New characterizations of EP, generalized normal and generalized Hermitian elements in rings

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Abstract

We present a number of new characterizations of EP elements in rings with involution in purely algebraic terms. Then, we study equivalent conditions for an element a in a ring with involution to satisfy $a^n a^* = a^* a^n$ or $a^n = (a^*)^n$ for arbitrary $n \in N$. For $n = 1$, we present some new characterizations of normal and Hermitian elements in rings with involution.

Key words and phrases: EP elements, Moore–Penrose inverse, group inverse, generalized normal elements, generalized Hermitian elements, ring with involution.

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1 Introduction

Complex matrices and Hilbert spaces operators with closed ranges A with the property that the ranges of A and A^* coincide, are known as EP or range-Hermitian (EP for equal projections onto $R(A)$ and $R(A^*)$). Hermitian, normal and nonsingular matrices (operators) are special cases of EP matrices (operators). EP, normal and Hermitian matrices, as well as EP, normal and Hermitian linear operators on Banach or Hilbert spaces have been investigated by many authors (see, for example, [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18, 20, 21, 24, 29]). In rings with involution EP elements are elements for which the Drazin and the Moore–Penrose inverse exist and coincide. In this paper we use the setting of rings with involution to investigate EP elements, and give new characterizations. We introduce and investigate generalized normal and generalized Hermitian elements in

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rings. As a consequence, we give several new characterizations for elements in rings with involution to be normal and Hermitian elements.

Let $\mathcal{R}$ be an associative ring, and let $a \in \mathcal{R}$. Then a is *group invertible* if there exists $a^\# \in \mathcal{R}$ such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad aa^\# = a^\#a;$$

$a^\#$ is uniquely determined by these equations. The group inverse $a^\#$ doubly commutes with a, that is, $ax = xa$ implies $a^\#x = xa^\#$ [2]. We use $\mathcal{R}^\#$ to denote the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ satisfying $aa^* = a^*a$ is called *normal*. An element $a \in \mathcal{R}$ satisfying $a = a^*$ is called *Hermitian* (or *symmetric*).

We say that $b = a^\dagger$ is the *Moore–Penrose inverse* (or *MP-inverse*) of a, if the following hold [28]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [6, 12, 16, 19]). The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

If $\delta \subset \{1,2,3,4\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denoted by $a\{\delta\}$. Notice that $a\{1,2,3,4\} = \{a^\dagger\}$.

Theorem 1.1. [11, 25] *For any $a \in \mathcal{R}^\dagger$, the following are satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;

(e) $a^* = a^\dagger aa^* = a^*aa^\dagger$;

(f) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger = (a^*a)^\#a^* = a^*(aa^*)^\#$;

(g) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

Now, we state the following useful result.

Lemma 1.1. [25] *If $a \in \mathcal{R}^\dagger$, then $aa^*a \in \mathcal{R}^\dagger$ and $(aa^*a)^\dagger = a^\dagger(a^*)^\dagger a^\dagger$.*

In this paper we will use the following definition of EP elements [22].

Definition 1.1. An element a of a ring $\mathcal{R}$ with involution is said to be EP if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and $a^\# = a^\dagger$.

The following result is well known for matrices, Hilbert space operators and elements of C^*-algebras, and it is equally true in rings with involution:

Lemma 1.2. *An element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\dagger$ and $aa^\dagger = a^\dagger a$.*

We observe that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ if and only if $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ (see [22]) and a is EP if and only if a^* is EP. If $a \in \mathcal{R}^\#$, then $a^\pi = 1 - aa^\#$ is the spectral idempotent of a [22]. The following result is proved in [22].

Theorem 1.2. *An element $a \in \mathcal{R}$ is EP if and only if a is group invertible and one of the following equivalent conditions holds:*

(a) *$a^\# a$ is symmetric;*

(b) *$(a^\#)^* = aa^\#(a^\#)^*$;*

(c) *$(a^\#)^* = (a^\#)^* a^\# a$;*

(d) *$a^\#(a^\pi)^* = a^\pi(a^\#)^*$.*

The next results are proved in [25].

Lemma 1.3. [25] *Let $a \in \mathcal{R}^\dagger$ and $b \in \mathcal{R}$. If $ab = ba$ and $a^*b = ba^*$, then $a^\dagger b = ba^\dagger$.*

Lemma 1.4. [25] *Let $a \in \mathcal{R}^\dagger$. Then a is normal if and only if $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$.*

Notice that the condition $aa^\dagger = a^\dagger a$ generalizes the notion of EP matrices, and the condition $a^*a^\dagger = a^\dagger a^*$ generalizes the notion of star-dagger matrices [17].

Lemma 1.5. *If $a \in \mathcal{R}^\dagger$ is normal, then a is EP.*

Various characterizations of EP, normal and Hermitian complex matrices are proved in [5] or in [1], using mostly the rank of a matrix, or other finite dimensional methods. These results are generalized in [8] and [9] for linear operators on Hilbert spaces, using operator matrices. In [25] and [26], applying a purely algebraic technique, it is showed that neither the rank of a matrix, nor the properties of operator matrices are necessary for the characterization of EP, normal and Hermitian elements in rings with involution. Thus, the results from [1, 5, 8, 9] are extended to more general settings.

An element $a \in \mathcal{R}$ satisfying $a^n a^* = a^* a^n$ for arbitrary $n \in N$ will be called generalized normal element. An element $a \in \mathcal{R}$ satisfying $a^n = (a^*)^n$ for arbitrary $n \in N$ will be called generalized Hermitian element. In this paper we generalize the results from [25] and [26].

2 EP elements

In this section EP elements in rings with involution are characterized by conditions involving powers of their group and Moore–Penrose inverse.

In the following theorem we present 30 necessary and sufficient conditions for an element a of a ring with involution to be EP.

Theorem 2.1. *Let $m, n \in N$. An element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and one of the following equivalent conditions holds:*

(i) $(a^{\#})^{n+m-1} = (a^{\dagger})^m (a^{\#})^{n-1}$ (*or* $(a^{\#})^{n+m-1} = (a^{\#})^{n-1}(a^{\dagger})^m$);

(ii) $(a^*)^n a a^{\#} = (a^*)^n$;

(iii) $a a^{\#}(a^*)^n = (a^*)^n a a^{\#}$;

(iv) $a(a^{\#})^n (a^{\dagger})^m = a^{\dagger} a (a^{\#})^{n+m-1}$;

(v) $(a^{\dagger})^2 (a^{\#})^n = a^{\dagger}(a^{\#})^n a^{\dagger}$ (*or* $a^{\dagger}(a^{\#})^n a^{\dagger} = (a^{\#})^n (a^{\dagger})^2$);

(vi) $a^{\dagger}(a^{\#})^n = (a^{\#})^n a^{\dagger}$;

(vii) $a(a^{\dagger})^{n+1} = (a^{\#})^n$;

(viii) $a^*(a^{\dagger})^n = a^*(a^{\#})^n$ (*or* $(a^{\dagger})^n a^* = (a^{\#})^n a^*$);

(ix) $(a^{\dagger})^{n+1} = (a^{\#})^n a^{\dagger}$ (*or* $(a^{\dagger})^{n+1} = a^{\dagger}(a^{\#})^n$);

(x) $(a^{\dagger})^n = (a^{\#})^n$;

(xi) $a a^{\dagger}(a^*)^n = (a^*)^n a a^{\dagger}$(*or* $(a^*)^n a^{\dagger} a = a^{\dagger} a (a^*)^n$);

(xii) $a a^{\dagger}(a^*)^n a^m = (a^*)^n a^m a a^{\dagger}$(*or* $a^{\dagger} a a^m (a^*)^n = a^m (a^*)^n a^{\dagger} a$);

(xiii) $a a^{\dagger}(a^m (a^*)^n - (a^*)^n a^m) = (a^m (a^*)^n - (a^*)^n a^m) a a^{\dagger}$ (*or* $a^{\dagger} a (a^m (a^*)^n - (a^*)^n a^m) = (a^m (a^*)^n - (a^*)^n a^m) a^{\dagger} a$);

(xiv) $(a^*)^n a^\# a + aa^\#(a^*)^n = 2(a^*)^n$;

(xv) $a^\dagger (a^\#)^n a + aa^\#(a^\dagger)^n = 2(a^\dagger)^n$;

(xvi) $a^n aa^\dagger + a^\dagger aa^n = 2a^n$;

(xvii) $a^n aa^\dagger + (a^n aa^\dagger)^* = a^n + (a^*)^n$ (*or* $a^\dagger aa^n + (a^\dagger aa^n)^* = a^n + (a^*)^n$);

(xviii) $a^n = a^n aa^\dagger$ (*or* $a^n = a^\dagger aa^n$);

(xix) $a^n a^\dagger = a^\dagger a^n$;

(xx) $[(a^\#)^*]^n = aa^\#[(a^\#)^*]^n$ (*or* $[(a^\#)^*]^n = [(a^\#)^*]^n a^\# a$).

Proof. If a is EP, then it commutes with its Moore–Penrose inverse and $a^\# = a^\dagger$. It is not difficult to verify that conditions (i)-(xx) hold.

Conversely, we assume that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$. To conclude that a is EP, we show that one of the conditions of Theorem 1.2 is satisfied, or that the element a is subject to one of the preceding already established conditions of this theorem.

(i) The hypothesis $(a^\#)^{n+m-1} = (a^\dagger)^m (a^\#)^{n-1}$ implies

$$\begin{aligned} a^\# a &= (a^\#)^{n+m-1} a^{n+m-1} = (a^\dagger)^m (a^\#)^{n-1} a^{n+m-1} \\ &= a^\dagger a((a^\dagger)^m (a^\#)^{n-1}) a^{n+m-1} = a^\dagger a (a^\#)^{n+m-1} a^{n+m-1} = a^\dagger a. \end{aligned}$$

Since $a^\dagger a$ is symmetric, we get that $a^\# a$ is also symmetric.

(ii) The assumption $(a^*)^n aa^\# = (a^*)^n$ gives

$$(aa^\#)^* = [a^n (a^\#)^n]^* = [(a^\#)^n]^* (a^*)^n = [(a^\#)^n]^* (a^*)^n aa^\# = (aa^\#)^* aa^\#.$$

Since the element $(aa^\#)^* aa^\#$ is symmetric, we get that $aa^\#$ is symmetric.

(iii) If $aa^\#(a^*)^n = (a^*)^n aa^\#$, then the condition (ii) holds:

$$(a^*)^n = a^\dagger aa^*(a^*)^{n-1} = a^\dagger a(aa^\#(a^*)^n) = a^\dagger a(a^*)^n aa^\# = (a^*)^n aa^\#.$$

(iv) The equality $a(a^\#)^n (a^\dagger)^m = a^\dagger a(a^\#)^{n+m-1}$ implies

$$(a^\#)^{n-1}(a^\dagger)^m = a^\# a(a(a^\#)^n (a^\dagger)^m) = a^\# aa^\dagger a(a^\#)^{n+m-1} = (a^\#)^{n+m-1}.$$

So, the condition (i) is satisfied.

(v) When we use $(a^\dagger)^2 (a^\#)^n = a^\dagger (a^\#)^n a^\dagger$, we have

$$\begin{aligned} a^\dagger (a^\#)^n a^\dagger &= (a^\dagger)^2 (a^\#)^n = ((a^\dagger)^2 (a^\#)^n) aa^\# = a^\dagger (a^\#)^n a^\dagger aa^\# \\ &= a^\dagger (a^\#)^{n+1} aa^\dagger aa^\# = a^\dagger (a^\#)^{n+1}. \end{aligned} \tag{1}$$

From (1), it follows

$$aa^\dagger = a^{n+1}aa^\dagger a(a^\#)^{n+1}a^\dagger = a^{n+2}(a^\dagger(a^\#)^n a^\dagger) = a^{n+2}a^\dagger(a^\#)^{n+1} = aa^\#.$$

So, the element $aa^\#$ is symmetric.

(vi) Using the assumption $a^\dagger(a^\#)^n = (a^\#)^n a^\dagger$, we have

$$a^\# a = (a^\#)^{n+1}aa^\dagger aa^n = ((a^\#)^n a^\dagger)a^{n+1} = a^\dagger(a^\#)^n a^{n+1} = a^\dagger a.$$

Therefore, $a^\# a$ is symmetric.

(vii) Assume that $a(a^\dagger)^{n+1} = (a^\#)^n$. Now

$$aa^\# = a^n(a^\#)^n = a^n a(a^\dagger)^{n+1} = a^n(a(a^\dagger)^{n+1})aa^\dagger = a^n(a^\#)^n aa^\dagger = aa^\dagger.$$

Hence, $aa^\#$ is symmetric.

(viii) The condition $a^*(a^\dagger)^n = a^*(a^\#)^n$ implies

$$\begin{aligned} a(a^\dagger)^{n+1} &= (aa^\dagger)^*(a^\dagger)^n = (a^\dagger)^*(a^*(a^\dagger)^n) = (a^\dagger)^*(a^*(a^\#)^n) \\ &= (aa^\dagger)^*(a^\#)^n = aa^\dagger(a^\#)^n = aa^\dagger a(a^\#)^{n+1} = (a^\#)^n, \end{aligned}$$

i.e. the equality (vii) holds.

(ix) Using the equality $(a^\dagger)^{n+1} = (a^\#)^n a^\dagger$, we get that (viii) is satisfied:

$$(a^\dagger)^n a^* = ((a^\dagger)^n a^\dagger)aa^* = (a^\#)^n a^\dagger aa^* = (a^\#)^n a^*.$$

(x) If we assume that $(a^\dagger)^n = (a^\#)^n$, then (vii) holds:

$$a(a^\dagger)^{n+1} = aa^\dagger(a^\dagger)^n = aa^\dagger(a^\#)^n = aa^\dagger a(a^\#)^{n+1} = (a^\#)^n.$$

(xi) The equality $aa^\dagger(a^*)^n = (a^*)^n aa^\dagger$ is equivalent to $aa^\dagger(a^*)^n = (a^*)^n$. Applying involution to the last expression, we have $a^n aa^\dagger = a^n$. If we multiply the previous equality by $(a^\#)^n$ from the left side, we get $aa^\dagger = a^\# a$ and $a^\# a$ is symmetric.

(xii) Applying the assumption $aa^\dagger(a^*)^n a^m = (a^*)^n a^m aa^\dagger$, we get

$$\begin{aligned} aa^\dagger(a^*)^n &= (aa^\dagger(a^*)^n a^m)(a^\#)^{m-1}a^\dagger = (a^*)^n a^m aa^\dagger(a^\#)^{m-1}a^\dagger \\ &= (a^*)^n a^m aa^\dagger a(a^\#)^m a^\dagger = (a^*)^n aa^\dagger. \end{aligned}$$

Therefore, the condition (xi) holds.

(xiii) The assumption (xiii) is equivalent to $a^m(a^*)^n - aa^\dagger(a^*)^n a^m = a^m(a^*)^n - (a^*)^n a^m aa^\dagger$. So, $aa^\dagger(a^*)^n a^m = (a^*)^n a^m aa^\dagger$, that is, the condition (xii) is satisfied.

(xiv) By the condition $(a^*)^n a^\# a + aa^\#(a^*)^n = 2(a^*)^n$, we obtain

$$\begin{aligned}
2(a^*)^n &= 2a^\dagger aa^*(a^*)^{n-1} = a^\dagger a 2(a^*)^n = a^\dagger a((a^*)^n a^\# a + aa^\#(a^*)^n) \\
&= (a^*)^n a^\# a + a^\dagger a(a^*)^n = (a^*)^n a^\# a + (a^*)^n.
\end{aligned}$$

Now, we have $(a^*)^n = (a^*)^n a^\# a = (a^*)^n aa^\#$, which is the equality (ii).

(xv) Multiplying $a^\dagger(a^\#)^n a + aa^\#(a^\dagger)^n = 2(a^\dagger)^n$ by $a^\dagger a$ from the left side, we get $a^\dagger(a^\#)^n a + a^\dagger a(a^\dagger)^n = 2a^\dagger a(a^\dagger)^n$, which yields $a^\dagger a(a^\#)^n = (a^\dagger)^n$. Now, we have

$$aa^\# = a^n(a^\dagger a(a^\#)^n) = a^n(a^\dagger)^n = a^n(a^\dagger)^n aa^\dagger = a^n a^\dagger a(a^\#)^n aa^\dagger = aa^\dagger.$$

Thus, $aa^\#$ is symmetric.

(xvi) Multiplying $a^n aa^\dagger + a^\dagger aa^n = 2a^n$ by $(a^\#)^n$ from the right side, we get

$$2aa^\# = a^n aa^\dagger(a^\#)^n + a^\dagger a = a^n aa^\dagger a(a^\#)^{n+1} + a^\dagger a = aa^\# + a^\dagger a,$$

i.e. $aa^\# = a^\dagger a$. Therefore, $aa^\#$ is symmetric.

(xvii) Multiplying $a^n aa^\dagger + (a^n aa^\dagger)^* = a^n + (a^*)^n$ by $aa^\dagger$ from the right side, we obtain that $a^n aa^\dagger + aa^\dagger(a^*)^n aa^\dagger = a^n aa^\dagger + (a^*)^n aa^\dagger$, which gives $aa^\dagger(a^*)^n = (a^*)^n aa^\dagger$. So, the condition (xi) holds.

(xviii) Using the assumption $a^n = a^n aa^\dagger$, we have $a^\# a = (a^\#)^n a^n = (a^\#)^n a^n aa^\dagger = aa^\dagger$. Thus, $a^\# a$ is symmetric.

(xix) By the hypothesis $a^n a^\dagger = a^\dagger a^n$, we conclude that condition (xviii) is satisfied: $a^n = (a^n a^\dagger)a = a^\dagger a^n a = a^\dagger aa^n$.

(xx) Applying involution to $[(a^\#)^*]^n = aa^\#[(a^\#)^*]^n$ we get $(a^\#)^n = (a^\#)^n(aa^\#)^*$ which yields $aa^\# = a^n(a^\#)^n = a^n(a^\#)^n(aa^\#)^* = aa^\#(aa^\#)^*$. The element $aa^\#(aa^\#)^*$ is symmetric and so is $aa^\#$. □

If $m = n = 1$, then the above conditions are known in rings with involution ([26]) and in special cases such as matrices ([1], [5]) and operators on Hilbert spaces ([8],[9]).

In the following result we present some equivalent conditions for an element in a unital C^*–algebra to be EP. If $m = n = k = 1$, then the conditions (i)-(iv) were established in [4].

Theorem 2.2. *Let $\mathcal{A}$ be a unital C^*–algebra, let $m, n \in N$ and let $k \in C \setminus \{0\}$. An element $a \in \mathcal{A}$ is EP if and only if $a \in \mathcal{A}^\# \cap \mathcal{A}^\dagger$ and one of the following equivalent conditions holds:*

(i) $aa^\dagger(a^n + k(a^\dagger)^m) = (a^n + k(a^\dagger)^m)aa^\dagger$;

(ii) $a^\dagger a(a^n + k(a^\dagger)^m) = (a^n + k(a^\dagger)^m)a^\dagger a$;

(iii) $aa^\dagger(a^n + k(a^*)^m) = (a^n + k(a^*)^m)aa^\dagger$;

(iv) $a^\dagger a(a^n + k(a^*)^m) = (a^n + k(a^*)^m)a^\dagger a$;

(iv) $a^m(a^n a^\dagger + k a^\dagger a^n) = (a^n a^\dagger + k a^\dagger a^n)a^m$.

Proof. If a is EP, it is easy to check that conditions (i)-(v) hold.

Conversely, we assume that $a \in \mathcal{A}^\# \cap \mathcal{A}^\dagger$.

(i) The condition $aa^\dagger(a^n + k(a^\dagger)^m) = (a^n + k(a^\dagger)^m)aa^\dagger$ is equivalent to

$$a^n + ka(a^\dagger)^{m+1} = a^n aa^\dagger + k(a^\dagger)^m. \tag{2}$$

Multiplying (2) from the left side first by $a^\dagger$ and then by a, we obtain

$$a^n + ka(a^\dagger)^{m+1} = a^n aa^\dagger + ka(a^\dagger)^{m+1},$$

which implies $a^n = a^n aa^\dagger$. Thus, the condition (xviii) of Theorem 2.1 holds.

(ii)-(iv) These parts of proof follow similarly as (i).

(v) The equality $a^m(a^n a^\dagger + k a^\dagger a^n) = (a^n a^\dagger + k a^\dagger a^n)a^m$ gives

$$a^{m+n}a^\dagger + ka^{m+n-1} = a^{m+n-1} + ka^\dagger a^{m+n}. \tag{3}$$

Multiplying (3) from the left side by $(a^\#)^{m+n-1}$, we see that $aa^\dagger = a^\# a$. Hence, $a^\# a$ is symmetric. □

The following theorem gives a characterization of EP elements in rings with involution. The assumption $a^n \in \mathcal{R}^\dagger$ is required.

Theorem 2.3. *Let $n \in N$ and $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$. If $a^n \in \mathcal{R}^\dagger$, then element a is EP if and only if $a(a^n)^\dagger = (a^n)^\dagger a$.*

Proof. First, we can verify that $a(a^n)^\dagger \in (a^n a^\dagger)\{1, 2, 3\}$.

Suppose that $a(a^n)^\dagger = (a^n)^\dagger a$. Then $a^{n-1}(a^n)^\dagger = (a^n)^\dagger a^{n-1}$ and

$$(a(a^n)^\dagger a^n a^\dagger)^* = (a(a^n)^\dagger a^{n-1}aa^\dagger)^* = (a^n(a^n)^\dagger aa^\dagger)^* = aa^\dagger a^n(a^n)^\dagger = a^n(a^n)^\dagger,$$

i.e. $a(a^n)^\dagger \in (a^n a^\dagger)\{4\}$. Thus, we conclude that $a^n a^\dagger \in \mathcal{R}^\dagger$ and $(a^n a^\dagger)^\dagger = a(a^n)^\dagger$. Similarly, we can verify that $a^\dagger a^n \in \mathcal{R}^\dagger$ and $(a^\dagger a^n)^\dagger = (a^n)^\dagger a$. Since $a(a^n)^\dagger = (a^n)^\dagger a$, we deduce that $a^n a^\dagger = a^\dagger a^n$ and, by Theorem 2.1, a is EP.

If a is EP, then

$$a(a^n)^\dagger a^n a^\dagger = (a^\#)^{n-1}(a^n(a^n)^\dagger a^n)a^\dagger = (a^\#)^{n-1}a^n a^\dagger = aa^\dagger.$$

So, we obtain again that $a(a^n)^\dagger \in (a^n a^\dagger)\{4\}$. Therefore, $a^n a^\dagger \in \mathcal{R}^\dagger$ and $(a^n a^\dagger)^\dagger = a(a^n)^\dagger$, and similarly, $a^\dagger a^n \in \mathcal{R}^\dagger$ and $(a^\dagger a^n)^\dagger = (a^n)^\dagger a$. From Theorem 2.1 we obtain $a^n a^\dagger = a^\dagger a^n$, and then $a(a^n)^\dagger = (a^n)^\dagger a$. □

3 Generalized normal elements

In this section a number of necessary and sufficient conditions for both MP-invertible and group invertible elements in rings with involution to be generalized normal elements are presented.

First, we formulate the next results which generalize Lemma 1.4 and Lemma 1.5.

Lemma 3.1. *Let $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $n \in N$. Then $a^n a^* = a^* a^n$ if and only if $a^\dagger a^n = a^n a^\dagger$ and $(a^*)^n a^\dagger = a^\dagger (a^*)^n$.*

Proof. $\Rightarrow$: From the assumption $a^n a^* = a^* a^n$ and the equality $a^n a = a a^n$ ($a^n (a^*)^* = (a^*)^* a^n$), by Lemma 1.3, we deduce that a^n commutes with $a^\dagger$ and $(a^*)^\dagger$, i.e. $a^\dagger a^n = a^n a^\dagger$ and $a^n (a^\dagger)^* = (a^\dagger)^* a^n$. Applying involution to $a^n (a^\dagger)^* = (a^\dagger)^* a^n$, we obtain $(a^*)^n a^\dagger = a^\dagger (a^*)^n$.

$\Leftarrow$: Using the condition $a^\dagger a^n = a^n a^\dagger$, we get

$$(4) \qquad aa^\# = a^n (a^\#)^n = (a^n a^\dagger) a (a^\#)^n = a^\dagger a^n a (a^\#)^n = a^\dagger a.$$

Since $a^\dagger a$ is symmetric, $aa^\#$ is symmetric too and a is EP, by Theorem 1.2. Multiplying the hypothesis $(a^*)^n a^\dagger = a^\dagger (a^*)^n$ by a from the left and from the right sides, we have $a (a^*)^n a^\dagger a = a a^\dagger (a^*)^n a$. Then, by $aa^\dagger = a^\dagger a$, $a(a^*)^n = (a^*)^n a$ and, applying the involution to this, $a^n a^* = a^* a^n$. □

From the proof of Lemma 3.1, we get the following result.

Lemma 3.2. *Let $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $n \in N$. Then $a^n a^* = a^* a^n$ if and only if a is EP and $(a^*)^n a^\dagger = a^\dagger (a^*)^n$.*

In the following theorem we study 21 conditions involving $a^\dagger$, $a^\#$, a^* and their powers to ensure that element a is generalized normal. When $m = n = 1$, the following result is known for characterizations of normal elements in rings with involution ([25]) and in special cases such as matrices ([1, 5]) and operators on Hilbert spaces ([8, 9]).

Theorem 3.1. *Let $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $m, n \in N$. Then $a^n a^* = a^* a^n$ if and only if one of the following equivalent conditions holds:*

(i) $a^n (aa^*a)^\dagger = (aa^*a)^\dagger a^n$;

(ii) $a^\dagger (a^n + (a^*)^n) = (a^n + (a^*)^n) a^\dagger$;

(iii) $a^m a^* (a^\#)^n = (a^\#)^n a^m a^*$ (*or* $a^* a^m (a^\#)^n = (a^\#)^n a^* a^m$);

(iv) $a^m a^n a^* = a^m a^* a^n$ (*or* $a^n a^* a^m = a^* a^n a^m$);

(v) $a^*(a^\#)^n = (a^\#)^n a^*$;

(vi) $a^*(a^\dagger)^n = (a^\#)^n a^*$ (*or* $a^*(a^\#)^n = (a^\dagger)^n a^*$);

(vii) $a(a^*)^n a^\dagger = (a^*)^n$ (*or* $a^\dagger(a^*)^n a = (a^*)^n$);

(viii) $a(a^*)^n a^\# = (a^*)^n$ (*or* $a^\#(a^*)^n a = (a^*)^n$);

(ix) $a^*(a^*)^n a^\# = a^* a^\#(a^*)^n$ (*or* $(a^*)^n a^\# a^* = a^\#(a^*)^n a^*$);

(x) $a^* a^*(a^\#)^n = a^*(a^\#)^n a^*$ (*or* $a^*(a^\#)^n a^* = (a^\#)^n a^* a^*$);

(xi) $(a^*)^n a^\dagger a^\# = a^\#(a^*)^n a^\dagger$ (*or* $a^\dagger(a^*)^n a^\# = a^\# a^\dagger(a^*)^n$);

(xii) $a^\dagger a^\#(a^*)^n = a^\#(a^*)^n a^\dagger$ (*or* $(a^*)^n a^\# a^\dagger = a^\dagger(a^*)^n a^\#$).

Proof. Suppose that $a^n a^* = a^* a^n$. By Lemma 3.2, a^n commutes with $(a^*)^\dagger$, a commutes with $a^\dagger$ and $a^\# = a^\dagger$. The group inverse $(a^n)^\# = (a^\#)^n$ doubly commutes with a^n, so $(a^\#)^n$ commutes with a^*. Since a^n commutes with $(a^*)^\dagger = (a^\#)^*$, then, applying the involution, $(a^*)^n$ commutes with $a^\#$. Now, we can easily verify that conditions (i)-(xii) hold.

Conversely, we will show that the condition $a^n a^* = a^* a^n$ is satisfied.

(i) By Lemma 1.1, the hypothesis $a^n(aa^*a)^\dagger = (aa^*a)^\dagger a^n$ can be written as $a^n a^\dagger(a^\dagger)^* a^\dagger = a^\dagger(a^\dagger)^* a^\dagger a^n$. Multiplying this equality by a from the left and from the right sides, we get $a^n aa^\dagger(a^\dagger)^* a^\dagger a = aa^\dagger(a^\dagger)^* a^\dagger aa^n$. Using Theorem 1.1, we have

$$a^n(a^\dagger)^* = (a^\dagger)^* a^n. \tag{5}$$

Applying the involution to (5), it follows

$$a^\dagger(a^*)^n = (a^*)^n a^\dagger. \tag{6}$$

The equalities $a^n a^\dagger(a^\dagger)^* a^\dagger = a^\dagger(a^\dagger)^* a^\dagger a^n$ and (5) imply

$$\begin{aligned} a^n a^\dagger &= (a^n a^\dagger(a^\dagger)^* a^\dagger)aa^* = a^\dagger(a^\dagger)^* a^\dagger a^n aa^* = a^\dagger((a^\dagger)^* a^\dagger a)a^n a^* \\ &= a^\dagger((a^\dagger)^* a^n)a^* = a^\dagger a^n(a^\dagger)^* a^* = a^*((a^\dagger)^* a^n)a^\dagger = a^* a^n(a^\dagger)^* a^\dagger \\ &= a^* a^n aa^\dagger(a^\dagger)^* a^\dagger = a^* a(a^n a^\dagger(a^\dagger)^* a^\dagger) = a^* aa^\dagger(a^\dagger)^* a^\dagger a^n = a^\dagger a^n. \end{aligned} \tag{7}$$

Then, by (6), (7) and Lemma 3.1, we have $a^* a^n = a^n a^*$.

(ii) The assumption $a^\dagger(a^n + (a^*)^n) = (a^n + (a^*)^n)a^\dagger$ is equivalent to

$$a^\dagger a^n + a^\dagger(a^*)^n = a^n a^\dagger + (a^*)^n a^\dagger. \tag{8}$$

Multiplying (8) by a from the left and from the right sides, we have

$$a^n a + aa^\dagger (a^*)^n a = aa^n + a(a^*)^n a^\dagger a,$$

which yields $aa^\dagger (a^*)^n a = a(a^*)^n a^\dagger a$. Multiplying the previous equality by $a^\dagger$ from the left and from the right sides, we get, by Theorem 1.1,

$$a^\dagger (a^*)^n = (a^*)^n a^\dagger. \tag{9}$$

From (8) and (9), we conclude $a^\dagger a^n = a^n a^\dagger$, which implies $a^* a^n = a^n a^*$, by Lemma 3.1.

(iii) Using the condition $a^m a^* (a^\#)^n = (a^\#)^n a^m a^*$, we obtain

$$aa^*(a^\#)^n = (a^\#)^{m-1} a^m a^* (a^\#)^n = (a^\#)^{m-1} (a^\#)^n a^m a^* = (a^\#)^n aa^*, \tag{10}$$

which gives

$$\begin{aligned} a^* &= a^\dagger a^n ((a^\#)^n aa^*) = a^\dagger a^n aa^* (a^\#)^n = a^\dagger a^n (aa^*(a^\#)^n) a^{n+1} a^\dagger (a^\#)^n \\ &= a^\dagger a^n (a^\#)^n aa^* a^n aa^\dagger a (a^\#)^{n+1} = a^* aa^\#. \end{aligned} \tag{11}$$

Now, by the previous equality, we have $aa^\dagger = (a^\dagger)^* a^* = (a^\dagger)^* a^* aa^\# = aa^\#$ and $a^* = (aaa^\#)^* = (aaa^\dagger)^* = aa^\dagger a^* = aa^\# a^*$. From this equality, (10) and (11), we observe $a^* a^n = aa^\# a^* a^n = a^{n-1}((a^\#)^n aa^*) a^n = a^{n-1} aa^* (a^\#)^n a^n = a^n a^*$.

(iv) The equality $a^m a^n a^* = a^m a^* a^n$ gives $a^\# aa^* = (a^\#)^{m+n}(a^m a^n a^*) = (a^\#)^{m+n} a^m a^* a^n = (a^\#)^n a^* a^n$ and

$$\begin{aligned} a^n a^* (a^\#)^n &= (a^\#)^m (a^m a^n a^*)(a^\#)^n = (a^\#)^m a^m a^* a^n (a^\#)^n = (a^\# aa^*) aa^\# \\ &= (a^\#)^n a^* a^n aa^\# = (a^\#)^n a^* a^n = a^\# aa^*. \end{aligned}$$

Now, by this, we prove that condition (iii) holds:

$$a^m a^* (a^\#)^n = a^m (a^\#)^n (a^n a^* (a^\#)^n) = a^m (a^\#)^n a^\# aa^* = (a^\#)^n a^m a^*.$$

(v) If $a^*(a^\#)^n = (a^\#)^n a^*$, then the equality (iii) is satisfied.

(vi) From $a^*(a^\dagger)^n = (a^\#)^n a^*$, it follows

$$\begin{aligned} a^\# a &= (a^\#)^{n+1} a (a^\dagger a)^* a^n = ((a^\#)^n a^*)(a^\dagger)^* a^n = a^* (a^\dagger)^n (a^\dagger)^* a^n \\ &= a^\dagger a (a^* (a^\dagger)^n)(a^\dagger)^* a^n = a^\dagger a (a^\#)^n a^* (a^\dagger)^* a^n = a^\dagger aa^\# a = a^\dagger a, \end{aligned}$$

and a is EP, by Theorem 1.2. The equalities $a^\dagger = a^\#$ and (vi) imply $a^*(a^\#)^n = (a^\#)^n a^*$. So, the condition (v) holds.

(vii) Multiplying the hypothesis $a(a^*)^n a^\dagger = (a^*)^n$ from the right side by $a^\dagger$, we obtain $(a^*)^n a^\dagger = a^\dagger (a^*)^n$. Hence, $(a^*)^n = a((a^*)^n a^\dagger) = aa^\dagger (a^*)^n$ and

$$a^\# a = (a^\#)^n [(a^*)^n]^* = (a^\#)^n (aa^\dagger (a^*)^n)^* = (a^\#)^n a^n aa^\dagger = aa^\dagger. \tag{12}$$

The equality (12) and Theorem 1.2 give a is EP. By Lemma 3.2 and $(a^*)^n a^\dagger = a^\dagger (a^*)^n$, we deduce $a^* a^n = a^n a^*$.

(viii) Using the equality $a(a^*)^n a^\# = (a^*)^n$, we get $aa^\dagger (a^*)^n = aa^\dagger a(a^*)^n a^\# = a(a^*)^n a^\# = (a^*)^n$ which implies (12) and, by Theorem 1.2, a is EP. Since $a^\dagger = a^\#$, from (viii), we conclude that (vii) holds.

(ix) By the assumption $a^*(a^*)^n a^\# = a^* a^\# (a^*)^n$, we have

$$\begin{aligned} (a^*)^n &= a^\dagger a^2 (aa^\dagger)^* a (a^\#)^2 (a^*)^n = a^\dagger a^2 (a^\dagger)^* (a^* a^\# (a^*)^n) \\ &= a^\dagger a^2 (a^\dagger)^* a^* (a^*)^n a^\# = a^\dagger a^2 (a^\dagger)^* (a^* (a^*)^n a^\#) aa^\# \\ &= a^\dagger a^2 (a^\dagger)^* a^* a^\# (a^*)^n aa^\# = a^\dagger a^2 aa^\dagger a (a^\#)^2 (a^*)^n aa^\# = (a^*)^n aa^\#. \end{aligned} \tag{13}$$

This equality gives $aa^\# = [(a^*)^n]^* (a^\#)^n = [(a^*)^n aa^\#]^* (a^\#)^n = (aa^\#)^* aa^\#$. Since $(aa^\#)^* aa^\#$ is symmetric element, then $aa^\#$ is symmetric and, by Theorem 1.2, a is EP. Now, by (ix), (13) and $aa^\dagger = a^\dagger a$, we observe

$$\begin{aligned} a^\# (a^*)^n a &= a (a^\#)^2 (a^*)^n a = aa^\dagger a (a^\#)^2 (a^*)^n a = (a^\dagger)^* (a^* a^\# (a^*)^n) a \\ &= (a^\dagger)^* a^* ((a^*)^n a^\# a) = aa^\dagger (a^*)^n = a^\dagger a (a^*)^n = (a^*)^n. \end{aligned}$$

Thus, the condition (viii) is satisfied.

(x) Suppose that $a^* a^* (a^\#)^n = a^* (a^\#)^n a^*$. Then

$$\begin{aligned} (a^\#)^n a^* &= aa^\dagger a (a^\#)^{n+1} a^* = (a^\dagger)^* (a^* (a^\#)^n a^*) = (a^\dagger)^* a^* a^* (a^\#)^n \\ &= (a^\dagger)^* (a^* a^* (a^\#)^n) aa^\# = (a^\dagger)^* a^* (a^\#)^n a^* aa^\# \\ &= aa^\dagger (a^\#)^n a^* aa^\# = aa^\dagger a (a^\#)^{n+1} a^* aa^\# = (a^\#)^n a^* aa^\#, \end{aligned}$$

which yields $aa^\# = (a^\dagger)^* a^\dagger a^{n+1} ((a^\#)^n a^* aa^\#) = (a^\dagger)^* a^\dagger a^{n+1} (a^\#)^n a^* = aa^\dagger$. Hence, by Theorem 1.2, a is EP. From $a^\dagger a = aa^\dagger$ and (x), we show that the condition (v) holds:

$$a^* (a^\#)^n = aa^\dagger a^* (a^\#)^n = (a^\dagger)^* (a^* a^* (a^\#)^n) = (a^\dagger)^* a^* (a^\#)^n a^* = (a^\#)^n a^*.$$

(xi) From the hypothesis $(a^*)^n a^\dagger a^\# = a^\# (a^*)^n a^\dagger$, we get

$$(a^*)^n a^\dagger = (a^*)^n a^\dagger aa^\dagger = ((a^*)^n a^\dagger a^\#) aaa^\dagger = a^\# (a^*)^n a^\dagger aaa^\dagger \tag{14}$$

and then

$$\begin{aligned} a^\# (a^*)^n a^\dagger &= (a^*)^n a^\dagger a^\# = a^\dagger a ((a^*)^n a^\dagger a^\#) = a^\dagger aa^\# ((a^*)^n a^\dagger) \\ &= a^\dagger aa^\# a^\# (a^*)^n a^\dagger aaa^\dagger = a^\dagger (a^\# (a^*)^n a^\dagger) aaa^\dagger \\ &= a^\dagger (a^*)^n a^\dagger a^\# aaa^\dagger = a^\dagger (a^*)^n a^\dagger aa^\dagger = a^\dagger (a^*)^n a^\dagger. \end{aligned}$$

By the previous equality, we obtain

$$a(a^*)^{n+1} = a^2(a^\#(a^*)^n a^\dagger)aa^* = a^2 a^\dagger(a^*)^n a^\dagger aa^* = aaa^\dagger(a^*)^{n+1}. \tag{15}$$

Now, the equality (15) gives

$$\begin{aligned} a^\# a^n a^* &= (a^\#)^2 a^{n+1} a^* = (a^\#)^2 (a(a^*)^{n+1})^* = (a^\#)^2 (aaa^\dagger(a^*)^{n+1})^* \\ &= (a^\#)^2 a^{n+1} aa^\dagger a^* = a^n a^\dagger a^* \end{aligned}$$

implying

$$a^\# a = (a^\#)^{n-1}(a^\# a^n a^*)(a^\dagger)^* = (a^\#)^{n-1} a^n a^\dagger a^* (a^\dagger)^* = aa^\dagger a^\dagger a. \tag{16}$$

Using (14) and (16), it follows

$$(a^*)^n a^\dagger = a^\#(a^*)^n a^\dagger aaa^\dagger = a^\#(aa^\dagger a^\dagger aa^n)^* = a^\#(a^\# aa^n)^* = a^\#(a^*)^n. \tag{17}$$

Therefore, by (17) and (xi),

$$(a^*)^n a^\# = a^\dagger a^2(a^\#(a^*)^n)a^\# = a^\dagger a^2((a^*)^n a^\dagger a^\#) = a^\dagger a^2 a^\#(a^*)^n a^\dagger = (a^*)^n a^\dagger. \tag{18}$$

Obviously, by (17) and (18), $(a^*)^n a^\# = a^\#(a^*)^n$ which yields $a^*(a^*)^n a^\# = a^* a^\#(a^*)^n$. So, the equality (ix) is satisfied.

(xii) Assume that $a^\dagger a^\#(a^*)^n = a^\#(a^*)^n a^\dagger$. Then

$$\begin{aligned} a^\dagger a^\#(a^*)^n &= a^\#(a^*)^n a^\dagger = a^\# a^\dagger a(a^*)^n a^\dagger = a^\# a^\dagger a^2(a^\#(a^*)^n a^\dagger) \\ &= (a^\#)^2 aa^\dagger a^2 a^\dagger a^\#(a^*)^n = a^\# aa^\dagger a(a^\#)^2(a^*)^n = a^\# a^\#(a^*)^n \end{aligned}$$

and consequently

$$\begin{aligned} a^\dagger a^\# a^* &= a^\dagger a^\#((a^\#)^{n-1}a^n)^* = (a^\dagger a^\#(a^*)^n)((a^\#)^{n-1})^* \\ &= a^\# a^\#(a^*)^n((a^\#)^{n-1})^* = a^\# a^\# a^*. \end{aligned}$$

This equality implies

$$\begin{aligned} a^\# a &= a^\#(a^\#)^2 aa^\dagger aa^2 = (a^\# a^\# a^*)(a^\dagger)^* a^2 = a^\dagger a^\# a^*(a^\dagger)^* a^2 \\ &= a^\dagger a^\# a^\dagger aa^2 = a^\dagger(a^\#)^2 aa^\dagger aa^2 = a^\dagger a. \end{aligned}$$

Thus, element $a^\# a$ is symmetric and a is EP, by Theorem 1.2. When we use $a^\dagger = a^\#$ and (xii), we obtain $a^\# a^\#(a^*)^n = a^\#(a^*)^n a^\#$ and

$$a(a^*)^n a^\# = a^2(a^\#(a^*)^n a^\#) = a^2 a^\# a^\#(a^*)^n = a^\# a(a^*)^n = a^\dagger a(a^*)^n = (a^*)^n.$$

Hence, the condition (viii) holds. □

For $n = 1$ in Theorem 3.1, we get new equivalent conditions for an element a of a ring with involution to be normal, extending some results from [25].

Theorem 3.2. *Let $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $m \in N$. Then a is normal if and only if one of the following equivalent conditions holds:*

(i) $a^m a^* a^\# = a^\# a^m a^*$ (*or* $a^* a^m a^\# = a^\# a^* a^m$);

(ii) $a^m a a^* = a^m a^* a$ (*or* $a a^* a^m = a^* a a^m$).

4 Generalized Hermitian elements

In this section we consider an element a in rings with involution which is generalized Hermitian. In the following theorem we assume again that element a is MP-invertible and group invertible and we investigate equivalent conditions for element a to be generalized Hermitian element. The following theorem is motivated by results about Hermitian elements in [1, 9, 25] which are recovered for $m = n = 1$.

Theorem 4.1. *Let $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $m, n \in N$. Then $a^n = (a^*)^n$ if and only if one of the following equivalent conditions holds:*

(i) $a^n a a^\dagger = (a^*)^n$ (*or* $a^n a^m = (a^*)^n a^m$ *or* $a^n (a^\#)^m = (a^*)^n (a^\#)^m$);

(ii) $a^n a^\dagger = (a^*)^n a^\dagger$ (*or* $a^n a^\# = (a^*)^n a^\dagger$ *or* $a^n a^\# = a^\dagger (a^*)^n$ *or* $(a^*)^n a^\dagger (a^\#)^n = a^\#$);

(iii) $a^\dagger a^n = a^\# (a^*)^n$;

(iv) $a^* (a^*)^n (a^\#)^n = a^*$;

(v) $(a^*)^n a^\dagger (a^\dagger)^n = a^\#$;

(vi) $(a^*)^n a^\dagger (a^\#)^n = a^\dagger$;

(vii) $a^\# (a^*)^n (a^\#)^n = a^\dagger$;

(viii) $a (a^*)^n a^\dagger = a^n$.

Proof. Assume that $a^n = (a^*)^n$. Then $a^n a^* = (a^*)^{n+1} = a^* a^n$ and, by Lemma 3.2, we conclude that a commutes with its Moore–Penrose inverse and $a^\# = a^\dagger$. Since the group inverse $a^\#$ doubly commutes with a, we deduce that $a^\#$ commutes with a^n. Obviously, the conditions (i)-(viii) hold.

Conversely, we prove that a satisfies the equality $a^n = (a^*)^n$.

(i) Using the condition $a^n aa^\dagger = (a^*)^n$, we obtain

$$a^n = [(a^*)^n]^* = (a^n aa^\dagger)^* = aa^\dagger (a^*)^n = aa^\dagger a^n aa^\dagger = a^n aa^\dagger = (a^*)^n.$$

(ii) By the assumption $a^n a^\dagger = (a^*)^n a^\dagger$, we have $a^n = a^n a^\dagger a = (a^*)^n a^\dagger a$, which gives $(a^*)^n = (a^n)^* = ((a^*)^n a^\dagger a)^* = a^\dagger a a^n = a^\dagger a (a^*)^n a^\dagger a = a^n$.

(iii) Suppose that $a^\dagger a^n = a^\# (a^*)^n$. Now, we have $a^n = a(a^\dagger a^n) = aa^\# (a^*)^n$ and

$$\begin{aligned} (a^*)^n &= a^\dagger a (a^*)^n = a^\dagger a (aa^\# (a^*)^n) = a^\dagger a a^n = (a^\dagger a^n) a \\ &= a^\# (a^*)^n a = a^\# (aa^\# (a^*)^n) a = a^\# a^n a = a^n. \end{aligned}$$

(iv) The equality $a^* (a^*)^n (a^\#)^n = a^*$ gives

$$aa^\dagger = (a^\dagger)^* a^* = (a^\dagger)^* a^* (a^*)^n (a^\#)^n = aa^\dagger (a^*)^n (a^\#)^n, \tag{19}$$

which yields $aa^\# = (aa^\dagger) aa^\# = aa^\dagger (a^*)^n (a^\#)^n aa^\# = aa^\dagger (a^*)^n (a^\#)^n = aa^\dagger$. Because $aa^\dagger$ is symmetric, $aa^\#$ is symmetric too. Thus, from Theorem 1.2, a is EP and $aa^\dagger = a^\dagger a = aa^\#$. So, by this equality and (19),

$$a^n = (aa^\dagger) a^n = (aa^\dagger)(a^*)^n (a^\#)^n a^n = a^\dagger a (a^*)^n aa^\# = (a^*)^n aa^\dagger = (a^*)^n.$$

(v) When we apply the assumption $(a^*)^n a^\dagger (a^\dagger)^n = a^\#$, we see that

$$aa^\# = a (a^*)^n a^\dagger (a^\dagger)^n = a((a^*)^n a^\dagger (a^\dagger)^n) aa^\dagger = aa^\# aa^\dagger = aa^\dagger.$$

Therefore, $aa^\#$ is symmetric and, by Theorem 1.2, a is EP. Then, by $a^\# = a^\dagger$ and (v), we get that the condition (i) holds:

$$(a^*)^n a^m = ((a^*)^n a^\dagger (a^\dagger)^n) a^{m+n+1} = a^\# a^{m+n+1} = a^{m+n}.$$

(vi) Assume that $(a^*)^n a^\dagger (a^\#)^n = a^\dagger$. Now

$$aa^\# = aa^\dagger aa^\# = a(a^*)^n a^\dagger (a^\#)^n aa^\# = a((a^*)^n a^\dagger (a^\#)^n) = aa^\dagger.$$

Since $aa^\#$ is symmetric, by Theorem 1.2, a is EP. Using $a^\# = a^\dagger$ and (vi), we show that condition (i) holds:

$$\begin{aligned} a^n (a^\#)^m &= a^\# a^n (a^\#)^{m-1} = a^\dagger a^n (a^\#)^{m-1} = (a^*)^n a^\dagger (a^\#)^n a^n (a^\#)^{m-1} \\ &= (a^*)^n a^\# a^\# a (a^\#)^{m-1} = (a^*)^n (a^\#)^m. \end{aligned}$$

(vii) If $a^\# (a^*)^n (a^\#)^n = a^\dagger$, then

$$a^\# a = a^\# aa^\dagger a = a^\# aa^\# (a^*)^n (a^\#)^n a = (a^\# (a^*)^n (a^\#)^n) a = a^\dagger a.$$

Therefore, $a^{\#}a$ is symmetric and a is EP, by Theorem 1.2. It follows, by $a^{\#} = a^{\dagger}$ and (vii), that a satisfies the condition (iii):

$$a^{\dagger}a^n = a^{\#}(a^*)^n(a^{\#})^n a^n = a^{\#}(a^*)^n aa^{\#} = a^{\#}(a^*)^n aa^{\dagger} = a^{\#}(a^*)^n.$$

(ix) From the hypothesis $a(a^*)^n a^{\dagger} = a^n$, we have

$$a^{\#}a = (a^{\#})^n a^n = (a^{\#})^n a(a^*)^n a^{\dagger} = (a^{\#})^n(a(a^*)^n a^{\dagger})aa^{\dagger} = a^{\#}aaa^{\dagger} = aa^{\dagger},$$

implying that $a^{\#}a$ is symmetric and a is EP, by Theorem 1.2. The equalities $aa^{\dagger} = a^{\dagger}a$ and (ix) give $a^n a^{\dagger} = a^{\dagger}a^n = a^{\dagger}a(a^*)^n a^{\dagger} = (a^*)^n a^{\dagger}$. Thus, the condition (ii) is satisfied. $\square$

When $n = 1$, Theorem 4.1 implies the next new conditions for characterizations of Hermitian elements in rings with involution and generalizes some recent results in [25].

Theorem 4.2. *Let $a \in \mathcal{R}^{\dagger} \cap \mathcal{R}^{\#}$ and $m \in N$. Then a is Hermitian if and only if $aa^m = a^*a^m$ (or $a(a^{\#})^m = a^*(a^{\#})^m$).*

5 Conclusions

In this paper we investigate necessary and sufficient conditions for Moore-Penrose invertible and group invertible element a in rings with involution to be EP, generalized normal and generalized Hermitian elements. As a corollary, we obtain some new characterization of normal and Hermitian elements in terms of equations involving their adjoints, Moore-Penorse and group inverse and their powers, extending some earlier results.

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