Some results on the reverse order law in rings with involution

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Abstract

We investigate some necessary and sufficient conditions for the hybrid reverse order law $(ab)^{\#} = b^{\dagger}a^{\dagger}$ in rings with involution. Assuming that a and b are Moore-Penrose invertible, we present equivalent condition for the product ab to be EP element.

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1 Introduction

We start with some standard notations. Let $\mathcal{R}$ be an associative ring with the unit 1, and let $a \in \mathcal{R}$. Then a is *group invertible* if there exists $x \in \mathcal{R}$ satisfying

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (5)\ ax = xa;$$

such x is the uniquely determined *group inverse* of a, written $x = a^{\#}$. The group inverse $a^{\#}$ double commutes with a, that is, $ax = xa$ implies $a^{\#}x = xa^{\#}$ [1]. Denote by $\mathcal{R}^{\#}$ the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ is *self-adjoint* (or *Hermitian*) if $a^* = a$.

An element $a \in \mathcal{R}$ is *Moore–Penrose invertible* if there exists $x \in \mathcal{R}$ satisfying the so-called Penrose conditions [11],

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa;$$

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such x is the uniquely determined *Moore–Penrose inverse* of a denoted by $x = a^\dagger$. We provide a short proof: if b, c are two candidates for a Moore–Penrose inverse of a, then $ab = (ac)(ab)$ is self-adjoint. The product of two self-adjoint elements ab and ac is self-adjoint if and only if they commute. Hence, $ab = acab = abac = ac$. In the same way we can prove that $ba = ca$. Now we have $b = bab = bac = cac = c$. Thus, the Moore–Penrose inverse of a is unique if it exists. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

If $\delta \subset \{1, 2, 3, 4, 5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1, 2, 5\} = \{a^\#\}$ and $a\{1, 2, 3, 4\} = \{a^\dagger\}$. If a is invertible, then $a^\#$ and $a^\dagger$ each coincide with the ordinary inverse of a. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

For $a \in \mathcal{R}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{R} : xa = 0\}.$$

An element $a \in \mathcal{R}$ is: *left *-cancellable* if $a^*ax = a^*ay$ implies $ax = ay$; it is *right *-cancellable* if $xaa^* = yaa^*$ implies $xa = ya$; and it is **-cancellable* if it is both left and right *-cancellable. We observe that a is left *-cancellable if and only if a^* is right *-cancellable. In C^*-algebras all elements are *-cancellable. A ring $\mathcal{R}$ is called **-reducing* if every element of $\mathcal{R}$ is *-cancellable. This is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$ for all $a \in \mathcal{R}$.

We recall the definition of EP elements [9].

Definition 1.1. An element a of a ring $\mathcal{R}$ with involution is said to be *EP* if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and $a^\# = a^\dagger$.

The EP elements are important since they are characterized by commutativity with their Moore–Penrose inverse.

In the theory of generalized inverses, one of fundamental procedures is to find generalized inverses of products. If $a, b \in \mathcal{R}$ are invertible, then ab is also invertible, and the inverse of the product ab satisfied $(ab)^{-1} = b^{-1}a^{-1}$. This equality is called the reverse order law, and it can be used to simplify various expressions that involve inverses of products. Since this formula cannot be trivially extended to various generalized inverse of the product ab, the reverse order law for various generalized inverses yields a class of interesting problems that are fundamental in the theory of generalized inverses. Many authors studied these problems [1, 2, 4, 5, 6, 7].

Greville [6] proved that $(ab)^\dagger = b^\dagger a^\dagger$ holds for complex matrices, if and only if: $a^\dagger a$ commutes with bb^*, and $bb^\dagger$ commutes with a^*a. In the case of linear bounded operators on Hilbert spaces, the analogous result was proved by Izumino [7]. The corresponding result in rings with involution was proved in [8].

C.Y. Deng [4] presented some necessary and sufficient conditions concerning the reverse order law $(ab)^\# = b^\# a^\#$ for the group invertible linear bounded operators a and b on Hilbert space. He used the matrix form of operators induced by some natural decomposition of Hilbert spaces.

In this paper we introduce some conditions equivalent to the hybrid reverse order law $(ab)^\# = b^\dagger a^\dagger$ in rings with involution, motivated by [4]. We prove that the assumption of inclusion $(ab)\{1,5\} \subseteq b\{1,3,4\} \cdot a\{1,3,4\}$ automatically implies equality. We also give several conditions equivalent to $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$ (that is, the product of elements a and b be EP). We also study conditions related to the reverse order laws $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\# = (a^* ab)^\dagger a^*$, $(ab)^\# = b^\dagger (abb^\dagger)^\dagger$ and $(ab)^\# = b^*(abb^*)^\dagger$.

To conclude this section, we state the following well-known results on the Moore-Penrose inverse, which be used later.

Lemma 1.1. *If $a \in \mathcal{R}^\dagger$, then*

(i) $a \cdot a\{1,3\} = \{aa^\dagger\}$;

(ii) $a\{1,4\} \cdot a = \{aa^\dagger\}$.

Proof. We only prove the statement (i), because (ii) follows similarly. For $a^{(1,3)} \in a\{1,3\}$, we have

$$aa^{(1,3)} = aa^\dagger aa^{(1,3)} = (aa^{(1,3)}aa^\dagger)^* = (aa^\dagger)^* = aa^\dagger.$$

Thus, $a \cdot a\{1,3\} \subseteq \{aa^\dagger\}$. By $a^\dagger \in a\{1,3\}$, $aa^\dagger \in a \cdot a\{1,3\}$. □

Lemma 1.2. *Let $a, b \in \mathcal{R}$.*

(i) *If $a, a^\dagger ab \in \mathcal{R}^\dagger$, then $(a^\dagger ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger a$.*

(ii) *If $b, abb^\dagger \in \mathcal{R}^\dagger$, then $(abb^\dagger)^\dagger = bb^\dagger (abb^\dagger)^\dagger$.*

Proof. The statement (i) follows from

$$\begin{aligned}(a^\dagger ab)^\dagger a^\dagger a &= (a^\dagger ab)^\dagger a^\dagger ab(a^\dagger ab)^\dagger a^\dagger a = (a^\dagger ab)^\dagger (a^\dagger aa^\dagger ab(a^\dagger ab)^\dagger)^* \\ &= (a^\dagger ab)^\dagger (a^\dagger ab(a^\dagger ab)^\dagger)^* = (a^\dagger ab)^\dagger.\end{aligned}$$

The second statement can be verified in the same way. □

By Remark after Theorem 2.4 in [11], [11, Theorem 2.1] can be formulated as follows.

Theorem 1.1. *Let $\mathcal{R}$ be a ring with involution, let $a, b \in \mathcal{R}^\dagger$ and let $(1-a^\dagger a)b$ be left *-cancellable. Then the following conditions are equivalent:*

(a) $abb^\dagger a^\dagger ab = ab$;

(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;

(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$.

2 Reverse order law for the group inverse

If a and b are any matrices such that the product ab is defined, Cline [3] has developed a representation for the Moore–Penrose inverse of the product of ab, as follows: $(ab)^\dagger = (a^\dagger ab)^\dagger (abb^\dagger)^\dagger$. In [17], Z. Xiong and Y. Qin generalized this result to the case of the weighted Moore–Penrose inverse.

The mixed-type reverse-order laws for matrix product ab like $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\dagger = b^\dagger (abb^\dagger)^\dagger$, $(ab)^\dagger = (a^*ab)^\dagger a^*$, $(ab)^\dagger = b^*(abb^*)^\dagger$ have also been considered, see [14, 15, 16].

The equations $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\# = (a^*ab)^\dagger a^*$, $(ab)^\# = b^\dagger(abb^\dagger)^\dagger$ and $(ab)^\# = b^*(abb^*)^\dagger$ which appear in this section can be seen as a special case of the above results.

In the following theorem, some necessary and sufficient conditions for $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$ to be satisfied are given.

Theorem 2.1. *If $b \in \mathcal{R}$, $a, a^\dagger ab \in \mathcal{R}^\dagger$, and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$,

(ii) $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{5\}$,

(iii) $abaa^\dagger = ab$ *and* $(a^\dagger ab)^\dagger ba = ab(a^\dagger ab)^\dagger$,

(iv) $(a^\dagger ab)\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{5\}$.

Proof. (i) $\Rightarrow$ (ii): This is clear.

(ii) $\Rightarrow$ (iii): The assumption $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{5\}$ give $(a^\dagger ab)^\dagger a^\dagger ab = ab(a^\dagger ab)^\dagger a^\dagger$. Then, $(a^\dagger ab)^\dagger a^\dagger aba = ab(a^\dagger ab)^\# a^\dagger a$ and, by Lemma 1.2(i), $(a^\dagger ab)^\dagger ba = ab(a^\dagger ab)^\dagger$. Notice that $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1\}$, by

$$ab(a^\dagger ab)^\dagger a^\dagger ab = a(a^\dagger ab(a^\dagger ab)^\dagger a^\dagger ab) = aa^\dagger ab = ab. \tag{2.1}$$

Now, we obtain $ab = abab(a^\dagger ab)^\dagger a^\dagger$ and

$$abaa^\dagger = abab(a^\dagger ab)^\dagger a^\dagger aa^\dagger = abab(a^\dagger ab)^\# a^\dagger = ab.$$

(iii) $\Rightarrow$ (iv): Let $abaa^\dagger = ab$ and $(a^\dagger ab)^\dagger ba = ab(a^\dagger ab)^\dagger$. For $(a^\dagger ab)^{(1,3,4)} \in (a^\dagger ab)\{1,3,4\}$ and $a^{(1,3,4)} \in a\{1,3,4\}$, by Lemma 1.1 and Lemma 1.2, we obtain that $(a^\dagger ab)^{(1,3,4)}a^{(1,3,4)} \in (ab)\{5\}$:

$$\begin{aligned} ab(a^\dagger ab)^{(1,3,4)}a^{(1,3,4)} &= a(a^\dagger ab(a^\dagger ab)^{(1,3,4)})a^{(1,3,4)} = aa^\dagger ab(a^\dagger ab)^\dagger a^{(1,3,4)} \\ &= (ab(a^\dagger ab)^\dagger)a^{(1,3,4)} = (a^\dagger ab)^\dagger b(aa^{(1,3,4)}) \\ &= (a^\dagger ab)^\dagger a^\dagger(abaa^\dagger) = (a^\dagger ab)^\dagger a^\dagger ab \\ &= (a^\dagger ab)^{(1,3,4)}a^\dagger ab = (a^\dagger ab)^{(1,3,4)}a^{(1,3,4)}ab. \end{aligned}$$

Hence, for any $(a^\dagger ab)^{(1,3,4)} \in (a^\dagger ab)\{1,3,4\}$ and $a^{(1,3,4)} \in a\{1,3,4\}$, we conclude that $(a^\dagger ab)^{(1,3,4)}a^{(1,3,4)} \in (ab)\{5\}$ and the statement (iv) holds.

(iv) $\Rightarrow$ (i): By the hypothesis (iv), $(a^\dagger ab)^\dagger \in (a^\dagger ab)\{1,3,4\}$ and $a^\dagger \in \{1,3,4\}$, we deduce $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{5\}$. The equalities (2.1) and

$$((a^\dagger ab)^\dagger a^\dagger ab(a^\dagger ab)^\dagger)a^\dagger = (a^\dagger ab)^\dagger a^\dagger$$

imply $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1,2\}$ and the condition (i) is satisfied. $\square$

Remark. The following results concerning $(ab)^\# = (a^*ab)^\dagger a^*$, $(ab)^\# = b^\dagger(abb^\dagger)^\dagger$ and $(ab)^\# = b^*(abb^*)^\dagger$ can be proved in the similar way as in Theorem 2.1. Let $a, b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$.

(1) If $a, a^*ab \in \mathcal{R}^\dagger$, then

$$\begin{aligned} (ab)^\# = (a^*ab)^\dagger a^* &\Leftrightarrow (a^*ab)^\dagger a^* \in (ab)\{5\} \\ &\Leftrightarrow abaa^\dagger = ab \text{ and } (a^*ab)^\dagger a^*aba = ab(a^\dagger ab)^\dagger a^*a \\ &\Leftrightarrow (a^*ab)\{1,3,4\} \cdot a^* \subseteq (ab)\{5\}. \end{aligned}$$

(2) If $b, abb^\dagger \in \mathcal{R}^\dagger$, then

$$\begin{aligned} (ab)^\# = b^\dagger(abb^\dagger)^\dagger &\Leftrightarrow b^\dagger(abb^\dagger)^\dagger \in (ab)\{5\} \\ &\Leftrightarrow b^\dagger bab = ab \text{ and } ba(abb^\dagger)^\dagger = (abb^\dagger)^\dagger ab \\ &\Leftrightarrow b\{1,3,4\} \cdot (abb^\dagger)\{1,3,4\} \subseteq (ab)\{5\}. \end{aligned}$$

(3) If $b, abb^* \in \mathcal{R}^\dagger$, then

$$\begin{aligned} (ab)^\# = b^*(abb^*)^\dagger &\Leftrightarrow b^*(abb^*)^\dagger \in (ab)\{5\} \\ &\Leftrightarrow b^\dagger bab = ab \text{ and } babb^*(abb^*)^\dagger = bb^*(abb^*)^\dagger ab \\ &\Leftrightarrow b^* \cdot (abb^*)\{1,3,4\} \subseteq (ab)\{5\}. \end{aligned}$$

Supposing that a and b are Moore–Penrose invertible elements in ring with involution, we study conditions which ensure that $b^\dagger = (ab)^\# a$ in the next theorem. As we will see, these conditions imply that product ab is an EP element.

Theorem 2.2. *If $a, b \in \mathcal{R}^{\dagger}$, and $ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{\dagger} = (ab)^{\#}a$,

(ii) $b = a^{\dagger}ab = baa^{\dagger}$ *and* $abb^{\dagger} = b^{\dagger}ba$,

(iii) $b\mathcal{R} \subset a^*\mathcal{R}$, $a^{\dagger}ab = baa^{\dagger}$ *and* $abb^{\dagger} = b^{\dagger}ba$,

(iv) $(a^*)^{\circ} \subset b^{\circ}$, $a^{\dagger}ab = baa^{\dagger}$ *and* $abb^{\dagger} = b^{\dagger}ba$,

(v) $ab \in \mathcal{R}^{\dagger}$, $(ab)^{\#} = b^{\dagger}a^{\dagger} = (ab)^{\dagger}$ *and* $b = a^{\dagger}ab$.

Proof. (i) $\Rightarrow$ (ii): Applying the hypothesis $b^{\dagger} = (ab)^{\#}a$, we obtain

$$abb^{\dagger} = (ab(ab)^{\#})a = ((ab)^{\#}a)ba = b^{\dagger}ba,$$

and

$$a^{\dagger}ab = a^{\dagger}abb^{\dagger}b = (bb^{\dagger}a^{\dagger}a)^*b = (b(ab)^{\#}aa^{\dagger}a)^*b = (bb^{\dagger})^*b = b.$$

Since

$$aa^{\dagger}(ab)^{\#} = aa^{\dagger}ab[(ab)^{\#}]^2 = ab[(ab)^{\#}]^2 = (ab)^{\#},$$

we have

$$baa^{\dagger} = bb^{\dagger}baa^{\dagger} = b(aa^{\dagger}b^{\dagger}b)^* = b(aa^{\dagger}(ab)^{\#}ab)^* = b((ab)^{\#}ab)^* = b(b^{\dagger}b)^* = b.$$

Thus, (ii) is satisfied.

(ii) $\Rightarrow$ (i): Let $b = a^{\dagger}ab = baa^{\dagger}$ and $abb^{\dagger} = b^{\dagger}ba$. By the equalities

$$b(ab)^{\#}ab = a^{\dagger}(ab(ab)^{\#}ab) = a^{\dagger}ab = b \tag{2.2}$$

and

$$(ab)^{\#}ab(ab)^{\#}a = (ab)^{\#}a, \tag{2.3}$$

we deduce that $(ab)^{\#}a \in b\{1,2\}$. Using

$$(ab)^{\#}b^{\dagger}b = [(ab)^{\#}]^2abb^{\dagger}b = [(ab)^{\#}]^2ab = (ab)^{\#},$$

we get

$$b(ab)^{\#}a = b(ab)^{\#}(b^{\dagger}ba) = b((ab)^{\#}a)bb^{\dagger} = bb^{\dagger}bb^{\dagger} = bb^{\dagger},$$

which gives that $b(ab)^{\#}a$ is selfadjoint, i.e. $(ab)^{\#}a \in b\{3\}$. Further, from

$$ab(ab)^{\#} = (abb^{\dagger})b(ab)^{\#} = b^{\dagger}b(ab(ab)^{\#}) = b^{\dagger}b((ab)^{\#}a)b = b^{\dagger}bb^{\dagger}b = b^{\dagger}b,$$

we have $(ab)^{\#}ab = ab(ab)^{\#} = b^\dagger b$ is selfadjoint, that is, $(ab)^{\#}a \in b\{4\}$. So, we conclude that $b^\dagger = (ab)^{\#}a$.

(ii) $\Leftrightarrow$ (iii): To show that $b = a^\dagger ab$ is equivalent to $b\mathcal{R} \subset a^*\mathcal{R}$, obviously, if $b = a^\dagger ab$, then $b\mathcal{R} \subset a^\dagger\mathcal{R} = a^*\mathcal{R}$. Conversely, $b\mathcal{R} \subset a^*\mathcal{R}$ gives $b = a^*x$, for some $x \in R$, which implies $b = a^\dagger a(a^*x) = a^\dagger ab$.

(ii) $\Leftrightarrow$ (iv): This equivalence follows from $b = baa^\dagger$ iff $b(1 - aa^\dagger) = 0$ iff $(1 - aa^\dagger)\mathcal{R} \subset b^\circ$ iff $(a^*)^\circ \subset b^\circ$.

(i) $\Rightarrow$ (v): Suppose that $b^\dagger = (ab)^{\#}a$. It can be check easily that $b^\dagger a^\dagger \in (ab)\{1, 2\}$. Observe that, $b^\dagger b = (ab)^{\#}ab = ab(ab)^{\#}$ and

$$aa^\dagger b^\dagger = aa^\dagger(ab)^{\#}a = aa^\dagger ab((ab)^{\#})^2a = ab((ab)^{\#})^2a = (ab)^{\#}a = b^\dagger.$$

Now, we obtain

$$abb^\dagger a^\dagger = (ab(ab)^{\#})aa^\dagger = b^\dagger baa^\dagger = ((aa^\dagger b^\dagger)b)^* = (b^\dagger b)^* = b^\dagger b$$

and

$$b^\dagger a^\dagger ab = (ab)^{\#}aa^\dagger ab = (ab)^{\#}ab = b^\dagger b.$$

Consequently, $b^\dagger a^\dagger \in (ab)\{3, 4, 5\}$, by $abb^\dagger a^\dagger = b^\dagger a^\dagger ab = b^\dagger b$. Hence, $ab \in \mathcal{R}^\dagger$ and $(ab)^{\#} = b^\dagger a^\dagger = (ab)^\dagger$. The equality $b = a^\dagger ab$ follows as in part (i) $\Rightarrow$ (ii).

(v) $\Rightarrow$ (i): Using $(ab)^{\#} = b^\dagger a^\dagger = (ab)^\dagger$ and $b = a^\dagger ab$, we obtain $(ab)^{\#}ab = b^\dagger a^\dagger ab$ is selfadjoint, (2.2) and (2.3). Thus, $(ab)^{\#}a \in b\{1, 2, 4\}$. From

$$b(ab)^{\#}a = bb^\dagger a^\dagger a = (a^\dagger abb^\dagger)^* = (bb^\dagger)^* = bb^\dagger,$$

we have $(ab)^{\#}a \in b\{3\}$ and the statement (i) holds. $\square$

Remark. Similarly to Theorem 2.2, if $a, b \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^{\#}$, then

$$\begin{aligned} a^\dagger = b(ab)^{\#} \quad &\Leftrightarrow \quad a^\dagger ab = baa^\dagger \text{ and } a = abb^\dagger = b^\dagger ba \\ &\Leftrightarrow \quad a\mathcal{R} \subset b^*\mathcal{R},\ a^\dagger ab = baa^\dagger \text{ and } abb^\dagger = b^\dagger ba \\ &\Leftrightarrow \quad (b^*)^\circ \subset a^\circ,\ a^\dagger ab = baa^\dagger \text{ and } abb^\dagger = b^\dagger ba \\ &\Leftrightarrow \quad ab \in \mathcal{R}^\dagger,\ (ab)^{\#} = b^\dagger a^\dagger = (ab)^\dagger \text{ and } a = abb^\dagger. \end{aligned}$$

Corollary 2.1. *Let $a, b \in \mathcal{R}^\dagger$, and $ab \in \mathcal{R}^{\#}$. If any of conditions (i)-(iv) of Theorem 2.2 (or any of conditions of Remark after Theorem 2.2) is satisfied, then $(ab)^{\#} = b^\dagger a^\dagger = (ab)^\dagger$.*

The next result contains equivalent condition for $(ab)^{\#} = b^\dagger a^\dagger$ to hold in a ring with involution.

Theorem 2.3. *If $a, b \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = b^\dagger a^\dagger$,

(ii) $(ab)^\# a = b^\dagger a^\dagger a$ *and* $a^* ab = a^* abaa^\dagger$,

(iii) $(ab)^\# a = b^\dagger a^\dagger a$ *and* $a^\dagger ab = a^\dagger abaa^\dagger$,

(iv) $b(ab)^\# = bb^\dagger a^\dagger$ *and* $abb^* = b^\dagger babb^*$,

(v) $b(ab)^\# = bb^\dagger a^\dagger$ *and* $abb^\dagger = b^\dagger babb^\dagger$.

Proof. (i) $\Rightarrow$ (ii): By the condition $(ab)^\# = b^\dagger a^\dagger$, we get $(ab)^\# a = b^\dagger a^\dagger a$ and

$$\begin{aligned} a^* ab &= a^* ab((ab)^\# ab) = a^* abab(ab)^\# = a^* ababb^\dagger a^\dagger \\ &= a^*(ababb^\dagger a^\dagger) aa^\dagger = a^* abaa^\dagger. \end{aligned}$$

Thus, the statement (ii) holds.

(ii) $\Rightarrow$ (iii): The equalities $(ab)^\# a = b^\dagger a^\dagger a$ and $a^* ab = a^* abaa^\dagger$ imply that the conditon (iii) is satisfied:

$$a^\dagger ab = a^\dagger (a^\dagger)^* (a^* ab) = a^\dagger (a^\dagger)^* a^* abaa^\dagger = a^\dagger abaa^\dagger.$$

(iii) $\Rightarrow$ (i): Let $(ab)^\# a = b^\dagger a^\dagger a$ and $a^\dagger ab = a^\dagger abaa^\dagger$. Then, from

$$ab = ab((ab)^\# a)b = abb^\dagger a^\dagger ab$$

and

$$\begin{aligned} b^\dagger a^\dagger &= (b^\dagger a^\dagger a) a^\dagger = (ab)^\# aa^\dagger = ((ab)^\# a) b ((ab)^\# a) a^\dagger \\ &= b^\dagger a^\dagger abb^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger abb^\dagger a^\dagger, \end{aligned}$$

we deduce that $b^\dagger a^\dagger \in (ab)\{1, 2\}$. Since

$$\begin{aligned} (b^\dagger a^\dagger a) b &= (ab)^\# ab = (ab)^\# a(a^\dagger ab) = (ab)^\# aa^\dagger abaa^\dagger \\ &= ((ab)^\# ab) aa^\dagger = ab((ab)^\# a) a^\dagger = abb^\dagger a^\dagger aa^\dagger \\ &= abb^\dagger a^\dagger, \end{aligned}$$

we notice that $b^\dagger a^\dagger \in (ab)\{5\}$. Hence, $(ab)^\# = b^\dagger a^\dagger$.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i): Similarly as (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). $\square$

The condition $a^\dagger ab = a^\dagger abaa^\dagger$ in Theorem 2.3 can be replaced with equivalent conditions $\mathcal{R}a^\dagger ab \subset \mathcal{R}a^*$ or $(a^*)^\circ \subset (a^\dagger ab)^\circ$. Also, the condition $abb^\dagger = b^\dagger babb^\dagger$ in Theorem 2.3 can be replaced with equivalent conditions $abb^\dagger\mathcal{R} \subset b^*\mathcal{R}$ or ${}^\circ(b^*) \subset^\circ (abb^\dagger)$.

Sufficient conditions for the reverse order law $(ab)^\# = b^\dagger a^\dagger$ are given in the following theorem.

Theorem 2.4. *Suppose that $a, b \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^\#$. Then each of the following conditions is sufficient for $(ab)^\# = b^\dagger a^\dagger$ to hold:*

(i) $(ab)^\# a = b^\dagger a^\dagger a$ *and* $a^\dagger ab = baa^\dagger$,

(ii) $b(ab)^\# = bb^\dagger a^\dagger$ *and* $b^\dagger ba = abb^\dagger$.

Proof. (i) Using $(ab)^\# a = b^\dagger a^\dagger a$ and $a^\dagger ab = baa^\dagger$, we have

$$\begin{aligned} abb^\dagger a^\dagger &= (ab(ab)^\#)abb^\dagger a^\dagger = ((ab)^\# a)bab(b^\dagger a^\dagger a)a^\dagger = b^\dagger a^\dagger(abab(ab)^\#)aa^\dagger \\ &= b^\dagger a^\dagger a(baa^\dagger) = b^\dagger a^\dagger aa^\dagger ab = b^\dagger a^\dagger ab. \end{aligned}$$

So, $b^\dagger a^\dagger \in (ab)\{5\}$. By the equalities

$$ab(b^\dagger a^\dagger a)b = ab(ab)^\# ab = ab$$

and

$$\begin{aligned} b^\dagger a^\dagger abb^\dagger a^\dagger &= (b^\dagger a^\dagger a)a^\dagger ab(b^\dagger a^\dagger a)a^\dagger = (ab)^\# aa^\dagger ab(ab)^\# aa^\dagger \\ &= ((ab)^\# a)a^\dagger = b^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger, \end{aligned}$$

we conclude that $b^\dagger a^\dagger \in (ab)\{1,2\}$. Thus, $(ab)^\# = b^\dagger a^\dagger$.

(ii) Similarly as item (i).

□

Adding the assumption that $(1 - a^\dagger a)b$ is left $*$-cancellable, the following theorem gives a characterization of $(ab)^\# = b^\dagger a^\dagger$.

Theorem 2.5. *Let $a, b \in \mathcal{R}^\dagger$, and let $(1 - a^\dagger a)b$ be left *-cancellable. Then the following conditions are equivalent:*

(i) $ab \in \mathcal{R}^\#$ *and* $(ab)^\# = b^\dagger a^\dagger$,

(ii) $b^\dagger a^\dagger \in (ab)\{1,5\}$,

(iii) $b\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{1,5\}$,

(iv) $ab \in \mathcal{R}^{\#}$ *and* $ab(ab)^{\#} = abb^{\dagger}a^{\dagger} = b^{\dagger}a^{\dagger}ab$.

Proof. (i) $\Rightarrow$ (ii) $\wedge$ (iv): This is obvious.

(ii) $\Rightarrow$ (iii): Assume that $b^{\dagger}a^{\dagger} \in (ab)\{1,5\}$. Let $a^{(1,3,4)} \in a\{1,3,4\}$ and $b^{(1,3,4)} \in b\{1,3,4\}$. Since $abb^{\dagger}a^{\dagger}ab = ab$, by Theorem 1.1 (parts (a) and (c)), we have $a^{\dagger}abb^{\dagger} = bb^{\dagger}a^{\dagger}a$. By this equality and Lemma 1.1, we get

$$abb^{(1,3,4)}a^{(1,3,4)} = a(a^{\dagger}abb^{\dagger})a^{(1,3,4)} = abb^{\dagger}a^{\dagger}aa^{(1,3,4)} = abb^{\dagger}a^{\dagger}aa^{\dagger} = abb^{\dagger}a^{\dagger} \tag{2.4}$$

and

$$b^{(1,3,4)}a^{(1,3,4)}ab = b^{(1,3,4)}(a^{\dagger}abb^{\dagger})b = b^{(1,3,4)}bb^{\dagger}a^{\dagger}ab = b^{\dagger}bb^{\dagger}a^{\dagger}ab = b^{\dagger}a^{\dagger}ab. \tag{2.5}$$

Since $abb^{\dagger}a^{\dagger} = b^{\dagger}a^{\dagger}ab$, we observe that $abb^{(1,3,4)}a^{(1,3,4)} = b^{(1,3,4)}a^{(1,3,4)}ab$, that is $b^{(1,3,4)}a^{(1,3,4)} \in (ab)\{5\}$. Using the equality (2.4) and the assumption $b^{\dagger}a^{\dagger} \in (ab)\{1\}$, we get

$$abb^{(1,3,4)}a^{(1,3,4)}ab = abb^{\dagger}a^{\dagger}ab = ab,$$

i.e. $b^{(1,3,4)}a^{(1,3,4)} \in (ab)\{1\}$. So, the condition (iii) is satisfied.

(iii) $\Rightarrow$ (i): If $b\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{1,5\}$, by $b^{\dagger} \in b\{1,3,4\}$ and $a^{\dagger} \in a\{1,3,4\}$, we notice that $b^{\dagger}a^{\dagger} \in (ab)\{1,5\}$. From Theorem 1.1 (parts (a) and (b)), we conclude that $b^{\dagger}a^{\dagger} \in (ab)\{2\}$. Hence, $ab \in \mathcal{R}^{\#}$ and $(ab)^{\#} = b^{\dagger}a^{\dagger}$.

(iv) $\Rightarrow$ (ii): The conditions $ab \in \mathcal{R}^{\#}$ and $ab(ab)^{\#} = abb^{\dagger}a^{\dagger} = b^{\dagger}a^{\dagger}ab$ give $b^{\dagger}a^{\dagger} \in (ab)\{5\}$ and

$$ab = ab(ab)^{\#}ab = abb^{\dagger}a^{\dagger}ab.$$

Thus, $b^{\dagger}a^{\dagger} \in (ab)\{1\}$ and the statement (ii) holds. □

Theorem 2.6. *Let* $a, b \in \mathcal{R}^{\dagger}$ *and let* $(1 - a^{\dagger}a)b$ *be left *-cancellable. If* $ab \in \mathcal{R}^{\#}$, *then the inclusion* $(ab)\{1,5\} \subseteq b\{1,3,4\} \cdot a\{1,3,4\}$ *is always an equality.*

Proof. Suppose that $(ab)\{1,5\} \subseteq b\{1,3,4\} \cdot a\{1,3,4\}$. Since $(ab)^{\#} \in (ab)\{1,5\}$, then there exist $a^{(1,3,4)} \in a\{1,3,4\}$ and $b^{(1,3,4)} \in b\{1,3,4\}$ such that $(ab)^{\#} = b^{(1,3,4)}a^{(1,3,4)}$. Further, by Lemma 1.1, we have

$$b^{\dagger}b(ab)^{\#}aa^{\dagger} = b^{\dagger}(bb^{(1,3,4)})(a^{(1,3,4)}a)a^{\dagger} = b^{\dagger}bb^{\dagger}a^{\dagger}aa^{\dagger} = b^{\dagger}a^{\dagger}$$

which yields

$$ab(b^{\dagger}a^{\dagger})ab = abb^{\dagger}b(ab)^{\#}aa^{\dagger}ab = ab(ab)^{\#}ab = ab.$$

Therefore, by Theorem 1.1 (parts (a) and (c)), we have $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$ implying (2.4) and (2.5). From the equalities $abb^{(1,3,4)}a^{(1,3,4)} = b^{(1,3,4)}a^{(1,3,4)}ab$, (2.4) and (2.5), we notice that $abb^\dagger a^\dagger = b^\dagger a^\dagger ab$. By Theorem 2.5, $b^\dagger a^\dagger \in (ab)\{1,5\}$ gives $b\{1,3,4\}\cdot a\{1,3,4\} \subseteq (ab)\{1,5\}$. So, $(ab)\{1,5\} = b\{1,3,4\}\cdot a\{1,3,4\}$.

$\square$

Remark. Theorem 2.5 and Theorem 2.6 hold in C^*-algebras and *-reducing rings without the hypothesis $(1-a^\dagger a)b$ is left *-cancellable, since this condition is automatically satisfied.

In the following theorems, necessary and sufficient conditions for the reverse order law $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$ are obtained. Thus, some equivalent conditions which ensure that the product ab is EP are considered.

Theorem 2.7. *Suppose that $a, b \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^\#$. Then $ab \in \mathcal{R}^\dagger$ and $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$ if and only if one of the following equivalent conditions holds:*

(i) $a^\dagger ab \in \mathcal{R}^\dagger$ *and* $b(ab)^\# = bb^\dagger a^\dagger = (abb^\dagger)^\dagger$,

(ii) $abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\# a = b^\dagger a^\dagger a = (a^\dagger ab)^\dagger$,

(iii) $ab, a^\dagger ab \in \mathcal{R}^\dagger$, $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$ *and* $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$,

(iv) $ab, abb^\dagger \in \mathcal{R}^\dagger$, $(ab)^\# = b^\dagger (abb^\dagger)^\dagger$ *and* $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$,

(v) $ab, a^*ab \in \mathcal{R}^\dagger$, $(ab)^\# = (a^*ab)^\dagger a^*$ *and* $(a^*ab)^\dagger = b^\dagger (a^*a)^\#$,

(vi) $ab, abb^* \in \mathcal{R}^\dagger$, $(ab)^\# = b^*(abb^*)^\dagger$ *and* $(abb^*)^\dagger = (bb^*)^\# a^\dagger$.

Proof. If $ab \in \mathcal{R}^\dagger$ and $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$, then the conditions (i)-(vi) hold, by [10, Theorem 2.1].

Conversely, we will show that each of the conditions (i)-(vi) is sufficient for $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$, or for one of the preceding already established conditions of this theorem.

(i) Suppose that $b(ab)^\# = bb^\dagger a^\dagger = (abb^\dagger)^\dagger$. Then, we observe that $b^\dagger a^\dagger \in (ab)\{1,2\}$, from

$$a(bb^\dagger a^\dagger)ab = ab(ab)^\# ab = ab,$$

and

$$\begin{aligned} b^\dagger a^\dagger abb^\dagger a^\dagger &= b^\dagger (bb^\dagger a^\dagger) a (bb^\dagger a^\dagger) = b^\dagger b(ab)^\# ab(ab)^\# \\ &= b^\dagger (b(ab)^\#) = b^\dagger bb^\dagger a^\dagger = b^\dagger a^\dagger. \end{aligned}$$

The hypothesis $bb^\dagger a^\dagger = (abb^\dagger)^\dagger$ implies that $abb^\dagger a^\dagger = abb^\dagger bb^\dagger a^\dagger$ is selfadjoint, that is $b^\dagger a^\dagger \in (ab)\{3\}$. Note that

$$(ab)^\# b^\dagger b = [(ab)^\#]^2 abb^\dagger b = [(ab)^\#]^2 ab = (ab)^\#.$$

Now, we get $b^\dagger a^\dagger \in (ab)\{5\}$, by

$$\begin{aligned} b^\dagger a^\dagger ab &= b^\dagger (bb^\dagger a^\dagger)ab = b^\dagger b(ab)^\# ab = b^\dagger ba(b(ab)^\#) = b^\dagger b(abb^\dagger a^\dagger)^* \\ &= (a(bb^\dagger a^\dagger)b^\dagger b)^* = (ab(ab)^\# b^\dagger b)^* = (ab(ab)^\#)^* \\ &= (abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger. \end{aligned}$$

Hence, the reverse order law $(ab)^\# = b^\dagger a^\dagger$ holds. Further, we observe that

$$b^\dagger a^\dagger ab = (ab)^\# ab = ab(ab)^\# = abb^\dagger a^\dagger$$

is selfadjoint and $(ab)^\dagger = b^\dagger a^\dagger$.

(ii) It follows in the same manner as (i).

(iii) Let $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$ and $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$. By Lema 1.2, we obtain $(ab)^\# a = (a^\dagger ab)^\dagger a^\dagger a = (a^\dagger ab)^\dagger$. Thus, the condition (ii) is satisfied.

(iv) Analogy as (iii) $\Rightarrow$ (ii), we get (iv) $\Rightarrow$ (i).

(v) Since $a \in \mathcal{R}^\dagger$, then $a^* a \in \mathcal{R}^\#$ and $a^\dagger = (a^* a)^\# a^*$ (see [9]). The equalities $(ab)^\# = (a^* ab)^\dagger a^*$ and $(a^* ab)^\dagger = b^\dagger (a^* a)^\#$ give

$$(ab)^\# = (a^* ab)^\dagger a^* = b^\dagger (a^* a)^\# a^* = b^\dagger a^\dagger.$$

From $(a^* ab)^\dagger = b^\dagger (a^* a)^\#$, we deduce that $b^\dagger a^\dagger ab = b^\dagger (a^* a)^\# a^* ab$ is selfadjoint. Therefore, $abb^\dagger a^\dagger = ab(ab)^\# = b^\dagger a^\dagger ab$ is selfadjoint also and $(ab)^\dagger = b^\dagger a^\dagger$.

(vi) Similarly as the condition (v). $\square$

Theorem 2.8. *If $a, b \in \mathcal{R}^\dagger$, then the following conditions are equivalent:*

(i) $ab \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ *and* $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$,

(ii) $(a^\dagger)^* b \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ *and* $[(a^\dagger)^* b]^\# = b^\dagger a^* = [(a^\dagger)^* b]^\dagger$,

(iii) $a(b^\dagger)^* \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ *and* $[a(b^\dagger)^*]^\# = b^* a^\dagger = [a(b^\dagger)^*]^\dagger$.

Proof. (i) $\Rightarrow$ (ii): By [10, Theorem 2.1], the conditions $ab \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $(ab)^\# = b^\dagger a^\dagger = (ab)^\dagger$ imply $(a^\dagger)^* b \in \mathcal{R}^\dagger$ and $[(a^\dagger)^* b]^\dagger = b^\dagger a^*$. Then, from

$$(a^\dagger)^* bb^\dagger a^* = (abb^\dagger a^\dagger)^* = (b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab = b^\dagger a^* (a^\dagger)^* b,$$

we deduce that $b^\dagger a^* \in (ab)\{5\}$. Thus, $(a^\dagger)^* b \in \mathcal{R}^\#$ and $[(a^\dagger)^* b]^\# = b^\dagger a^*$.

((ii) $\Rightarrow$ (i)) $\wedge$ ((i) $\Leftrightarrow$ (iii)): Analogously as (i) $\Rightarrow$ (ii).

$\square$

An element $a \in \mathcal{R}^\dagger$ is *bi-dagger* if $a^2 \in \mathcal{R}^\dagger$ and $(a^2)^\dagger = (a^\dagger)^2$.

Applying Theorem 2.7 and Theorem 2.8, if $a = b$, then equivalent conditions can be obtained such that a is *bi-dagger* and a^2 is *EP* (that is $(a^2)^\dagger = (a^\dagger)^2 = (a^2)^\# = (a^\#)^2$).

In a unital C^*–algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. Recall that $a \in \mathcal{A}$ is Moore–Penrose invertible if and only if a is regular. Thus, anything with a group inverse automatically has a Moore–Penrose inverse. So, the results presented in this section hold in a C^*–algebra with conditions a, b are regular and $ab \in \mathcal{A}^\#$ instead of $a, b \in \mathcal{A}^\dagger$ and $ab \in \mathcal{A}^\dagger \cap \mathcal{A}^\#$. Also, notice that if a is regular and $ab \in \mathcal{A}^\#$, then $a^\dagger ab$ is regular too. In the ring of square complex matrices, since every complex matrix has a Moore–Penrose inverse, we only need to assume that the product ab is a group invertible and we obtain that results of this section are valid. If a is an $m \times n$ complex matrix and b is an $n \times m$ complex matrix, all equations are applicable to these rectangular matrices.

3 Conclusions

In this paper we present necessary and sufficient conditions related to the reverse order law $(ab)^\# = b^\dagger a^\dagger$ to hold in rings with involution, applying purely algebraic techniques. When we suppose that a is group invertible and b is Moore-Penrose invertible (or a is Moore-Penrose invertible and b is group invertible), we get similar results for the reverse order laws $(ab)^\# = b^\dagger a^\#$ (or $(ab)^\# = b^\# a^\dagger$) [12]. In the case of bounded linear operators on Hilbert spaces, where the method of operator matrices is very useful, similar results for the reverse order law $(ab)^\# = b^\# a^\#$ are given. It could be interesting to extend this work to the reverse order law of a triple product.

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