GENERAL REPRESENTATIONS OF PSEUDO INVERSES

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ABSTRACT. In this paper we investigate general representations of various classes of generalized inverses for bounded operators over Hilbert and Banach spaces. These representations are expressed by means of the full-rank decomposition of bounded operators and adequately selected operators.

1. Introduction

Let $X_1$ and $X_2$ denote arbitrary Banach spaces and $B(X_1, X_2)$ denote the set of all bounded operators from $X_1$ into $X_2$. For an arbitrary operator $A \in B(X_1, X_2)$, we use $\mathcal{N}(A)$ to denote its kernel, and $\mathcal{R}(A)$ to denote its image.

For $A \in B(X_1, X_2)$ we say that an operator $X \in B(X_2, X_1)$ is a generalized inverse of $A$, provided that some of the following equations are satisfied:

\begin{align*}
(1) \quad AXA &= A, \\
(2) \quad XAX &= X
\end{align*}

If $X$ satisfies the equation (1), then $X$ is called a $g$-inverse of $A$. If $X$ satisfies the equations (1) and (2), then it is called a reflexive $g$-inverse of $A$.

It is well-known that an operator $A \in B(X_1, X_2)$ has a $g$-inverse if and only if $\mathcal{R}(A)$ is closed, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, are complemented subspaces of $X_1$ and $X_2$.

The notion of the full-rank decomposition for complex matrices is well-known and frequently used. Recall the definition of the full rank factorization for a bounded operator acting on Banach spaces from [2] and [3]:

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Let $A \in B(\mathcal{X}_1, \mathcal{X}_2)$. If there exist: a Banach space $\mathcal{X}_3$ and operators $Q \in B(\mathcal{X}_1, \mathcal{X}_3)$ and $P \in B(\mathcal{X}_3, \mathcal{X}_2)$, such that $P$ is left invertible, $Q$ is right invertible and

$$A = PQ,$$

then we say that (1.1) is the full-rank decomposition of $A$.

It is well-known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has the full-rank decomposition, if and only if $A$ is $g$-invertible. In this case $\mathcal{X}_3$ is isomorphic to $\mathcal{R}(A)$, and $\mathcal{R}(A) = \mathcal{R}(P)$ [3].

We say that $A \in B(\mathcal{X})$ has the Drazin inverse, if there exists an operator $A^D \in B(\mathcal{X})$, such that $A^D$ satisfies the equation (2) and the equations

$$(1^k) \quad A^{k+1}A^D = A^k, \quad (5) \quad A^DA = AA^D,$$

for some non-negative integer $k$. Let us mention that the Drazin inverse, if it exists, is unique. The smallest $k$ in the previous definition is called the index of $A$ and denoted by $\text{ind}(A)$. In the case $\text{ind}(A) = 1$ the Drazin inverse is known as the group inverse of $A$, denoted by $A^\#$.

Recall that $\text{asc}(A)$ (respectively $\text{des}(A)$), the ascent (respectively descent) of $A$, is the smallest non-negative integer $n$, such that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ (respectively $\mathcal{R}(A^n) = \mathcal{R}(A^{n+1})$). If no such $n$ exists, then $\text{asc}(A) = \infty$ (respectively $\text{des}(A) = \infty$) [4]. It is well-known that $A$ has the Drazin inverse, if and only if the ascent and descent of $A$ are finite (hence, equal to $\text{ind}(A)$) [3], [4].

In the case when $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, it is well-known that an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has a $g$-inverse if and only if $\mathcal{R}(A)$ is closed. Among the equations (1), (2) we also consider the following equations in $X$:

$$\quad (3) \quad (AX)^* = AX, \quad (4) \quad (XA)^* = XA.$$  

For a subset $\mathcal{S}$ of the set $\{1, 2, 3, 4\}$, the set of operators obeying the conditions contained in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. An operator in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of $A$ and is denoted by $A^{(\mathcal{S})}$. If $\mathcal{R}(A)$ is closed, the set $A\{1, 2, 3, 4\}$ consists of a single element, the Moore-Penrose inverse of $A$, denoted by $A^\dagger$.

We also consider the following equations, which define the weighted Moore-Penrose inverse:

$$\quad (3M) \quad (MAX)^* = MAX \quad (4N) \quad (NXA)^* = NXA;$$

$$\quad (3M') \quad (AX)^*M = MAX \quad (4N') \quad (XA)^*N = NXA,$$
where $M \in B(H_2)$, $N \in B(H_1)$ are positive or invertible. Any solution of the equations (1), (2), (3M) and (4N), when it exists, will be denoted by $A_{M,N}^\dagger$. Similarly, any solution of the equations (1), (2), (3M') and (4N'), when it exists, will be denoted by $A_{M',N'}^\dagger$.

We investigate general representations and conditions for the existence of generalized inverses of bounded linear operators on Hilbert spaces, arising from the factorization (1.1). As a related result we investigate some representations of a generalized inverse $A_{T,S}^{(2)}$. Obtained representations are generalizations of the analogous results available in the literature for matrices. We also introduce a general representation and conditions for the existence of the Drazin inverse of a bounded operator on a Banach space. There representations are based on the full-rank decomposition of $A^l$, where $l \geq \text{ind}(A)$. Such an approach in representation of the Drazin inverse is not employed before even for complex matrices.

2. Results

Firstly, we investigate general representations of $\{1,2\}$-inverses, $\{1,2,3\}$, $\{1,2,4\}$-inverses, the Moore-Penrose and the weighted Moore-Penrose for operators in arbitrary Hilbert spaces.

We shall frequently use the following observation. If $S \in B(H_1,H_2)$ is onto, then $SS^*$ is invertible and $S^*$ is the right inverse of $S$. Analogously, if $T \in B(H_1,H_2)$ is one-to-one with closed range, then $T^*T$ is invertible and $T^*$ is the left inverse of $T$.

In the beginning, we state an analogy of the well-known result from [10, p. 20, 28].

**Lemma 2.1.** If $A = PQ$ is the full-rank decomposition of $A \in B(H_1,H_2)$ according to (1.1), then:

(a) Any right inverse of $Q$ can be represented in the following form: $Q^{-1}_r = W_1(QW_1)^{-1}$, for an arbitrary operator $W_1 \in B(H_3,H_1)$ such that $QW_1$ is invertible.

(b) Any left inverse of $P$ can be represented in the following form: $P^{-1}_l = (W_2P)^{-1}W_2$, for an arbitrary operator $W_2 \in B(H_2,H_3)$ such that $W_2P$ is invertible.

(c) Any reflexive generalized inverse $X$ of $A$ has the form $X = Q^{-1}_rP^{-1}_l$ for an arbitrary right inverse $Q^{-1}_r$ of $Q$ and an arbitrary left inverse $P^{-1}_l$ of $P$. 
In the literature there are known general representations for various classes of generalized inverses, for the set of complex matrices. The general representation of \( \{1, 2\} \) inverses for matrices is investigated in [9] and [10, p. 20, 28]. The general representations of \( \{1, 2, 3\} \) and \( \{1, 2, 4\} \) inverses for matrices are investigated in [9]. In [6] there is given a general representation and conditions for the existence of the group inverse for a given complex matrix. The general representation of the Moore-Penrose inverse is given in [2], for arbitrary Hilbert spaces.

In the following theorem we give general representations of \( \{1, 2\} \), \( \{1, 2, 3\} \) and \( \{1, 2, 4\} \) inverses for an arbitrary bounded operator on Hilbert spaces. As a consequence we obtain the known representation of the Moore-Penrose inverse from [2].

**Theorem 2.1.** Let \( A = PQ \) be a full–rank decomposition of \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) according to (1.1). Then:

\( (a) \) \( X \in A \{1, 2\} \) if and only if there exist operators \( W_1 \in B(\mathcal{H}_3, \mathcal{H}_1) \) and \( W_2 \in B(\mathcal{H}_2, \mathcal{H}_3) \), such that \( QW_1 \) and \( W_2P \) are invertible in \( B(\mathcal{H}_3) \). In such a case, \( X \) possesses the following general representation

\[
X = W_1(QW_1)^{-1}(W_2P)^{-1}W_2
\]

\( (b) \) \( X \in A \{1, 2, 3\} \) if and only if there exists an operator \( W_1 \in B(\mathcal{H}_3, \mathcal{H}_1) \), such that \( QW_1 \) is invertible in \( B(\mathcal{H}_3) \). In the case when it exists, a general representation for \( X \) is as follows:

\[
X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*.
\]

\( (c) \) \( X \in A \{1, 2, 4\} \) if and only if there exists an operator \( W_2 \in B(\mathcal{H}_2, \mathcal{H}_3) \), such that \( W_2P \) is invertible in \( B(\mathcal{H}_3) \). In this case

\[
X = Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2.
\]

\( (d) \) \( A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^* \).

**Proof.** (a) Follows from Lemma 2.1.

(b) If \( X \) has the form (2.2), then it is easy to verify \( X \in A \{1, 2, 3\} \). We need to prove that the form (2.2) holds for all \( \{1, 2, 3\} \) inverses of \( A \). Indeed, if \( X \in A \{1, 2, 3\} \), then \( X = Q_rP_r^{-1} \), and from the equation (3) it follows that \( (PP_r^{-1})^* = PP_r^{-1} \). Thus \( P^*PP_r^{-1} = P^* \). Operator \( P^*P \) is invertible,
so that \( P_l^{-1} = (P^*P)^{-1}P^* \). The right inverse of \( Q \) retains the general form \( Q_r^{-1} = W_1(QW_1)^{-1} \) from Lemma 2.1. Consequently,
\[
X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*.
\]

The proof of the statement (c) is similar as the proof of (b). Also, (d) follows from (b) and (c). For the part (d) see also [2]. □

Now, we shall consider the weighted Moore-Penrose inverse under the various hypothesis. The weighted Moore-Penrose inverse is investigated in [1], [7] and [10] for the set of complex matrices and in [8] for matrices over an integral domain. If \( M \) and \( N \) are positive, then \( A_{M,N}^{\dagger} \) and \( A_{M',N'}^{\dagger} \) always exist [1], [10], and \( A_{M,N}^{\dagger} = A_{M',N'}^{\dagger} \). In [8] and [7] it is derived a representation and conditions for the existence of \( A_{M,N}^{\dagger} \) and \( A_{M',N'}^{\dagger} \), respectively, under the more general assumptions that the matrices \( M \) and \( N \) are invertible (not necessary positive).

**Theorem 2.2.** Let \( A = PQ \) be a full–rank decomposition of \( A \) according to (1.1). Then:

(a) If \( M \in B(H_2) \) and \( N \in B(H_1) \) are invertible operators, then \( A_{M,N}^{\dagger} \) exists if and only if \( P^*MP \) and \( QN^{-1}Q^* \) are invertible selfadjoint operators. In that case

\[
A_{M,N}^{\dagger} = N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M^* = N^{-1}Q^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*.
\]

(b) Let \( M \in B(H_2) \) and \( N \in B(H_1) \) be invertible operators, such that \( QN^{-1}Q^* \) is left invertible and \( P^*MP \) right invertible. Then \( A_{M',N'}^{\dagger} \) exists if and only if \( QN^{-1}Q^* \) and \( P^*MP \) are invertible and

\[
E = N^{-1}Q^*(QN^{-1}Q^*)^{-1} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1},
\]

\[
F = (P^*MP)^{-1}P^*M = ((MP)^*P)^{-1}(MP)^*.
\]

In this case is \( A_{M',N'}^{\dagger} = Q_r^{-1}P_l^{-1} \), where \( Q_r^{-1} = E \) and \( P_l^{-1} = F \).

(c) If \( M \in B(H_2) \) and \( N \in B(H_1) \) are positive and invertible operators, then

\[
A_{M,N}^{\dagger} = A_{M',N'}^{\dagger} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* = N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M.
\]
Proof. (a) If $M$ and $N$ are invertible operators and $A_{M,N}^\dagger$ exists, using the principles from [8], from (3M) and (4N) we get

$$P^*MP = P^*M^*P, \quad QN^{-1}Q^* = Q(N^{-1})^*Q^*.$$  

From (2.7) we conclude that $P^*MP$ and $QN^{-1}Q^*$ are selfadjoint operators. We now prove that $P^*MP = P^*M^*P$ and $QN^{-1}Q^* = Q(N^{-1})^*Q^*$ are invertible. Indeed, from (1) and (3M) we get the following equation (see [8]):

$$(QXM^{-1}X^*Q^*)(P^*M^*P) = I.$$  

This means that $P^*M^*P$ is left invertible. Also, since $P^*M^*P$ is selfadjoint, we conclude that $P^*M^*P$ is invertible. Similarly, (1) and (4N) imply the following

$$(QN^{-1}Q^*)(P^*X^*NXP) = I,$$  

which means that $QN^{-1}Q^*$ is right invertible, so it is also invertible.

Using invertibility of $QN^{-1}Q^*$ and $P^*M^*P$, from (2.6) it follows

$$P_l^{-1} = (P^*M^*P)^{-1}(MP)^*, \quad Q_r^{-1} = N^{-1}Q^*(Q(N^{-1})^*Q^*)^{-1}.$$  

Now, the representations (2.3) follows from $A_{M,N}^\dagger = Q_r^{-1}P_l^{-1}$, (2.8) and (2.7).

(b) Suppose that $M \in B(H_2)$ and $N \in B(H_1)$ are invertible operators, such that $QN^{-1}Q^*$ is left invertible and $P^*MP$ right invertible and $A_{M',N'}^\dagger$ exists. From the equations (1) and (3M') in the same way as in [7] we get

$$(QXM^{-1}X^*Q^*)(P^*M^*P) = I,$$  

which means that $P^*MP$ is left invertible. Similarly, from (1) and (4N') we obtain

$$(QN^{-1}Q^*)(P^*X^*NXP) = I,$$  

which implies the right invertibility of $QN^{-1}Q^*$. According to the assumptions, we conclude that $P^*MP$ and $QN^{-1}Q^*$ are invertible. The identities (2.4) can be proved using the method from [7].

On the other hand, if $P^*MP$ and $QN^{-1}Q^*$ are invertible and (2.4) holds, one can verify that $Q_r^{-1}P_l^{-1} \quad (Q_r^{-1} = E$ and $P_l^{-1} = F)$ satisfies the equations which define $A_{M',N'}^\dagger$.

(c) Firstly, we prove that $Q(QN^{-1})^*x$ and $(MP)^*P$ are positive and invertible in $B(H_3)$. If $x \in H_3$ and $\|x\| = 1$, then

$$(Q(QN^{-1})^*x,x) = (N^{-1}Q^*x,Q^*x) > 0.$$
Suppose that \( \inf_{\|x\|=1} (Q(QN^{-1})^*x, x) = 0 \). Then there exists a sequence of unit vectors \((x_n)\) in \( \mathcal{H}_3 \), such that \( \lim_{n} (Q^{-1}Q^*x_n, Q^*x_n) = 0 \). Since \( N^{-1} \) is positive and invertible, it follows that there exists a subsequence \((x_{n_k})\) of \((x_n)\), such that \( \lim_{k} Q^*x_{n_k} = 0 \). Now, it follows that \( Q^* \) is not one-to-one with closed range, so \( Q \) is not onto. We get the contradiction, so \( Q(QN^{-1})^* \) is positive and invertible in \( B(\mathcal{H}_3) \). Analogously, we can prove that \( (MP)^*P \) is positive and invertible in \( B(\mathcal{H}_3) \).

The rest of the proof follows from parts (a) and (b). \( \square \)

Now, we consider \{2\}-generalized inverses with prescribed range and kernel. Fundamental results for matrices can be found in [1] and [5].

Let \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) and \( X \in B(\mathcal{H}_2, \mathcal{H}_1) \) be a \{2\}-inverse of \( A \), such that \( \mathcal{R}(X) = T \) is a closed subspace of \( \mathcal{H}_1 \) and \( \mathcal{N}(X) = S \) is a closed subspace of \( \mathcal{H}_2 \). Then we write \( X = A_{T,S}^{(2)} \). For given closed subspaces \( T \) of \( \mathcal{H}_1 \) and \( S \) of \( \mathcal{H}_2 \), it is a natural question when \( A_{T,S}^{(2)} \) exists? The answer in the case of arbitrary Hilbert spaces is given in the following theorem.

We state the following elementary result.

**Lemma 2.2.** Let \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \), \( T \) and \( S \) be closed subspaces of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively. Then the following statements are equivalent:

- (a) \( A \) has a \{2\}-inverse \( X \in B(\mathcal{H}_2, \mathcal{H}_1) \) such that \( \mathcal{R}(X) = T \) and \( \mathcal{N}(X) = S \);

- (b) \( A : T \to A(T) \) is invertible and \( A(T) \oplus S = \mathcal{H}_2 \).

In the case when (a) or (b) holds, \( X \) is unique and is denoted by \( A_{T,S}^{(2)} \).

Now, we generalize the result from [5].

**Theorem 2.3.** Suppose that \( A, T \) and \( S \) satisfy the condition (a) or (b) from Lemma 2.2 and let \( Y \in B(\mathcal{H}_2, \mathcal{H}_1) \) be such that \( \mathcal{R}(Y) = T \) and \( \mathcal{N}(Y) = S \). If there exists a Hilbert space \( \mathcal{H}_3 \) and a left invertible operator \( E \in B(\mathcal{H}_3, \mathcal{H}_1) \) such that \( \mathcal{R}(E) = T \), then

\[
W = E^*YA E \in B(\mathcal{H}_3)
\]

is invertible in \( B(\mathcal{H}_3) \) and

\[
A_{T,S}^{(2)} = EW^{-1}E^*Y.
\]

**Proof.** With the help of the proof of Theorem 2.3, notice that \( E : \mathcal{H}_3 \to T \) is invertible, \( A : T \to AT \) is invertible and \( Y : AT \to T \) is invertible.
Since $\mathcal{N}(E^*)^\perp = \mathcal{R}(E) = T$, it follows that $E^*: T \to \mathcal{H}_3$ is invertible, so $W$ is invertible. Now, it is easy to verify that $EW^{-1}E^*Y$ is a $\{2\}$-inverse of $A$. Also, $\mathcal{N}(Y) = S$, $W^{-1}E^*Y : AT \to \mathcal{H}_3$ is invertible and $\mathcal{R}(EW^{-1}E^*Y) = \mathcal{R}(E) = T$, $\mathcal{N}(EW^{-1}E^*Y) = S$, so

$$A_{T,S}^{(2)} = EW^{-1}E^*Y.$$ □

**Theorem 2.4.** Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $X : \mathcal{H}_2 \to \mathcal{H}_1$. Then $X \in A\{2\}$ if and only if there exist Hilbert spaces $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_5$ and operators

$$C \in B(\mathcal{H}_4, \mathcal{H}_1), \ D \in B(\mathcal{H}_2, \mathcal{H}_3), \ W_1 \in B(\mathcal{H}_5, \mathcal{H}_4), \ W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$$

such that $DAC$ is $g$-invertible and $W_2DACW_1$ is invertible. In this case:

$$X = CW_1(W_2DACW_1)^{-1}W_2D.$$ (2.9)

**Proof.** If $X$ possesses the form (2.9), it is not difficult to verify $X \in A\{2\}$. On the other hand, using the method from [10], it is easy to verify that $X \in A\{2\}$ if and only if there exist operators $C$ and $D$, such that

$$X = C(DAC)^{(1,2)}D, \ C \in B(\mathcal{H}_4, \mathcal{H}_1), \ D \in B(\mathcal{H}_2, \mathcal{H}_3).$$

According to part (a) of Theorem 2.1, $X \in A\{2\}$ if and only if there exist operators $W_1$ and $W_2$, such that $W_2DACW_1$ is invertible, and $X$ possesses the form (2.9). □

We introduce a general representation of the Drazin inverse based on an arbitrary full-rank factorization of $A^l$, $l \geq k = \text{asc}(A) = \text{des}(A)$. The following theorem is a natural generalization of a Cline’s result from [6], introduced for complex matrices. We shall assume that $A$ is not a nilpotent operator, i.e. $A^D \neq 0$.

**Theorem 2.5.** Let $\mathcal{X}$ be a Banach space. If $A \in B(\mathcal{X})$, $l \geq k = \text{asc}(A) = \text{des}(A) < \infty$ and $A^l = P_{A^l}Q_{A^l}$ is the full-rank decomposition of $A^l$, then

$$A^D = P_{A^l}(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}.$$ (2.10)

**Proof.** If asc($A$) = des($A$) = $k < \infty$, then it is well-known that $\mathcal{N}(A^l) = \mathcal{N}(A^k)$ and $\mathcal{R}(A^l) = \mathcal{R}(A^k)$ for all $l \geq k$,
where \( \mathcal{X}_1 = \mathcal{N}(A^i) \) and \( \mathcal{X}_2 = \mathcal{R}(A^i) \), \( A(\mathcal{X}_i) \subset \mathcal{X}_i \) for \( i = 1, 2 \), \( A_1 = A|_{\mathcal{X}_1} \) is nilpotent and \( A_2 = A|_{\mathcal{X}_2} \) is invertible (\( A \) is not nilpotent) [3], [4]. We can write

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix}
\]

with respect to the decomposition (2.10) ([3],[4]). Since \( \mathcal{N}(A_1) \) and \( \mathcal{R}(A_1) \) are complementary and closed subspaces of \( \mathcal{X} \), it follows that \( A_1 \) is \( g \)-invertible, so there exists the full-rank decomposition \( A_1 = P_AQ_A \), where \( P_A \in B(\mathcal{X}, \mathcal{X}) \) is left invertible and \( Q_A \in B(\mathcal{X}, \mathcal{X}) \) is right invertible, for some Banach space \( \mathcal{Z} \). By the isomorphism theorem [3], we can take that \( \mathcal{Z} = \mathcal{X}_2 \). We conclude that \( P_A \) and \( Q_A \) have the following representations with respect to (2.10):

\[
P_A = \begin{bmatrix} M \\ \tilde{P} \end{bmatrix} \quad \text{and} \quad Q_A = \begin{bmatrix} N & \tilde{Q} \end{bmatrix},
\]

where \( \tilde{P}, \tilde{Q} \in B(\mathcal{X}_2), M \in B(\mathcal{X}_2, \mathcal{X}_1), N \in B(\mathcal{X}_1, \mathcal{X}_2) \). Now, \( P_A \) is left invertible and \( Q_A \) is right invertible, so \( P_A \) and \( Q_A \) are \( g \)-invertible operators, \( \mathcal{N}(P_A) = \{0\} \) and \( \mathcal{R}(Q_A) = \mathcal{X}_2 \). It follows that \( \mathcal{R}(P_A) = \mathcal{R}(A^i) = \mathcal{X}_2 \) and \( \mathcal{N}(Q_A) = \mathcal{N}(A^i) = \mathcal{X}_1 \), so \( M = 0 \), \( N = 0 \) and

\[
P_A = \begin{bmatrix} 0 \\ \tilde{P} \end{bmatrix} \quad \text{and} \quad Q_A = \begin{bmatrix} 0 & \tilde{Q} \end{bmatrix}.
\]

It is easy to verify that \( \tilde{P} \) is left invertible and \( \tilde{Q} \) is right invertible in \( B(\mathcal{X}_2) \). But

\[
\begin{bmatrix} 0 & 0 \\ 0 & A_2^i \end{bmatrix} = A^i = P_A Q_A = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \tilde{Q} \end{bmatrix},
\]

so \( A_2^i = \tilde{P} \tilde{Q} \). Since \( A_2^i \) is invertible, it follows that \( \tilde{P} \) and \( \tilde{Q} \) are invertible in \( B(\mathcal{X}_2) \).

Now, \( Q_A A_2^i = \tilde{Q} A_2^i \tilde{P} \) is invertible in \( B(\mathcal{X}_2) \), so

\[
A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} (\tilde{Q} A_2^i \tilde{P})^{-1} \tilde{Q} \end{bmatrix} = P_A(A^i Q_A A_2^i)^{-1} Q_A.
\]

As a corollary, we get the following result.
Corollary 2.1. If $\mathcal{X}$ is a Banach space, $A \in B(\mathcal{X})$ and $\text{asc}(A) = \text{des}(A) = k < \infty$ and $A^l = P_{A^l}Q_{A^l}$ is an arbitrary full-rank decomposition of $A^l$, $l \geq k$, then

(a) $(A^D)^l = P_{A^l}(Q_{A^l}A^lP_{A^l})^{-1}Q_{A^l} = P_{A^l}(Q_{A^l}P_{A^l})^{-2}P_{A^l}$;

(b) $AA^D = P_{A^l}(Q_{A^l}P_{A^l})^{-1}Q_{A^l}$;

(c) If $\mathcal{X}$ is a Hilbert space, then $(A^D)^\dagger = (Q_{A^l})^\dagger Q_{A^l}AP_{A^l}(P_{A^l})^\dagger$.

Proof. (a) Follows from $(A^D)^l = (A^l)^\#$ and Theorem 2.6.

(b) According to Theorem 2.6 it follows that $Q_{A^l}P_{A^l} = \tilde{Q}\tilde{P}$, so an easy computation shows that

$$P_{A^l}(Q_{A^l}P_{A^l})^{-1}Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = AA^D.$$

(c) Follows from Theorem 2.1 (d) and Theorem 2.6. $\square$

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