;

(h9) $(a_1^\dagger a a_n^*)^\dagger = (a_n^\dagger)^* a^\dagger a_1$;

(h10) $(a_1^* a a_n^\dagger)^\dagger = a_n a^\dagger (a_1^\dagger)^*$;

(i1) $a_n^\dagger (a_1^* a a_n^*)^\dagger a_1^\dagger = (a_n^* a_n)^\dagger a^\dagger (a_1 a_1^*)^\dagger$;

(i2) $a_n^\dagger [(a_1^* a_1)^\dagger a_2 \dots a_n a_n^*]^\dagger a_1^* = (a_n^* a_n)^\dagger [(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1 a_1^*$;

(i3) $a_n^* [a_1^* a_1 \dots a_{n-1} (a_n a_n^*)^\dagger]^\dagger a_1^\dagger = a_n^* a_n [a_1 \dots a_{n-1} (a_n^\dagger)^*]^\dagger (a_1 a_1^*)^\dagger$;

(i4) $(a_1^\dagger)^* [(a_n a_n^*)^\dagger a_{n-1}^* \dots a_2^* (a_1^* a_1)^\dagger]^\dagger (a_n^\dagger)^* = a_1 a_1^\dagger (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n^\dagger a_n$;

(i5) $a_n^* (a_1^\dagger a a_n^\dagger)^\dagger a_1^* = a_n^* a_n a^\dagger a_1 a_1^*$;

(i6) $a_n^* (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger = a_n^* a_n [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1 a_1^*)^\dagger$;

(i7) $a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* = (a_n^* a_n)^\dagger [a_1 \dots a_{n-1} (a_n^\dagger)^*]^\dagger a_1 a_1^*$;

(i8) $a_1 (a_n a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1)^\dagger a_n = a_1 a_1^\dagger (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger a_n^\dagger a_n$;

(i9) $a_n^\dagger (a_1^\dagger a a_n^*)^\dagger a_1^* = (a_n^* a_n)^\dagger a^\dagger a_1 a_1^*$;

(i10) $a_n^* (a_1^* a a_n^\dagger)^\dagger a_1^\dagger = a_n^* a_n a^\dagger (a_1 a_1^*)^\dagger$;

(i11) $a_n^* a_n (a_1 a_1^* a a_n^* a_n)^\dagger a_1 a_1^* = a_n^* (a_1^* a a_n^*)^\dagger a_1^*$;

(i12) $(a_n^* a_n)^\dagger [(a_1 a_1^*)^\dagger a (a_n^* a_n)^\dagger]^\dagger (a_1 a_1^*)^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$.

Proof. By Lemma 2.1 and Theorem 1.1, the hypothesis $a \in \mathcal{A}^-$ implies MP-invertibility of various elements appearing in the statements of this theorem.

(a1) $\Rightarrow$ (b1): Using Theorem 1.2, we get

$$\begin{aligned}(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* (a_1^\dagger)^* a_2 \dots a_n &= (a_1^\dagger)^* (a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a a_n^\dagger) a_n \\ &= (a_1^\dagger)^* a_1^\dagger a a_n^\dagger a_n = (a_1^\dagger)^* a_2 \dots a_n,\end{aligned}$$

$$\begin{aligned}a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* (a_1^\dagger)^* a_2 \dots a_n a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* &= a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* \\ &= a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^*,\end{aligned}$$

i.e. $a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* \in [(a_1^\dagger)^* a_2 \dots a_n]\{1,2\}$. The hypothesis $a^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$ implies that the elements $a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$, $a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a$ are selfadjoint. Now, from

$$\begin{aligned}((a_1^\dagger)^* a_2 \dots a_n a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^*)^* &= ((a_1^\dagger)^* a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^*)^* \\ &= a_1 a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger \\ &= a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger,\end{aligned}$$

$$a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger a_1^*(a_1^\dagger)^* a_2 \dots a_n \quad = \quad a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a,$$

we deduce $a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger a_1^* \in [(a_1^\dagger)^* a_2 \dots a_n]\{3,4\}$. Hence, $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger = a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger a_1^*$.

Analogously we can prove (b1) $\Rightarrow$ (a1), (a1) $\Leftrightarrow$ (c1), by these equivalences, (c1) $\wedge$ (b1) $\Rightarrow$ (d1) $\Rightarrow$ (a1) and also (a3) $\Leftrightarrow$ (b3), (a4) $\Leftrightarrow$ (c4).

(b1) $\Rightarrow$ (g2): It is easy to check that $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^* \in (a_1^\dagger a)\{1,2,4\}$ and $a_n[(a_1^\dagger)^* a_2 \dots a_n]^\dagger \in [(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger]\{1,2,3\}$. Applying the assumption $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger = a_n^\dagger(a_1^\dagger a a_n^\dagger)^\dagger a_1^*$, we have

$$\begin{aligned}
(a_1^\dagger a[(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^*)^* &= (a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* (a_1^\dagger)^*)^* \\
&= a_1^\dagger a_1 a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger \\
&= a_1^\dagger a a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger \text{ is selfadjoint,}
\end{aligned}$$

$$\begin{aligned}
(a_n[(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^* a_2 \dots a_n a_n^\dagger)^* &= (a_n a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^* (a_1^\dagger)^* a_2 \dots a_n a_n^\dagger)^* \\
&= (a_n a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a a_n^\dagger)^* \\
&= (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a a_n^\dagger a_n a_n^\dagger \\
&= (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger a a_n^\dagger \text{ is selfadjoint.}
\end{aligned}$$

So, we deduce $[(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^* \in (a_1^\dagger a)\{3\}$ and $a_n[(a_1^\dagger)^* a_2 \dots a_n]^\dagger \in [(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger]\{4\}$. Thus, we obtain $(a_1^\dagger a)^\dagger = [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^*$ and $[(a_1^\dagger)^* a_2 \dots a_n a_n^\dagger]^\dagger = a_n[(a_1^\dagger)^* a_2 \dots a_n]^\dagger$.

(g2) $\Rightarrow$ (h2): Let us remark that the relation $(a_n^\dagger)^*[(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1 \in [(a_1^* a_1)^\dagger a_2 \dots a_n a_n^*]\{1,2\}$ holds. By the condition (g2), the elements

$$\begin{aligned}
&((a_1^* a_1)^\dagger a_2 \dots a_n a_n^* (a_n^\dagger)^* [(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1)^* \\
&= (a_1^\dagger (a_1^\dagger)^* a_2 \dots a_n [(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1)^* \\
&= a_1^* (a_1^\dagger)^* a_2 \dots a_n [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^* \\
&= a_1^\dagger a [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^*,
\end{aligned}$$

$$\begin{aligned}
&((a_n^\dagger)^* [(a_1^\dagger)^* a_2 \dots a_n]^\dagger a_1 (a_1^* a_1)^\dagger a_2 \dots a_n a_n^*)^* \\
&= ((a_n^\dagger)^* [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^* a_2 \dots a_n a_n^*)^* \\
&= a_n [(a_1^\dagger)^* a_2 \dots a_n]^\dagger (a_1^\dagger)^* a_2 \dots a_n a_n^\dagger
\end{aligned}$$

are selfadjoint. Therefore, $[(a_1^*a_1)^\dagger a_2\dots a_na_n^*]^\dagger=(a_n^\dagger)^*[(a_1^\dagger)^*a_2\dots a_n]^\dagger a_1$.

(h2) $\Rightarrow$ (b2): Notice $a_n^*[(a_1^*a_1)^\dagger a_2\dots a_na_n^*]^\dagger a_1^\dagger \in [(a_1^\dagger)^*a_2\dots a_n]\{1,2\}$. The equality $[(a_1^*a_1)^\dagger a_2\dots a_na_n^*]^\dagger=(a_n^\dagger)^*[(a_1^\dagger)^*a_2\dots a_n]^\dagger a_1$ gives

$$\begin{aligned}
&((a_1^\dagger)^*a_2\dots a_na_n^*[(a_1^*a_1)^\dagger a_2\dots a_na_n^*]^\dagger a_1^\dagger)^*\\
&=((a_1^\dagger)^*a_2\dots a_na_n^*(a_n^\dagger)^*[(a_1^\dagger)^*a_2\dots a_n]^\dagger a_1a_1^\dagger)^*\\
&=((a_1^\dagger)^*a_2\dots a_n[(a_1^\dagger)^*a_2\dots a_n]^\dagger a_1a_1^\dagger)^*\\
&=a_1a_1^\dagger(a_1^\dagger)^*a_2\dots a_n[(a_1^\dagger)^*a_2\dots a_n]^\dagger\\
&=(a_1^\dagger)^*a_2\dots a_n[(a_1^\dagger)^*a_2\dots a_n]^\dagger \text{ is selfadjoint,}
\end{aligned}$$

$$\begin{aligned}
&(a_n^*[(a_1^*a_1)^\dagger a_2\dots a_na_n^*]^\dagger a_1^\dagger(a_1^\dagger)^*a_2\dots a_n)^*\\
&=(a_n^*(a_n^\dagger)^*[(a_1^\dagger)^*a_2\dots a_n]^\dagger a_1a_1^\dagger(a_1^\dagger)^*a_2\dots a_n)^*\\
&=(a_n^*(a_n^\dagger)^*[(a_1^\dagger)^*a_2\dots a_n]^\dagger(a_1^\dagger)^*a_2\dots a_n)^*\\
&=[(a_1^\dagger)^*a_2\dots a_n]^\dagger(a_1^\dagger)^*a_2\dots a_na_n^\dagger a_n\\
&=[(a_1^\dagger)^*a_2\dots a_n]^\dagger(a_1^\dagger)^*a_2\dots a_n \text{ is selfadjoint.}
\end{aligned}$$

Thus, the condition (b2) $[(a_1^\dagger)^*a_2\dots a_n]^\dagger=a_n^*[(a_1^*a_1)^\dagger a_2\dots a_na_n^*]^\dagger a_1^\dagger$ is satisfied.

In the same way we can get (b2) $\Rightarrow$ (g6) $\Rightarrow$ (h6) $\Rightarrow$ (b1) and the following implications:

(a1) $\Rightarrow$ (g1) $\Rightarrow$ (h1) $\Rightarrow$ (a2) $\Rightarrow$ (g5) $\Rightarrow$ (h5) $\Rightarrow$ (a1);
(c1) $\Rightarrow$ (g3) $\Rightarrow$ (h3) $\Rightarrow$ (c2) $\Rightarrow$ (g7) $\Rightarrow$ (h7) $\Rightarrow$ (c1);
(d1) $\Rightarrow$ (g4) $\Rightarrow$ (h4) $\Rightarrow$ (d2)$\Rightarrow$ (g8) $\Rightarrow$ (h8) $\Rightarrow$ (d1);
(a1) $\Rightarrow$ (h9) $\Rightarrow$ (a3) $\Rightarrow$ (g9) $\Rightarrow$ (h5) $\Rightarrow$ (a1);
(a1) $\Rightarrow$ (h10) $\Rightarrow$ (a4) $\Rightarrow$ (g10) $\Rightarrow$ (h5) $\Rightarrow$ (a1).

(d2) $\Rightarrow$ (f8): The elementary computations show that $(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]^\dagger \in [(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*a_1^\dagger]\{1,2,3\}$ and $[(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^* \in [a_n^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]\{1,2,4\}$. By the assumption $(a_n^\dagger a_{n-1}^*\dots a_2^*a_1^\dagger)^\dagger=(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^*$, we have that the elements $(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^*a_n^\dagger a_{n-1}^*\dots a_2^*a_1^\dagger$, $a_n^\dagger a_{n-1}^*\dots a_2^*a_1^\dagger(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^*$ are selfadjoint, that is $(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*(a_1^*a_1)^\dagger]^\dagger \in [(a_na_n^*)^\dagger a_{n-1}^*\dots a_2^*a_1^\dagger]\{4\}$ and

$[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^* \in [a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]\{3\}$. Therefore, $[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger = (a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger$ and $[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger = [(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^*$.

(f8) $\Rightarrow$ (e8): From (f8), we obtain

$$[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger(a_n^\dagger)^* = (a_1^\dagger)^*[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger. \tag{1}$$

Applying involution to (1), we get (e8).

(e8) $\Rightarrow$ (d2): First, to prove the equality $[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger = a_1^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger$, we can easy check that $a_1^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger \in [(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]\{1,2,3\}$. Applying involution to (e8), we obtain (1), which yields

$$\begin{aligned}
&(a_1^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger)^* \\
&= (a_1^*(a_1^\dagger)^*[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger)^* \\
&= [a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger a_1^\dagger a_1 \\
&= [a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger.
\end{aligned}$$

Thus, we conclude $[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger = a_1^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger$. In order to show (d2), we easy verify $(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^* \in (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)\{1,2\}$. Using the above equality and (1), we get the relation $(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^* \in (a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)\{3,4\}$ by

$$\begin{aligned}
&a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger(a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^* \\
&= a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger(a_1^\dagger)^* a_1^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger(a_n^\dagger)^* \\
&= a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger(a_1^\dagger)^*[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger \\
&= a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger[a_n^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger,
\end{aligned}$$

$$\begin{aligned}
&((a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^* a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^* \\
&= ((a_1^\dagger)^* a_1^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^* \\
&= [(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger a_1 a_1^\dagger \\
&= [(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger]^\dagger(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger.
\end{aligned}$$

Hence, $(a_n^\dagger a_{n-1}^* \dots a_2^* a_1^\dagger)^\dagger = (a_1^\dagger)^*[(a_na_n^*)^\dagger a_{n-1}^* \dots a_2^*(a_1^*a_1)^\dagger]^\dagger(a_n^\dagger)^*$.

Similarly, we can prove the following chains of implications:

(a1) $\Rightarrow$ (f1) $\Rightarrow$ (e1) $\Rightarrow$ (a1);
(b1) $\Rightarrow$ (f2) $\Rightarrow$ (e2) $\Rightarrow$ (b1);
(c1) $\Rightarrow$ (f3) $\Rightarrow$ (e3) $\Rightarrow$ (c1);
(d1) $\Rightarrow$ (f4) $\Rightarrow$ (e4) $\Rightarrow$ (d1);
(a2) $\Rightarrow$ (f5) $\Rightarrow$ (e5) $\Rightarrow$ (a2);
(b2) $\Rightarrow$ (f6) $\Rightarrow$ (e6) $\Rightarrow$ (b2);
(c2) $\Rightarrow$ (f7) $\Rightarrow$ (e7) $\Rightarrow$ (c2)
(a3) $\Rightarrow$ (f9) $\Rightarrow$ (e9) $\Rightarrow$ (a3);
(a4) $\Rightarrow$ (f10) $\Rightarrow$ (e10) $\Rightarrow$ (a4).

(a2) $\Rightarrow$ (e11): Obviously, $a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger \in (a_1a_1^*aa_n^*a_n)\{1,2\}$. By the equality

$$\begin{aligned}(a_1a_1^*aa_n^*a_na_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger)^* &= (a_1a_1^*aa_n^*(a_1^*aa_n^*)^\dagger a_1^\dagger)^* \\ &= (a_1^\dagger)^*a_1^*aa_n^*(a_1^*aa_n^*)^\dagger a_1^* \\ &= aa_n^*(a_1^*aa_n^*)^\dagger a_1^*,\end{aligned}$$

$$\begin{aligned}(a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger a_1a_1^*aa_n^*a_n)^* &= (a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^*aa_n^*a_n)^* \\ &= a_n^*(a_1^*aa_n^*)^\dagger a_1^*aa_n^*(a_n^\dagger)^* \\ &= a_n^*(a_1^*aa_n^*)^\dagger a_1^*a\end{aligned}$$

and the assumption $a^\dagger = a_n^*(a_1^*aa_n^*)^\dagger a_1^*$, we observe that $a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger \in (a_1a_1^*aa_n^*a_n)\{3,4\}$. Consequently, $(a_1a_1^*aa_n^*a_n)^\dagger = a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger$.

(e11) $\Rightarrow$ (a2): We can easy get that $a_n^*(a_1^*aa_n^*)^\dagger a_1^* \in a\{1,2\}$. From the hypothesis $(a_1a_1^*aa_n^*a_n)^\dagger = a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger$, we obtain

$$\begin{aligned}(aa_n^*(a_1^*aa_n^*)^\dagger a_1^*)^* &= ((a_1^\dagger)^*a_1^*aa_n^*(a_1^*aa_n^*)^\dagger a_1^*)^* \\ &= a_1a_1^*aa_n^*(a_1^*aa_n^*)^\dagger a_1^\dagger \\ &= a_1a_1^*aa_n^*a_na_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger \text{ is selfadjoint}\end{aligned}$$

and

$$\begin{aligned}(a_n^*(a_1^*aa_n^*)^\dagger a_1^*a)^* &= (a_n^*(a_1^*aa_n^*)^\dagger a_1^*aa_n^*(a_n^\dagger)^*)^* \\ &= a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^*aa_n^*a_n \\ &= a_n^\dagger(a_1^*aa_n^*)^\dagger a_1^\dagger a_1a_1^*aa_n^*a_n \text{ is selfadjoint.}\end{aligned}$$

Thus, $a^\dagger = a_n^*(a_1^*aa_n^*)^\dagger a_1^*$, i.e., the statements (a2) is satisfied.

In the same manner, we can prove that (e12) $\Leftrightarrow$ (a1).

(f1) $\Rightarrow$ (f11): First, to prove that $(a_1^\dagger aa_n^*)^\dagger = (a_n^\dagger)^* a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger$, we can show $(a_n^\dagger)^* a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger \in (a_1^\dagger aa_n^*)\{1,2,3\}$. Using the assumption $(a_1^\dagger a)^\dagger = a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger$, we get $a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger a$ is selfadjoint and

$$\begin{aligned}(a_n^\dagger)^* a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger aa_n^* &= (a_n a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger aa_n^\dagger)^* = (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger aa_n^\dagger a_n a_n^\dagger \\ &= (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger aa_n^\dagger \text{ is selfadjoint.}\end{aligned}$$

Therefore, $(a_1^\dagger aa_n^*)^\dagger = (a_n^\dagger)^* a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger$. By this equality and (f1), we have

$$(a_1^\dagger a)^\dagger = a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger = a_n^* (a_n^\dagger)^* a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger = a_n^* (a_1^\dagger aa_n^*)^\dagger.$$

In the same way, from the hypothesis $(aa_n^\dagger)^\dagger = (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger$, we can obtain $(a_1^* aa_n^\dagger)^\dagger = (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger (a_1^\dagger)^*$ and

$$(aa_n^\dagger)^\dagger = (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger = (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger (a_1^\dagger)^* a_1^* = (a_1^* aa_n^\dagger)^\dagger a_1^*.$$

So, the condition (f2) is satisfied.

(f11) $\Rightarrow$ (f1): Observe that $a_n a_n^* (a_1^\dagger aa_n^*)^\dagger \in (a_1^\dagger aa_n^\dagger)\{1,2,3\}$. The equality $(a_1^\dagger a)^\dagger = a_n^* (a_1^\dagger aa_n^*)^\dagger$ implies that $a_n^* (a_1^\dagger aa_n^*)^\dagger a_1^\dagger a$ is selfadjoint and then

$$\begin{aligned}a_n a_n^* (a_1^\dagger aa_n^*)^\dagger a_1^\dagger aa_n^\dagger &= ((a_n^\dagger)^* a_n^* (a_1^\dagger aa_n^*)^\dagger a_1^\dagger aa_n^*)^* = (a_1^\dagger aa_n^*)^\dagger a_1^\dagger aa_n^* a_n a_n^\dagger \\ &= (a_1^\dagger aa_n^*)^\dagger a_1^\dagger aa_n^* \text{ is selfadjoint.}\end{aligned}$$

Hence, $(a_1^\dagger aa_n^\dagger)^\dagger = a_n a_n^* (a_1^\dagger aa_n^*)^\dagger$. Now, by (f2) and the last equality,

$$(a_1^\dagger a)^\dagger = a_n^* (a_1^\dagger aa_n^*)^\dagger = a_n^\dagger a_n a_n^* (a_1^\dagger aa_n^*)^\dagger = a_n^\dagger (a_1^\dagger aa_n^\dagger)^\dagger.$$

Similarly, by the assumption $(aa_n^\dagger)^\dagger = (a_1^* aa_n^\dagger)^\dagger a_1^*$, we can verify $(a_1^\dagger aa_n^\dagger)^\dagger = (a_1^* aa_n^\dagger)^\dagger a_1^* a_1$ and

$$(aa_n^\dagger)^\dagger = (a_1^* aa_n^\dagger)^\dagger a_1^* = (a_1^* aa_n^\dagger)^\dagger a_1^* a_1 a_1^\dagger = (a_1^\dagger aa_n^\dagger)^\dagger a_1^\dagger.$$

Thus, the condition (f1) holds.

(h1) $\Rightarrow$ (i1): Obviously.

(i1) $\Rightarrow$ (h1): It can be checked easy that $(a_n^\dagger)^* a^\dagger (a_1^\dagger)^* \in (a_1^* aa_n^*)\{1,2\}$. From $a_n^\dagger (a_1^* aa_n^*)^\dagger a_1^\dagger = (a_n^* a_n)^\dagger a^\dagger (a_1 a_1^*)^\dagger$, we get

$$\begin{aligned}(a_1^* aa_n^* (a_n^\dagger)^* a^\dagger (a_1^\dagger)^*)^* &= (a_1^* aa_n^* a_n (a_n^* a_n)^\dagger a^\dagger (a_1 a_1^*)^\dagger a_1)^* \\ &= (a_1^* aa_n^* a_n a_n^\dagger (a_1^* aa_n^*)^\dagger a_1^\dagger a_1)^* \\ &= (a_1^* aa_n^* (a_1^* aa_n^*)^\dagger a_1^\dagger a_1)^* \\ &= a_1^\dagger a_1 a_1^* aa_n^* (a_1^* aa_n^*)^\dagger \\ &= a_1^* aa_n^* (a_1^* aa_n^*)^\dagger \text{ is selfadjoint,}\end{aligned}$$

$$
\begin{aligned}
((a_n^\dagger)^* a^\dagger (a_1^\dagger)^* a_1^* a a_n^*)^* &= (a_n (a_n^* a_n)^\dagger a^\dagger (a_1 a_1^*)^\dagger a_1 a_1^* a a_n^*)^* \\
&= (a_n a_n^\dagger (a_1^* a a_n^*)^\dagger a_1^\dagger a_1 a_1^* a a_n^*)^* \\
&= (a_n a_n^\dagger (a_1^* a a_n^*)^\dagger a_1^* a a_n^*)^* \\
&= (a_1^* a a_n^*)^\dagger a_1^* a a_n^* a_n a_n^\dagger \\
&= (a_1^* a a_n^*)^\dagger a_1^* a a_n^* \text{ is selfadjoint,}
\end{aligned}
$$

and conclude that $(a_1^* a a_n^*)^\dagger = (a_n^\dagger)^* a^\dagger (a_1^\dagger)^*$.

Analogously, we can show (h2) $\Leftrightarrow$ (i2), (h3) $\Leftrightarrow$ (i3), (h4) $\Leftrightarrow$ (i4), (h5) $\Leftrightarrow$ (i5), (h6) $\Leftrightarrow$ (i6), (h7) $\Leftrightarrow$ (i7), (h8) $\Leftrightarrow$ (i8), (h9) $\Leftrightarrow$ (i9), (h10) $\Leftrightarrow$ (i10), (e11) $\Leftrightarrow$ (i11), (e12) $\Leftrightarrow$ (i12). □

In the following result we see the relation between reverse order law $a^\dagger = a_n^\dagger \dots a_2^\dagger a_1^\dagger$ and mixed-type reverse order law $a^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$.

Theorem 2.2. *Let $\mathcal{A}$ be a unital C^*–algebra, let $a_1, a_2, \dots, a_n, a = a_1 a_2 \dots a_n \in \mathcal{A}^-$. If $a^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$ and $(a_1^\dagger a a_n^\dagger)^\dagger = a_n a_n^\dagger \dots a_2^\dagger a_1^\dagger a_1$, then $a^\dagger = a_n^\dagger \dots a_2^\dagger a_1^\dagger$.*

Proof. The assumptions give $a^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger = a_n^\dagger a_n a_n^\dagger \dots a_2^\dagger a_1^\dagger a_1 a_1^\dagger = a_n^\dagger \dots a_2^\dagger a_1^\dagger$. □

The condition $a^\dagger = a_n^\dagger (a_1^\dagger a a_n^\dagger)^\dagger a_1^\dagger$ in previous theorem can be replaced by some equivalent conditions from Theorem 2.1.

For $n = 2$ and $n = 3$, as a consequence of Theorem 2.1, we get some special results. In order to avoid repetition, we mention only some of them.

Theorem 2.3. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following statements are equivalent:*

(a1) $(ab)^\dagger = b^\dagger (a^\dagger a b b^\dagger)^\dagger a^\dagger$;

(b1) $[(a^\dagger)^* b]^\dagger = b^\dagger (a^\dagger a b b^\dagger)^\dagger a^*$;

(c1) $[a (b^\dagger)^*]^\dagger = b^* (a^\dagger a b b^\dagger)^\dagger a^\dagger$;

(d1) $(b^\dagger a^\dagger)^\dagger = a (b b^\dagger a^\dagger a)^\dagger b$;

(e1) $(a^\dagger a b)^\dagger a^\dagger = b^\dagger (a b b^\dagger)^\dagger$;

(f1) $(a^\dagger a b)^\dagger = b^\dagger (a^\dagger a b b^\dagger)^\dagger$ *and* $(a b b^\dagger)^\dagger = (a^\dagger a b b^\dagger)^\dagger a^\dagger$;

(g1) $(a^\dagger ab)^\dagger = (ab)^\dagger a$ *and* $(abb^\dagger)^\dagger = b(ab)^\dagger$;

(h1) $(a^* abb^*)^\dagger = (b^\dagger)^*(ab)^\dagger(a^\dagger)^*$;

(i1) $b^\dagger(a^* abb^*)^\dagger a^\dagger = (b^* b)^\dagger(ab)^\dagger(aa^*)^\dagger$.

Some of the statements in the previous theorem are proved in [20] by Tian for products of two matrices. In the case of triple matrix products, we list some generalizations from [24].

Theorem 2.4. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, c \in \mathcal{A}$. If $a, c, abc \in \mathcal{A}^-$, then the following statements are equivalent:*

(a1) $(abc)^\dagger = c^\dagger(a^\dagger abcc^\dagger)^\dagger a^\dagger$;

(b1) $[(a^\dagger)^* bc]^\dagger = c^\dagger(a^\dagger abcc^\dagger)^\dagger a^*$;

(c1) $[ab(c^\dagger)^*]^\dagger = c^*(a^\dagger abcc^\dagger)^\dagger a^\dagger$;

(d1) $(c^\dagger b^* a^\dagger)^\dagger = a(cc^\dagger b^* a^\dagger a)^\dagger c$;

(e1) $(a^\dagger abc)^\dagger a^\dagger = c^\dagger(abcc^\dagger)^\dagger$;

(f1) $(a^\dagger abc)^\dagger = c^\dagger(a^\dagger abcc^\dagger)^\dagger$ *and* $(abcc^\dagger)^\dagger = (a^\dagger abcc^\dagger)^\dagger a^\dagger$;

(g1) $(a^\dagger abc)^\dagger = (abc)^\dagger a$ *and* $(abcc^\dagger)^\dagger = c(abc)^\dagger$;

(h1) $(a^* abcc^*)^\dagger = (c^\dagger)^*(abc)^\dagger(a^\dagger)^*$;

(i1) $c^\dagger(a^* abcc^*)^\dagger a^\dagger = (c^* c)^\dagger(abc)^\dagger(aa^*)^\dagger$.

Notice that the previous results hold in rings with involution where regularity of elements must be change with Moore-Penrose invertibility.

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