Factorization of weighted–EP elements in C^*-algebras

Dijana Mosić, Dragan S. Djordjević*

Abstract

We present characterizations of weighted–EP elements in C^*-algebras using different kinds of factorizations.

Key words and phrases: weighted–EP elements, Moore–Penrose inverse, group inverse, C^*-algebra.
2010 *Mathematics subject classification*: 46L05, 47A05.

1 Introduction

The weighted–EP matrices are characterized by commutativity with their weighted Moore–Penrose inverse. They were introduced and investigated by Tian and Wang in [26]. The notion of weighted–EP matrices was extended to elements of C^*-algebras in [23].

Generalized inverses have lots of applications in numerical linear algebra, as well as in approximation methods in general Hilbert spaces. Hence, we characterize weighted–EP elements of C^*-algebras through various factorizations.

Let $\mathcal{A}$ be a unital C^*–algebra with the unit 1. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. The set of all regular elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^-$. An element $a \in \mathcal{A}$ satisfying $a^* = a$ is called *symmetric* (or *Hermitian*). An element $x \in \mathcal{A}$ is positive if $x = y^*y$ for some $y \in \mathcal{A}$. Notice that positive elements are self-adjoint.

An element $a^\dagger \in \mathcal{A}$ is the *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{A}$, if the following hold [25]:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that above conditions hold (see [13, 15]).

*The authors are supported by the Ministry of Education and Science, Serbia, grant no. 174007.

Theorem 1.1. [13] *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

Let e, f be invertible positive elements in $\mathcal{A}$. The element $a \in \mathcal{A}$ has the weighted MP-inverse with weights e, f, if there exists $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^\dagger_{e,f}$ if it exists [7].

Theorem 1.2. [7] *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then the unique weighted MP-inverse $a^\dagger_{e,f}$ exists.*

Define the mapping $(*, e, f) : x \mapsto x^{*e,f} = e^{-1}x^*f$, for all $x \in \mathcal{A}$. Notice that $(*, e, f) : \mathcal{A} \to \mathcal{A}$ is not an involution, because in general $(xy)^{*e,f} \neq y^{*e,f}x^{*e,f}$. The following result is frequently used in the rest of the paper.

Theorem 1.3. [23] *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be positive invertible elements of $\mathcal{A}$. For any $a \in \mathcal{A}^-$, the following is satisfied:*

(a) $(a^\dagger_{e,f})^\dagger_{f,e} = a$;

(b) $(a^{*f,e})^\dagger_{f,e} = (a^\dagger_{e,f})^{*e,f}$;

(c) $a^{*f,e} = a^\dagger_{e,f}aa^{*f,e} = a^{*f,e}aa^\dagger_{e,f}$;

(d) $a^{*f,e}(a^\dagger_{e,f})^{*e,f} = a^\dagger_{e,f}a$;

(e) $(a^\dagger_{e,f})^{*e,f}a^{*f,e} = aa^\dagger_{e,f}$;

(f) $(a^{*f,e}a)^\dagger_{f,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*e,f}$;

(g) $(aa^{*f,e})^\dagger_{e,e} = (a^\dagger_{e,f})^{*e,f}a^\dagger_{e,f}$;

(h) $a^\dagger_{e,f} = (a^{*f,e}a)^\dagger_{f,f}a^{*f,e} = a^{*f,e}(aa^{*f,e})^\dagger_{e,e}$;

(i) $(a^{*e,f})^\dagger_{f,e} = a(a^{*f,e}a)^\dagger_{f,f} = (aa^{*f,e})^\dagger_{e,e}a$.

For $a \in \mathcal{A}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{A} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{A} : xa = 0\}.$$

Observe that,

$$(a^*)^\circ = a^\circ \Leftrightarrow {}^\circ(a^*) = {}^\circ a, \qquad\qquad a\mathcal{A} = a^*\mathcal{A} \Leftrightarrow \mathcal{A}a = \mathcal{A}a^*.$$

Lemma 1.1. [11] *The following hold for $a \in \mathcal{A}$.*

(i) $a \in \mathcal{A}^{-1} \iff a\mathcal{A} = \mathcal{A}$ *and* $a^\circ = \{0\}$.

(ii) $a \in \mathcal{A}^{-} \iff \mathcal{A} = (a^*\mathcal{A}) \oplus a^\circ$.

(iii) $a^*\mathcal{A} = \mathcal{A} \iff a \in \mathcal{A}^{-}$ *and* $a^\circ = \{0\}$.

The following lemmas related to weighted MP-inverse are very useful.

Lemma 1.2. [23] *Let $a \in \mathcal{A}^{-}$ and let e, f be invertible positive elements in $\mathcal{A}$. Then*

(i) $a^\dagger_{e,f}\mathcal{A} = a^\dagger_{e,f}a\mathcal{A} = f^{-1}a^*\mathcal{A} = a^{*f,e}\mathcal{A}$;

(ii) $(a^\dagger_{e,f})^*\mathcal{A} = (aa^\dagger_{e,f})^*\mathcal{A} = ea\mathcal{A} = (a^{*f,e})^*\mathcal{A}$;

(iii) $a^\circ = (ea)^\circ$;

(iv) $(a^*)^\circ = (f^{-1}a^*)^\circ$;

(v) $(a^\dagger_{e,f})^\circ = [(ea)^*]^\circ = (a^{*f,e})^\circ$;

(vi) $[(a^\dagger_{e,f})^*]^\circ = (af^{-1})^\circ$.

Lemma 1.3. [23] *Let $a \in \mathcal{A}^{-}$, and let e, f be invertible positive elements in $\mathcal{A}$. Then*

$$a^\dagger_{e,f} = (a^{*f,e}a + 1 - a^\dagger_{e,f}a)^{-1}a^{*f,e} = a^{*f,e}(aa^{*f,e} + 1 - aa^\dagger_{e,f})^{-1}, \tag{1}$$

$$a^{*f,e}\mathcal{A}^{-1} = a^\dagger_{e,f}\mathcal{A}^{-1} \text{ and } \mathcal{A}^{-1}a^{*f,e} = \mathcal{A}^{-1}a^\dagger_{e,f}, \tag{2}$$

$$(a^{*f,e})^\circ = (a^\dagger_{e,f})^\circ \text{ and } {}^\circ(a^{*f,e}) = {}^\circ(a^\dagger_{e,f}). \tag{3}$$

We recall the definition of EP elements.

Definition 1.1. An element $a \in \mathcal{A}^{-}$ is EP if $aa^\dagger = a^\dagger a$.

Lemma 1.4. [17] *An element $a \in \mathcal{A}$ is EP, if $a \in \mathcal{A}^{-}$ and $a\mathcal{A} = a^*\mathcal{A}$ (or, equivalently, if $a \in \mathcal{A}^{-}$ and $a^\circ = (a^*)^\circ$).*

Many authors have investigated various characterizations of EP elements in a ring and C^*-algebra (see, for example, [15, 17, 18, 20, 21, 24]), many more still for Banach or Hilbert space operators and matrices (see [1, 2, 4, 5, 6, 8, 9, 10, 14, 16, 19, 22]). In [12], Drivaliaris, Karanasios and Pappas and in [11] Djordjević, J.J. Koliha and I. Straškraba have characterized EP Hilbert space operators and EP C^*-algebra elements respectively trough several different factorizations. Boasso [3] have recently characterized EP Banach space operators and EP Banach algebra elements using factorizations, extending results of [11, 12].

Now, we state the definition of weighted–EP elements and some characterizations of weighted–EP elements.

Definition 1.2. [23] An element $a \in \mathcal{A}$ is said to be weighted–EP with respect to two invertible positive elements $e, f \in \mathcal{A}$ (or weighted–EP w.r.t. (e,f)) if both ea and af^{-1} are EP, that is $a \in \mathcal{A}^-$, $ea\mathcal{A} = (ea)^*\mathcal{A}$ and $af^{-1}\mathcal{A} = (af^{-1})^*\mathcal{A}$.

Theorem 1.4. [23] *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:*

(i) *a is weighted–EP w.r.t. (e,f);*

(ii) $aa^\dagger_{e,f} = a^\dagger_{e,f}a$;

(iii) $a^\dagger_{e,f} = a(a^\dagger_{e,f})^2 = (a^\dagger_{e,f})^2a$;

(iv) $a \in a^\dagger_{e,f}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}a^\dagger_{e,f}$

(v) $a \in a^\dagger_{e,f}\mathcal{A} \cap \mathcal{A}a^\dagger_{e,f}$;

(vi) $a\mathcal{A} = a^{*f,e}\mathcal{A}$ *and* $\mathcal{A}a = \mathcal{A}a^{*f,e}$;

(vi) $a^\circ = (a^{*f,e})^\circ$ *and* ${}^\circ a = {}^\circ(a^{*f,e})$.

We turn our attention for characterizing weighted–EP elements in terms of factorizations, motivated by papers [3, 11, 12], which are related to similar characterizations of EP elements.

2 Factorization $a = ba^{*f,e}$

In this section we characterize weighted–EP elements of C^*–algebras through factorizations of the form $a = ba^{*f,e}$.

Theorem 2.1. *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:*

(i) *a is weighted–EP w.r.t. (e,f);*

(ii) *$a = ba^{*f,e} = a^{*f,e}c$ for some $b, c \in \mathcal{A}$;*

(iii) *$a^{*f,e}a = b_1a^{*f,e} = ac_1$ and $aa^{*f,e} = a^{*f,e}b_2 = c_2a$ for some $b_1, b_2, c_1, c_2 \in \mathcal{A}$;*

(iv) *$a^{*f,e}a = b_3a^\dagger_{e,f}$, $aa^{*f,e} = a^\dagger_{e,f}b_4$ and $a^\dagger_{e,f} = c_3a = ac_4$ for some $b_3, b_4, c_3, c_4 \in \mathcal{A}$.*

Proof. (i) $\Leftrightarrow$ (ii): By Theorem 1.4, a is weighted–EP w.r.t. (e,f) if and only if $a \in a^\dagger_{e,f}\mathcal{A} \cap \mathcal{A}a^\dagger_{e,f}$, which is equivalent to $a \in a^{*f,e}\mathcal{A} \cap \mathcal{A}a^{*f,e}$, by Lemma 1.2. Thus, the equivalence (i) $\Leftrightarrow$ (ii) holds.

(i) $\Leftrightarrow$ (iii): Notice that, by Theorem 1.3, $a\mathcal{A} = aa^{*f,e}\mathcal{A}$, $\mathcal{A}a = \mathcal{A}a^{*f,e}a$, $a^{*f,e}\mathcal{A} = a^{*f,e}a\mathcal{A}$ and $\mathcal{A}a^{*f,e} = \mathcal{A}aa^{*f,e}$. Now (iii) is equivalent to $a\mathcal{A} = a^{*f,e}\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{*f,e}$. By Theorem 1.4, these equalities hold if and only if a is weighted–EP w.r.t. (e,f).

(i) $\Leftrightarrow$ (iv): Similarly as the previous part. $\square$

3 Factorization $a^{*f,e} = sa$

In this section, the weighted–EP elements of the form $a^{*f,e} = sa$ or $a^\dagger_{e,f} = sa$ will be characterized.

We start with characterizations of weighted–EP elements via factorizations of the form $a^{*f,e} = sa$.

Theorem 3.1. *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:*

(i) *a is weighted–EP w.r.t. (e,f);*

(ii) *$\exists\ s, t \in \mathcal{A} : s^\circ = {}^\circ t = \{0\}$ and $a^{*f,e} = sa = at$;*

(iii) $\exists\, s_1, s_2, t_1, t_2 \in \mathcal{A} : a^{*f,e} = s_1 a = a t_1$ *and* $a = s_2 a^{*f,e} = a^{*f,e} t_2$;

(iv) $\exists\, u, v \in \mathcal{A} : u\mathcal{A} = \mathcal{A} = \mathcal{A}v$ *and* $a^{*f,e} = au = va$;

(v) $\exists\, x, y \in \mathcal{A}^{-1} : a^{*f,e}a = xaa^{*f,e} = aa^{*f,e}y$;

(vi) $\exists\, x_1, y_1 \in \mathcal{A} : x_1^{\circ} = {}^{\circ}y_1 = \{0\}$ *and* $a^{*f,e}a = x_1 aa^{*f,e} = aa^{*f,e}y_1$;

(vii) $\exists\, x_2, y_2 \in \mathcal{A} : \mathcal{A}x_2 = \mathcal{A} = y_2\mathcal{A}$ *and* $a^{*f,e}a = x_2 aa^{*f,e} = aa^{*f,e}y_2$;

(viii) $\exists\, x_3, x_4, y_3, y_4 \in \mathcal{A} : a^{*f,e}a = x_3 aa^{*f,e} = aa^{*f,e}y_3$ *and* $aa^{*f,e} = x_4 a^{*f,e}a = a^{*f,e}ay_4$;

(ix) $\exists\, z_1, z_2 \in \mathcal{A} : a^{*f,e}a = az_1 a^{*f,e}$ *and* $aa^{*f,e} = a^{*f,e}z_2 a$;

(x) $\exists\, g_1, h_1 \in \mathcal{A}^{-1} : a^{*f,e}a = ah_1 h_1^{*e,f} a^{*f,f}$ *and* $aa^{*f,e} = a^{*e,f} g_1^{*f,f} g_1 a$;

(xi) $\exists\, g_2, h_2 \in \mathcal{A} : g_2^{\circ} = {}^{\circ}h_2 = \{0\}$, $a^{*f,e}a = ah_2 h_2^{*e,f} a^{*f,f}$ *and* $aa^{*f,e} = a^{*e,f} g_2^{*f,f} g_2 a$;

(xii) $\exists\, g_3, h_3 \in \mathcal{A} : \mathcal{A}g_3 = \mathcal{A} = h_3\mathcal{A}$, $a^{*f,e}a = ah_3 h_3^{*e,f} a^{*f,f}$ *and* $aa^{*f,e} = a^{*e,f} g_3^{*f,f} g_3 a$.

Proof. (i) $\Rightarrow$ (ii): If a is weighted–EP w.r.t. (e,f), by Theorem 1.4, $a \in a^{\dagger}_{e,f}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}a^{\dagger}_{e,f}$, i.e. $a \in a^{*f,e}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}a^{*f,e}$, by Lemma 1.3. So, there exist $s, t \in \mathcal{A}^{-1}$ such that $a^{*f,e} = sa = at$ and the statement (ii) holds.

Similarly, we can prove that (i) implies (iii) and (iv).

(ii) $\Rightarrow$ (i): The condition (ii) implies $a^{\circ} \subseteq (a^{*f,e})^{\circ}$ and ${}^{\circ}a \subseteq {}^{\circ}(a^{*f,e})$. Let $x \in (a^{*f,e})^{\circ}$, then $sax = a^{*f,e}x = 0$, by $s^{\circ} = \{0\}$, gives $ax = 0$. Hence, $a^{\circ} = (a^{*f,e})^{\circ}$ and, analogy, ${}^{\circ}a = {}^{\circ}(a^{*f,e})$. By Theorem 1.4, a is weighted–EP w.r.t. (e,f).

In the similar way, we can check (iii) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (i): From the assumption (iv), we deduce that $a^{*f,e}\mathcal{A} = au\mathcal{A} = a\mathcal{A}$ and $\mathcal{A}a^{*f,e} = \mathcal{A}va = \mathcal{A}a$ which gives that the condition (i) is satisfied, by Theorem 1.4.

(i) $\Rightarrow$ (v): Let $x = (a^{\dagger}_{e,f})^{*e,f}a^{\dagger}_{e,f} + 1 - aa^{\dagger}_{e,f} (= (aa^{*f,e} + 1 - aa^{\dagger}_{e,f})^{-1})$ and $y = a^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*e,f} + 1 - a^{\dagger}_{e,f}a (= (a^{*f,e}a + 1 - a^{\dagger}_{e,f}a)^{-1})$. Then $x, y \in \mathcal{A}^{-1}$, $a^{*f,e} = y^{-1}a^{\dagger}_{e,f}$ and $a^{\dagger}_{e,f} = a^{*f,e}x$, by Lemma 1.3. Now, we can verify that $aa^{*f,e}x = xaa^{*f,e} = aa^{\dagger}_{e,f}$ and $a^{*f,e}ay = ya^{*f,e}a = a^{\dagger}_{e,f}a$. Further,

$$a^{*f,e}a = y^{-1}(a^{\dagger}_{e,f}a) = y^{-1}aa^{\dagger}_{e,f} = y^{-1}(aa^{*f,e}x) = y^{-1}xaa^{*f,e}$$

and

$$aa^{*f,e} = (aa^{\dagger}_{e,f})x^{-1} = a^{\dagger}_{e,f}ax^{-1} = (ya^{*f,e}a)x^{-1} = a^{*f,e}ayx^{-1},$$

i.e. $a^{*f,e}a = taa^{*f,e} = aa^{*f,e}z^{-1}$, for $t = y^{-1}x$ and $z = yx^{-1}$. Therefore, the condition (v) holds.

It is clear that the condition (v) implies (vi)-(viii).

(vi) $\Rightarrow$ (i): Using (vi), we obtain $(a^{*f,e}a)^{\circ} = (aa^{*f,e})^{\circ}$ and ${}^{\circ}(a^{*f,e}a) = {}^{\circ}(aa^{*f,e})$. Observe that, by Theorem 1.3, $(a^{*f,e}a)^{\circ} = a^{\circ}$, $(aa^{*f,e})^{\circ} = (a^{*f,e})^{\circ}$, ${}^{\circ}(a^{*f,e}a) = {}^{\circ}(a^{*f,e})$, ${}^{\circ}(aa^{*f,e}) = {}^{\circ}a$. Hence, $a^{\circ} = (a^{*f,e})^{\circ}$ and ${}^{\circ}a = {}^{\circ}(a^{*f,e})$ and, by Theorem 1.4, a is weighted–EP w.r.t. (e,f).

Analogy, we check that (viii) $\vee$ (ix) $\Rightarrow$ (i).

(vii) $\Rightarrow$ (i): Applying the hypothesis (vii) and the equalities $a\mathcal{A} = aa^{*f,e}\mathcal{A}$, $\mathcal{A}a = \mathcal{A}a^{*f,e}a$, $a^{*f,e}\mathcal{A} = a^{*f,e}a\mathcal{A}$, $\mathcal{A}a^{*f,e} = \mathcal{A}aa^{*f,e}$, we get $a\mathcal{A} = a^{*f,e}\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{*f,e}$. Thus, by Theorem 1.4, a is weighted–EP w.r.t. (e,f).

(i) $\Rightarrow$ (ix): It is well-known that (i) gives that $a = a^{*f,e}x_1 = x_2a^{*f,e}$ and $a^{*f,e} = ax_3 = x_4a$, for some $x_1, x_2, x_3, x_4 \in \mathcal{A}$. Now, we conclude that $a^{*f,e}a = a(x_3x_2)a^{*f,e}$ and $aa^{*f,e} = a^{*f,e}(x_1x_4)a$. So, (ix) holds.

(i) $\Rightarrow$ (x): The condition (i) implies that there exist $g_1, h_1 \in \mathcal{A}^{-1}$ such that $a^{*f,e} = ah_1 = g_1a$ which gives $a = h_1^{*e,f}a^{*f,f} = a^{*e,f}g_1^{*f,f}$. Therefore, (x) is satisfied.

Obviously, (x) $\Rightarrow$ (xi) $\wedge$ (xii).

(xi) $\Rightarrow$ (i): Since ${}^{\circ}(a^{*f,e}a) = {}^{\circ}(ah_2h_2^{*e,f}a^{*f,f}) = {}^{\circ}(ah_2(ah_2)^{*e,f})$, then ${}^{\circ}(a^{*f,e}) = {}^{\circ}(ah_2)$ and, by ${}^{\circ}h_2 = \{0\}$, ${}^{\circ}(a^{*f,e}) = {}^{\circ}a$. Similarly, from $(aa^{*f,e})^{\circ} = ((g_2a)^{*e,f}g_2a)^{\circ}$ and $g_2^{\circ} = \{0\}$, we have $(a^{*f,e})^{\circ} = a^{\circ}$. Hence, a is weighted–EP w.r.t. (e,f), by Theorem 1.4.

(xii) $\Rightarrow$ (i): The assumption (xii) gives $a^{*f,e}a\mathcal{A} = ah_3h_3^{*e,f}a^{*f,f}\mathcal{A} = ah_3(ah_3)^{*e,f}\mathcal{A}$. Then $a^{*f,e}\mathcal{A} = ah_3\mathcal{A} = a\mathcal{A}$, by $h_3\mathcal{A} = \mathcal{A}$. In the same way, $\mathcal{A}aa^{*f,e} = \mathcal{A}a^{*e,f}g_3^{*f,f}g_3a$ and $\mathcal{A}g_3 = \mathcal{A}$ imply $\mathcal{A}a^{*f,e} = Aa$. Therefore, a is weighted–EP w.r.t. (e,f), by Theorem 1.4. $\square$

We continue with characterizations of weighted–EP elements via factorizations of the form $a^{\dagger}_{e,f} = sa$.

Theorem 3.2. *Let $\mathcal{A}$ be a unital C^*–algebra, and let e, f be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:*

(i) *a is weighted–EP w.r.t. (e,f);*

(ii) *$\exists\ s, t \in \mathcal{A} : s^{\circ} = {}^{\circ}t = \{0\}$ and $a^{\dagger}_{e,f} = sa = at$;*

(iii) $\exists\, s_1, s_2, t_1, t_2 \in \mathcal{A} : a^{\dagger}_{e,f} = s_1 a = a t_1$ *and* $a = s_2 a^{\dagger}_{e,f} = a^{\dagger}_{e,f} t_2$*;*

(iv) $\exists\, u, v \in \mathcal{A} : u\mathcal{A} = \mathcal{A} = \mathcal{A}v$ *and* $a^{\dagger}_{e,f} = au = va$*;*

(v) $\exists\, x, y \in \mathcal{A}^{-1} : a^{\dagger}_{e,f} a = x a a^{\dagger}_{e,f} = a a^{\dagger}_{e,f} y$*;*

(vi) $\exists\, x_1, y_1 \in \mathcal{A} : x_1^{\circ} = {}^{\circ}y_1 = \{0\}$ *and* $a^{\dagger}_{e,f} a = x_1 a a^{\dagger}_{e,f} = a a^{\dagger}_{e,f} y_1$*;*

(vii) $\exists\, x_2, y_2 \in \mathcal{A} : \mathcal{A}x_2 = \mathcal{A} = y_2\mathcal{A}$ *and* $a^{\dagger}_{e,f} a = x_2 a a^{\dagger}_{e,f} = a a^{\dagger}_{e,f} y_2$*;*

(viii) $\exists\, x_3, x_4, y_3, y_4 \in \mathcal{A} : a^{\dagger}_{e,f} a = x_3 a a^{\dagger}_{e,f} = a a^{\dagger}_{e,f} y_3$ *and* $a a^{\dagger}_{e,f} = x_4 a^{\dagger}_{e,f} a = a^{\dagger}_{e,f} a y_4$*;*

(ix) $\exists\, z_1, z_2 \in \mathcal{A} : a^{\dagger}_{e,f} a = a z_1 a^{\dagger}_{e,f}$ *and* $a a^{\dagger}_{e,f} = a^{\dagger}_{e,f} z_2 a$*.*

Proof. Similarly as the proof of Theorem 3.1, using Lemma 1.2 and Lemma 1.3. □

4 Factorization $a = e^{-1}ucvf$

In this section, we give characterizations of weighted–EP elements through factorizations of the form $a = e^{-1}ucvf$.

Theorem 4.1. *Let* e*,* f *be invertible positive elements in* $\mathcal{A}$*. If* $a \in \mathcal{A}^{-}$*, then the following statements are equivalent:*

(i) a *is weighted–EP w.r.t.* (e,f)*;*

(ii) $\exists\, c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf = e^{-1}fv^*d^*u^*e^{-1}f$, $v\mathcal{A} = \mathcal{A} = \mathcal{A}u$, $c\mathcal{A} = d\mathcal{A}$ *and* $\mathcal{A}c = \mathcal{A}d$*;*

(iii) $\exists\, c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf = e^{-1}fv^*d^*u^*e^{-1}f$, $u^{\circ} = \{0\} = {}^{\circ}v$, $c^{\circ} = d^{\circ}$ *and* ${}^{\circ}c = {}^{\circ}d$*;*

(iv) $\exists\, c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf$, $a^{\dagger}_{e,f} = e^{-1}udvf$, $v\mathcal{A} = \mathcal{A} = \mathcal{A}u$, $c\mathcal{A} = d\mathcal{A}$ *and* $\mathcal{A}c = \mathcal{A}d$*;*

(v) $\exists\, c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf$, $a^{\dagger}_{e,f} = e^{-1}udvf$, $u^{\circ} = \{0\} = {}^{\circ}v$, $c^{\circ} = d^{\circ}$ *and* ${}^{\circ}c = {}^{\circ}d$*;*

(vi) $\exists\ c,d,u,v\in\mathcal{A}: a^{*f,e}a=ucv,\ aa^{*f,e}=udv,\ v\mathcal{A}=\mathcal{A}=\mathcal{A}u,\ c\mathcal{A}=d\mathcal{A}$ *and* $\mathcal{A}c=\mathcal{A}d$*;*

(vii) $\exists\ c,d,u,v\in\mathcal{A}: a^{*f,e}a=ucv,\ aa^{*f,e}=udv,\ u^\circ=\{0\}={}^\circ v,\ c^\circ=d^\circ$ *and* ${}^\circ c={}^\circ d$*.*

Proof. (ii) $\Rightarrow$ (i): If $a=e^{-1}ucvf=e^{-1}fv^*d^*u^*e^{-1}f$, for some $c,d,u,v\in\mathcal{A}$ satisfying $v\mathcal{A}=\mathcal{A}=\mathcal{A}u$, $c\mathcal{A}=d\mathcal{A}$ and $\mathcal{A}c=\mathcal{A}d$, then $a^*=fe^{-1}udvfe^{-1}$. Further

$$\begin{aligned} a\mathcal{A} &= e^{-1}ucvf\mathcal{A}=e^{-1}ucv\mathcal{A}=e^{-1}uc\mathcal{A}=e^{-1}ud\mathcal{A}\\ &= e^{-1}udv\mathcal{A}=e^{-1}udvf\mathcal{A}=f^{-1}a^*e\mathcal{A}=a^{*f,e}\mathcal{A}\end{aligned}$$

and

$$\begin{aligned} \mathcal{A}a &= \mathcal{A}e^{-1}ucvf=\mathcal{A}ucvf=\mathcal{A}cvf=\mathcal{A}dvf\\ &= \mathcal{A}udvf=\mathcal{A}e^{-1}udvf=\mathcal{A}f^{-1}a^*e=\mathcal{A}a^{*f,e}.\end{aligned}$$

By Theorem 1.4, we deduce that a is weighted–EP w.r.t. (e,f).

(i) $\Rightarrow$ (ii): Since a is weighted–EP w.r.t. (e,f), we have $a\mathcal{A}=a^{*f,e}\mathcal{A}$ and $\mathcal{A}a=\mathcal{A}a^{*f,e}$. Let $u=v=1$, $c=eaf^{-1}$ and $d=ea^{*f,e}f^{-1}$. Now, we obtain

$$e^{-1}c\mathcal{A}=af^{-1}\mathcal{A}=a\mathcal{A}=a^{*f,e}\mathcal{A}=a^{*f,e}f^{-1}\mathcal{A}=e^{-1}d\mathcal{A},$$

and

$$\mathcal{A}cf=\mathcal{A}ea=\mathcal{A}a=\mathcal{A}a^{*f,e}=\mathcal{A}ea^{*f,e}=\mathcal{A}df,$$

implying $c\mathcal{A}=d\mathcal{A}$ and $\mathcal{A}c=\mathcal{A}d$. The rest is obviously.

(iii) $\Rightarrow$ (i): Assume that there exist $c,d,u,v\in\mathcal{A}$ satisfying $a=e^{-1}ucvf=e^{-1}fv^*d^*u^*e^{-1}f$, $u^\circ=\{0\}={}^\circ v$, $c^\circ=d^\circ$ and ${}^\circ c={}^\circ d$. To prove that $a^\circ=(a^{*f,e})^\circ$, let $x\in a^\circ$, i.e. $e^{-1}ucvfx=0$. Now, $ucvfx=0$ and, by $u^\circ=\{0\}$, $cvfx=0$. So, $vfx\in c^\circ=d^\circ$, that is, $dvfx=0$ which gives $a^{*f,e}x=e^{-1}udvfx=0$. Hence, $a^\circ\subseteq(a^{*f,e})^\circ$. The reverse inclusion follows similarly. The conditions $\{0\}={}^\circ v$ and ${}^\circ c={}^\circ d$ imply ${}^\circ a={}^\circ(a^{*f,e})$, analogy. Thus, a is weighted–EP w.r.t. (e,f), by Theorem 1.4.

(i) $\Rightarrow$ (iii): Because a is weighted–EP w.r.t. (e,f), then $a^\circ=(a^{*f,e})^\circ$ and ${}^\circ a={}^\circ(a^{*f,e})$. We can show that $(af^{-1})^\circ=(a^{*f,e}f^{-1})^\circ$ and ${}^\circ(ea)={}^\circ(ea^{*f,e})$. For $u=v=1$, $c=eaf^{-1}$ and $d=ea^{*f,e}f^{-1}$, we obtain

$$c^\circ=(e^{-1}c)^\circ=(af^{-1})^\circ=(a^{*f,e}f^{-1})^\circ=(e^{-1}d)^\circ=d^\circ$$

and

$${}^\circ c={}^\circ(cf)={}^\circ(ea)={}^\circ(ea^{*f,e})={}^\circ(df)={}^\circ d.$$

(iv) $\Rightarrow$ (i): We can verify that (iv) gives $a\mathcal{A} = a^{\dagger}_{e,f}\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{\dagger}_{e,f}$ in the same way as in the part (ii) $\Rightarrow$ (i). By the equality (1), we conclude that $a\mathcal{A} = a^{*f,e}\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{*f,e}$ and, by Theorem 1.4, (i) holds.

(i) $\Rightarrow$ (iv): The statements (i) implies $a\mathcal{A} = a^{*f,e}\mathcal{A} = a^{\dagger}_{e,f}\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{*f,e} = \mathcal{A}a^{\dagger}_{e,f}$. The condition (iv) follows on choosing $u = v = 1$, $c = eaf^{-1}$ and $d = ea^{\dagger}_{e,f}f^{-1}$.

(v) $\Rightarrow$ (i): As in the part (iii) $\Rightarrow$ (i), we get $a^{\circ} = (a^{\dagger}_{e,f})^{\circ}$ and ${}^{\circ}a = {}^{\circ}(a^{\dagger}_{e,f})$ which yields (i), by (1) and Theorem 1.4.

(i) $\Rightarrow$ (v): By the choose $u = v = 1$, $c = eaf^{-1}$ and $d = ea^{\dagger}_{e,f}f^{-1}$.

(vi) $\Rightarrow$ (i): From the hypothesis (vi), we can check that $a^{*f,e}a\mathcal{A} = aa^{*f,e}\mathcal{A}$ and $\mathcal{A}a^{*f,e}a = \mathcal{A}aa^{*f,e}$. This equalities give $a^{*f,e}\mathcal{A} = a\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{*f,e}$, i.e. (i) is satisfied.

(i) $\Rightarrow$ (vi) $\wedge$ (vii): It follows for $u = v = 1$, $c = a^{*f,e}a$ and $d = aa^{*f,e}$.

(vii) $\Rightarrow$ (i): Using (vii), we have $(a^{*f,e}a)^{\circ} = (aa^{*f,e})^{\circ}$ and ${}^{\circ}(a^{*f,e}a) = {}^{\circ}(aa^{*f,e})$ which yields $a^{\circ} = (a^{*f,e})^{\circ}$ and ${}^{\circ}(a^{*f,e}) = {}^{\circ}a$. So, (i) holds. $\square$

5 Factorization $a = bc$

For an invertible positive element $f \in \mathcal{A}$, we consider a factorization of $a \in \mathcal{A}$ of the form

$$a = bc, \quad f^{-1}b^{*}\mathcal{A} = \mathcal{A} = c\mathcal{A}. \tag{4}$$

Lemma 5.1. *Let e, f, h be invertible positive elements in $\mathcal{A}$. If $a \in \mathcal{A}$ has a factorization* (4)*, then a is regular and $a^{\dagger}_{e,h} = c^{\dagger}_{f,h}b^{\dagger}_{e,f}$.*

Proof. Since $f^{-1}b^{*}\mathcal{A} = \mathcal{A} = c\mathcal{A}$, by Lemma 1.1, $bf^{-1}, c^{*} \in \mathcal{A}^{-}$ and $(bf^{-1})^{\circ} = \{0\} = (c^{*})^{\circ}$. Thus, the elements b and c are regular. Also, by the hypothesis $f^{-1}b^{*}\mathcal{A} = \mathcal{A} = c\mathcal{A}$, there exist $x, y \in \mathcal{A}$ such that $f^{-1}b^{*}y = 1 = cx$. Then,

$$b^{\dagger}_{e,f}b = f^{-1}(fb^{\dagger}_{e,f}b)^{*}1 = f^{-1}b^{*}(b^{\dagger}_{e,f})^{*}ff^{-1}b^{*}y = f^{-1}(bb^{\dagger}_{e,f}b)^{*}y = f^{-1}b^{*}y = 1 \tag{5}$$

and

$$cc^{\dagger}_{f,h} = cc^{\dagger}_{f,h}1 = cc^{\dagger}_{f,h}cx = cx = 1. \tag{6}$$

Now, we can easy check that $(bc)^{\dagger}_{e,h} = c^{\dagger}_{f,h}b^{\dagger}_{e,f}$. $\square$

Lemma 5.2. *Let e, f, h be invertible positive elements in $\mathcal{A}$. If $a \in \mathcal{A}$ has a factorization* (4), *then*

(i) $b\mathcal{A} = a\mathcal{A}$;

(ii) $c^*\mathcal{A} = a^*\mathcal{A}$;

(iii) $c^\circ = a^\circ$;

(iv) $(b^*)^\circ = (a^*)^\circ$;

(v) $[(eb)^*]^\circ = [(ea)^*]^\circ$;

(vi) $(ch^{-1})^\circ = (ah^{-1})^\circ$;

(vii) $b^\dagger_{e,f}(b^\dagger_{e,f})^{*e,f} \in \mathcal{A}^{-1}$ *and* $(b^\dagger_{e,f}(b^\dagger_{e,f})^{*e,f})^{-1} = b^{*f,e}b$;

(viii) $(c^\dagger_{f,h})^{*f,h}c^\dagger_{f,h} \in \mathcal{A}^{-1}$ *and* $((c^\dagger_{f,h})^{*f,h}c^\dagger_{f,h})^{-1} = cc^{*h,f}$;

(ix) $b^*eb \in \mathcal{A}^{-1}$ *and* $b^\dagger_{e,f} = (b^*eb)^{-1}b^*e$;

(x) $ch^{-1}c^* \in \mathcal{A}^{-1}$ *and* $c^\dagger_{f,h} = h^{-1}c^*(ch^{-1}c^*)^{-1}$.

Proof. (i) The condition $c\mathcal{A} = \mathcal{A}$ implies $b\mathcal{A} = bc\mathcal{A} = a\mathcal{A}$.

(ii) From the equality $f^{-1}b^*\mathcal{A} = \mathcal{A}$, we get

$$c^*\mathcal{A} = c^*f\mathcal{A} = c^*ff^{-1}b^*\mathcal{A} = (bc)^*\mathcal{A} = a^*\mathcal{A}.$$

(iii) Notice that, $c^\circ \subseteq a^\circ$. If $x \in a^\circ$, then $bf^{-1}fcx = 0$. By Lemma 1.1, we observe that $(bf^{-1})^\circ = \{0\}$ which gives $fcx = 0$. Now, we deduce that $cx = 0$ and $a^\circ \subseteq c^\circ$. Hence, $c^\circ = a^\circ$.

(iv) Because $(c^*)^\circ = \{0\}$, by Lemma 1.1, then

$$x \in (a^*)^\circ \Leftrightarrow a^*x = 0 \Leftrightarrow c^*b^*x = 0 \Leftrightarrow b^*x = 0 \Leftrightarrow x \in (b^*)^\circ.$$

In the similar way, we can show conditions (v)-(vi).

By (5) and (6), it follows (vii)-(viii).

(ix) Since

$$bc = a = aa^\dagger_{e,h}aa^\dagger_{e,h}a = aa^\dagger_{e,h}e^{-1}(a^\dagger_{e,h})^*a^*ea = bca^\dagger_{e,h}e^{-1}(a^\dagger_{e,h})^*a^*ebc,$$

then

$$ca^\dagger_{e,h}e^{-1}(a^\dagger_{e,h})^*c^*b^*eb = b^\dagger_{e,f}(bca^\dagger_{e,h}e^{-1}(a^\dagger_{e,h})^*a^*ebc)c^\dagger_{f,h} = b^\dagger_{e,f}bcc^\dagger_{f,h} = 1$$

implies $b^*eb \in \mathcal{A}^{-1}$. We can easily check that $b^\dagger_{e,f} = (b^*eb)^{-1}b^*e$.

Considering a^* we verify (x) similarly as in the proof of part (ix). □

In the following result, we characterize weighted–EP elements through their factorizations of the form $a = bc$.

Theorem 5.1. *Let e, f, h be invertible positive elements in $\mathcal{A}$. If $a \in \mathcal{A}$ has a factorization (4), then $a \in \mathcal{A}^-$ and the following conditions are equivalent*

(i) a *is weighted–EP w.r.t.* (e,h);

(ii) $bb^\dagger_{e,f} = c^\dagger_{f,h}c$;

(iii) $c^\circ = [(eb)^*]^\circ$ *and* $(b^*)^\circ = (ch^{-1})^\circ$;

(iv) ${}^\circ c^* = {}^\circ(eb)$ *and* ${}^\circ b = {}^\circ(h^{-1}c^*)$;

(v) $c^*\mathcal{A} = eb\mathcal{A}$ *and* $b\mathcal{A} = h^{-1}c^*\mathcal{A}$;

(vi) $\mathcal{A}c = \mathcal{A}b^*e$ *and* $\mathcal{A}b^* = \mathcal{A}ch^{-1}$;

(vii) $\exists\ u \in \mathcal{A}^{-1} : c = ub^\dagger_{e,f}$ *and* $b = c^\dagger_{f,h}u$;

(viii) $\exists\ x, y \in \mathcal{A}^{-1} : c = xb^*e$ *and* $b^* = ych^{-1}$;

(ix) $\mathcal{A}^{-1}c = \mathcal{A}^{-1}b^*e$ *and* $\mathcal{A}^{-1}b^* = \mathcal{A}^{-1}ch^{-1}$;

(x) $c^*\mathcal{A}^{-1} = eb\mathcal{A}^{-1}$ *and* $b\mathcal{A}^{-1} = h^{-1}c^*\mathcal{A}^{-1}$;

(xi) $\exists\ x, y \in \mathcal{A} : x^\circ = y^\circ = \{0\}$, $c = xb^*e$ *and* $b^* = ych^{-1}$;

(xii) $\exists\ x, x_1, y, y_1 \in \mathcal{A} : c = xb^*e$, $b^*e = x_1c$, $b^* = ych^{-1}$ *and* $ch^{-1} = y_1b^*$;

(xiii) $\exists\ x, y \in \mathcal{A} : x\mathcal{A} = y\mathcal{A} = \mathcal{A}$, $c^* = ebx$ *and* $b = h^{-1}c^*y$;

(xiv) $a \in h^{-1}c^*\mathcal{A} \cap \mathcal{A}b^*e$ *(or* $a \in c^\dagger_{f,h}\mathcal{A} \cap \mathcal{A}b^\dagger_{e,f}$*)*;

(xv) $a^\dagger_{e,h} \in b\mathcal{A} \cap \mathcal{A}c$;

(xvi) $b(b^*eb)^{-1}b^*e = h^{-1}c^*(ch^{-1}c^*)^{-1}c$

(xvii) $b = c^\dagger_{f,h}cb$, $c = cbb^\dagger_{e,f}$, $b^\dagger_{e,f} = b^\dagger_{e,f}c^\dagger_{f,h}c$ *and* $c^\dagger_{f,h} = bb^\dagger_{e,f}c^\dagger_{f,h}$;

(xviii) $\mathcal{A}^{-1}c = \mathcal{A}^{-1}b^\dagger_{e,f}$ *and* $b\mathcal{A}^{-1} = c^\dagger_{f,h}\mathcal{A}^{-1}$;

(xix) $\exists\ u \in \mathcal{A} : u^\circ = {}^\circ u = \{0\}$, $c = ub^\dagger_{e,f}$ *and* $b = c^\dagger_{f,h}u$;

(xx) $\exists\ u \in \mathcal{A} : \mathcal{A}u = u\mathcal{A} = \mathcal{A}$, $c = ub^\dagger_{e,f}$ *and* $b = c^\dagger_{f,h}u$;

(xxi) $\exists\ v \in \mathcal{A} : v^\circ = {}^\circ v = \{0\}$, $b^\dagger_{e,f} = vc$ *and* $c^\dagger_{f,h} = bv$;

(xxii) $\exists\ v \in \mathcal{A} : \mathcal{A}v = v\mathcal{A} = \mathcal{A}$, $b^\dagger_{e,f} = vc$ *and* $c^\dagger_{f,h} = bv$;

(xxiii) $\exists\ u, u_1, v, v_1 \in \mathcal{A} : c = ub^\dagger_{e,f}$, $b^\dagger_{e,f} = vc$, $b = c^\dagger_{f,h}u_1$ *and* $c^\dagger_{f,h} = bv_1$.

Proof. (i) $\Leftrightarrow$ (ii): By Theorem 1.4, a is weighted–EP w.r.t. (e,h) if and only if $aa^\dagger_{e,h} = a^\dagger_{e,h}a$ which is equivalent to $bb^\dagger_{e,f} = c^\dagger_{f,h}c$, by Lemma 5.1, (5) and (6).

(i) $\Leftrightarrow$ (iii): The element a is weighted–EP w.r.t. (e,h) if and only if ea and af^{-1} are EP, that is, $(ea)^\circ = [(ea)^*]^\circ$ and $(ah^{-1})^\circ = [(ah^{-1})^*]^\circ$. Notice that, by Lemma 1.2 and Lemma 5.2, these equalities are equivalent to $c^\circ = [(eb)^*]^\circ$ and $(ch^{-1})^\circ = (b^*)^\circ$.

(iv) $\Leftrightarrow$ (iii): This part can be check using involution.

(i) $\Leftrightarrow$ (v): It is well-known that a is weighted–EP w.r.t. (e,h) if and only if $ea\mathcal{A} = (ea)^*\mathcal{A}$ and $ah^{-1}\mathcal{A} = (ah^{-1})^*\mathcal{A}$, i.e. $ea\mathcal{A} = a^*\mathcal{A}$ and $a\mathcal{A} = h^{-1}a^*\mathcal{A}$. Observe that, from Lemma 5.2, $eb\mathcal{A} = ea\mathcal{A}$ and $h^{-1}c^*\mathcal{A} = h^{-1}a^*\mathcal{A}$. Now, we conclude that $ea\mathcal{A} = a^*\mathcal{A}$ and $a\mathcal{A} = (ah^{-1})^*\mathcal{A}$ is equivalent to $eb\mathcal{A} = c^*\mathcal{A}$ and $b\mathcal{A} = h^{-1}c^*\mathcal{A}$.

(vi) $\Leftrightarrow$ (v): Applying the involution we verify this equivalence.

(ii) $\Rightarrow$ (vii): Suppose that $bb^\dagger_{e,f} = c^\dagger_{f,h}c$. Let $u = cb$ and let $v = b^\dagger_{e,f}c^\dagger_{f,h}$. Then

$$c = cc^\dagger_{f,h}c = cbb^\dagger_{e,f} = ub^\dagger_{e,f}, \qquad b = bb^\dagger_{e,f}b = c^\dagger_{f,h}cb = c^\dagger_{f,h}u,$$

and

$$uv = ub^\dagger_{e,f}c^\dagger_{f,h} = cc^\dagger_{f,h} = 1 = b^\dagger_{e,f}b = b^\dagger_{e,f}c^\dagger_{f,h}u = vu.$$

Hence, $u \in \mathcal{A}^{-1}$ and the condition (vii) holds.

(vii) $\Rightarrow$ (viii): If there exists $u \in \mathcal{A}^{-1}$ such that $c = ub^\dagger_{e,f}$ and $b = c^\dagger_{f,h}u$, then $c = x'b^{*f,e}$ and $b = c^{*h,f}y'$, for $x' = ub^\dagger_{e,f}(b^\dagger_{e,f})^{*e,f}$ and $y' = (c^\dagger_{f,h})^{*f,h}c^\dagger_{f,h}u$. For $x = x'f^{-1}$ and $y = (y')^*f$, we see that $c = xb^*e$, $b^* = ych^{-1}$ and, by Lemma 5.2, $x, y \in \mathcal{A}^{-1}$.

The following implications can be proved easily:

(viii) $\Rightarrow$ (vi);

(viii) $\Leftrightarrow$ (ix) $\Leftrightarrow$ (x);

(viii) $\Rightarrow$ (xi) $\Rightarrow$ (iii);

(viii) $\Rightarrow$ (xii) $\Rightarrow$ (iii);

(viii) $\Rightarrow$ (xiii) $\Rightarrow$ (v).

(i) $\Leftrightarrow$ (xiv): Notice that, by Theorem 1.4, a is weighted–EP w.r.t. (e,h) if and only if $a \in a^{\dagger}_{e,h}\mathcal{A} \cap \mathcal{A}a^{\dagger}_{e,h}$, which is equivalent to $a \in h^{-1}a^*\mathcal{A} \cap \mathcal{A}a^*e = h^{-1}c^*\mathcal{A} \cap \mathcal{A}b^*e$.

(i) $\Rightarrow$ (xv): By Theorem 1.4, a is weighted–EP w.r.t. (e,h) $\Leftrightarrow$ $a \in a^{\dagger}_{e,f}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}a^{\dagger}_{e,f}$. Consequently, $a^{\dagger}_{e,f} \in a\mathcal{A} \cap \mathcal{A}a = b\mathcal{A} \cap \mathcal{A}c$.

(xv) $\Rightarrow$ (i): Since $a^{\dagger}_{e,h} \in b\mathcal{A} \cap \mathcal{A}c$, then $a^{\dagger}_{e,f} \in a\mathcal{A} \cap \mathcal{A}a$. Therefore, for some $x, y \in a\mathcal{A}$, $a^{\dagger}_{e,f} = ax = ya$, which gives

$$a^{\dagger}_{e,f} - aa^{\dagger}_{e,f}a^{\dagger}_{e,f} = (a - aa^{\dagger}_{e,f}a)x = 0$$

and

$$a^{\dagger}_{e,f} - a^{\dagger}_{e,f}a^{\dagger}_{e,f}a = y(a - aa^{\dagger}_{e,f}a) = 0.$$

By Theorem 1.4, $a^{\dagger}_{e,f} = aa^{\dagger}_{e,f}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\dagger}_{e,f}a$ implies that a is weighted–EP w.r.t. (e,h).

(xvi) $\Leftrightarrow$ (ii): Obviously, by statements (ix) and (x) of Lemma 5.2.

(ii) $\Rightarrow$ (xvii): By elementary computations.

(xvii) $\Rightarrow$ (i): The assumption (xvii) can be written as $(1 - c^{\dagger}_{f,h}c)b = 0$, $c(1 - bb^{\dagger}_{e,f}) = 0$, $b^{\dagger}_{e,f}(1 - c^{\dagger}_{f,h}c) = 0$ and $(1 - bb^{\dagger}_{e,f})c^{\dagger}_{f,h} = 0$ implying

$$a\mathcal{A} = b\mathcal{A} \subseteq (1 - c^{\dagger}_{f,h}c)^{\circ} = h^{-1}c^*\mathcal{A} = h^{-1}a^*\mathcal{A},$$

$$(a^*e)^{\circ} = (b^*e)^{\circ} = (1 - bb^{\dagger}_{e,f})\mathcal{A} \subseteq c^{\circ} = a^{\circ},$$

$$a^{\circ} = c^{\circ} = (1 - c^{\dagger}_{f,h}c)\mathcal{A} \subseteq (b^{\dagger}_{e,f})^{\circ} = (b^*e)^{\circ} = (a^*e)^{\circ},$$

and

$$h^{-1}a^*\mathcal{A} = h^{-1}c^*\mathcal{A} = c^{\dagger}_{f,h}\mathcal{A} \subseteq (1 - bb^{\dagger}_{e,f})^{\circ} = b\mathcal{A} = a\mathcal{A}.$$

Thus, $ah^{-1}\mathcal{A} = a\mathcal{A} = h^{-1}a^*\mathcal{A}$ and $(ea)^{\circ} = a^{\circ} = (a^*e)^{\circ}$ which gives that ah^{-1} and ea are EP elements, that is, a is weighted–EP w.r.t. (e,h).

Note that for $u = cb$ and $v = b^{\dagger}_{e,f}c^{\dagger}_{f,h}$, we can show:

(vii) $\Leftrightarrow$ (xviii);

(vii) $\Rightarrow$ (xix) $\vee$ (xxi) $\vee$ (xxiii) $\Rightarrow$ (iii);

(vii) $\Rightarrow$ (xx) $\vee$ (xxii) $\Rightarrow$ (vi). $\square$

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Address:

Dijana Mosić
University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia
E-mail: dijana@pmf.ni.ac.rs

Dragan S. Djordjević
University of Niš, Faculty of Sciences and Mathematics, Višegradska 33,
P.O. Box 224, 18000 Niš, Serbia
E-mail: dragan@pmf.ni.ac.rs dragandjordjevic70@gmail.com