BASIC REVERSE ORDER LAW AND ITS EQUIVALENCIES

Nebojša Č. Dinčić[1] **and Dragan S. Djordjević** [1]

Abstract

In this paper we present new results related to the various equivalencies of the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ for the Moore-Penrose inverse for operators on Hilbert spaces. Some finite dimensional results given by Tian [13] are extended to infinite dimensional settings; also some new more general relations are proved.

2000 *Mathematics Subject Classification*: 47A05, 15A09.

Keywords and phrases: Moore-Penrose inverse, reverse order law.

1 Introduction

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X,Y)$ be the set of all linear bounded operators from X to Y. For $A \in \mathcal{L}(X,Y)$ we use, respectively, $\mathcal{N}(A)$, $\mathcal{R}(A)$, A^* for the null space, the range space and the adjoint of A.

The Moore-Penrose inverse of $A \in \mathcal{L}(X,Y)$ (if it exists) is the unique operator $A^\dagger \in \mathcal{L}(Y,X)$ satisfying the following four Penrose equations:

$$(I)\ AA^\dagger A = A,\ (II)\ A^\dagger AA^\dagger = A^\dagger,\ (III)\ (AA^\dagger)^* = AA^\dagger,\ (IV)\ (A^\dagger A)^* = A^\dagger A.$$

It is well-known that $A^\dagger$ exists for given A if and only if $\mathcal{R}(A)$ is closed.

We assume that the reader is familiar with the generalized invertibility and the Moore-Penrose inverse (see, for example, [1], [4], [8]).

The reverse order law of the form $(AB)^\dagger = B^\dagger A^\dagger$ does not hold in general for the Moore-Penrose inverse. The classical equivalent condition (A^*A commutes with $BB^\dagger$, and BB^* commutes with $AA^\dagger$) is proved in [9] for complex matrices, in [2], [3] and [11] for closed-range linear bounded operators on Hilbert spaces, and in [12] for rings with involutions. A lot of papers concerning various forms and conditions for the reverse order law are also investigated, for example [6], [10], [14].

In this paper we present a set of equivalencies of the reverse-order law $(AB)^\dagger = B^\dagger A^\dagger$ for the Moore-Penrose inverse of bounded linear operators on Hilbert spaces. Some finite dimensional results, given by Tian [13], are

[1]The authors are supported by the Ministry of Science, Republic of Serbia, grant No. 174007.

extended to infinite dimensional settings. Also, some further generalizations are obtained. As a corollaries, we get some results for generalized inverses for the positive integer powers of the operators. Connection with mixed-type reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ is presented.

We continue with several auxiliary results.

Lemma 1.1. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof is straightforward.

Lemma 1.2. *[7] Let $A \in \mathcal{L}(X,Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

The reader should notice the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A, B] = AB - BA$ denotes the commutator of A and B. On the other hand, if $U \in \mathcal{L}(X, Z)$ and $V \in \mathcal{L}(Y, Z)$, then $[U \quad V] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Z$ denote the matrix form of the corresponding operator.

The following result is Proposition 2.1. from [11] (also can be found in [4], pp 127), and it will be useful tool for proving the existence of Moore-Penrose inverses of some terms.

Lemma 1.3. *Let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ have closed ranges. Then AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.*

The following result is proved in [7], Lemma 2.1.

Lemma 1.4. *Let X, Y be Hilbert spaces, let $C \in \mathcal{L}(X, Y)$ have a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, CC^\dagger] = 0$.*

In the next proposition, a lot of well-known and important facts and properties concerning Moore-Penrose inverse are collected, especially those we are using in the proofs.

Proposition 1.1. *Let $A \in \mathcal{L}(X, Y)$ be closed-range operator and let $M \in \mathcal{L}(Y)$ and $N \in \mathcal{L}(X)$ be positive definite invertible operators. Then:*

(1) $A^* = A^\dagger AA^* = A^*AA^\dagger,\ A = AA^*(A^*)^\dagger = (A^*)^\dagger A^*A$;

(2) $A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^*,\ (AA^*)^\dagger = (A^*)^\dagger A^\dagger,\ (A^*A)^\dagger = A^\dagger(A^*)^\dagger$;

(3) $\mathcal{R}(A) = \mathcal{R}(AA^\dagger) = \mathcal{R}(AA^*)$;

(4) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$;

(5) $\mathcal{R}(I - A^\dagger A) = \mathcal{N}(A^\dagger A) = \mathcal{N}(A) = \mathcal{R}(A^*)^\perp$;

(6) $\mathcal{R}(I - AA^\dagger) = \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$.

Lemma 1.5. *Let H be Hermitian bounded linear operator. Then:*

$$(\forall n \in \mathbb{N})\ \ (H^n)^\dagger = (H^\dagger)^n.$$

Proof. For $n = 1$ we actually have well-known identity for Moore-Penrose inverse. For other values of n, it is easy to check all four Penrose equation, using the following fact:

$$H = H^\dagger H^2 = H^2 H^\dagger,$$

which follows from Proposition 1.1 for any Hermitian operator H. $\square$

Remark 1.1. *According to the Lemma 1.5, if the operator T has the form:*

$$T = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

where "$*$" denotes arbitrary component, then:

$$((T^*T)^\dagger)^n = (T^\dagger(T^\dagger)^*)^n = T^\dagger((T^\dagger)^*T^\dagger)^{n-1}(T^*)^\dagger = T^\dagger((TT^*)^\dagger)^{n-1}(T^\dagger)^*,$$

where TT^* has the following form ("inv." means some invertible operator):

$$TT^* = \begin{pmatrix} inv. & 0 \\ 0 & 0 \end{pmatrix},$$

which provides us with simplified computations.
If the operator S has the form:

$$S = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix},$$

then:

$$((SS^*)^\dagger)^n = ((S^\dagger)^*S^\dagger)^n = (S^\dagger)^*(S^\dagger(S^\dagger)^*)^{n-1}S^\dagger = (S^*)^\dagger((S^*S)^\dagger)^{n-1}S^\dagger,$$

where S^*S has the following form:

$$S^*S = \begin{pmatrix} inv. & 0 \\ 0 & 0 \end{pmatrix},$$

which provides us with simplified computations.

Those facts will be used in the proof of our main result.

Proposition 1.2. *Let X and Y be arbitrary Hilbert spaces and let $A \in \mathcal{L}(X, Y)$. For any $m \in \mathbb{N}$,*

(a) $((AA^*)^\dagger)^m(AA^*)^m = ((A^*)^\dagger A^\dagger)^m(AA^*)^m = AA^\dagger$;

(b) $(AA^*)^m((AA^*)^\dagger)^m = (AA^*)^m((A^*)^\dagger A^\dagger)^m = AA^\dagger$;

(c) $((A^*A)^\dagger)^m(A^*A)^m = (A^\dagger(A^*)^\dagger)^m(A^*A)^m = A^\dagger A$;

(d) $(A^*A)^m((A^*A)^\dagger)^m = (A^*A)^m(A^\dagger(A^*)^\dagger)^m = A^\dagger A$.

Proof.

$$\begin{aligned}
(a)\ ((AA^*)^\dagger)^m(AA^*)^m &= ((AA^*)^\dagger)^{m-1}(AA^*)^\dagger AA^*(AA^*)^{m-1} = \\
&= ((AA^*)^\dagger)^{m-1}(A^*)^\dagger A^*(AA^*)^{m-1} = \\
&= ((AA^*)^\dagger)^{m-1}(AA^*)^{m-1} = \ldots = \\
&= (AA^*)^\dagger AA^* = (A^*)^\dagger A^* = (AA^\dagger)^* = AA^\dagger;
\end{aligned}$$

by using Proposition 1.1(2), we have desired result. On the completely analogous way other three statements can be proved. Let us prove, for example, statement (c).

$$\begin{aligned}
(c)\ ((A^*A)^\dagger)^m(A^*A)^m &= ((A^*A)^\dagger)^{m-1}(A^*A)^\dagger A^*A(A^*A)^{m-1} = \\
&= ((A^*A)^\dagger)^{m-1}A^\dagger A(A^*A)^{m-1} = \\
&= ((A^*A)^\dagger)^{m-1}(A^*A)^{m-1} = \ldots = \\
&= (A^*A)^\dagger A^*A = A^\dagger A.
\end{aligned}$$

□

Lemma 1.6. *Let X and Y be arbitrary Hilbert spaces and let $A \in \mathcal{L}(X,Y)$. For any $m \in \mathbb{N}$,*

$$((AA^*)^mA)^\dagger = A^\dagger((A^*)^\dagger A^\dagger)^m, \quad (A(A^*A)^m)^\dagger = (A^\dagger(A^*)^\dagger)^mA^\dagger.$$

Proof. We prove the first identity by checking all four Penrose equations, using the Proposition 1.2 for simpler computation. The second one can be proved analogously.

(I) $(AA^*)^mAA^\dagger((A^*)^\dagger A^\dagger)^m(AA^*)^mA = (AA^*)^mAA^\dagger AA^\dagger A = (AA^*)^mA$;

$$\begin{aligned}
(II)\ A^\dagger((A^*)^\dagger A^\dagger)^m(AA^*)^mAA^\dagger((A^*)^\dagger A^\dagger)^m &= A^\dagger AA^\dagger AA^\dagger((A^*)^\dagger A^\dagger)^m = \\
&= A^\dagger((A^*)^\dagger A^\dagger)^m;
\end{aligned}$$

$$
\begin{aligned}
(III)\ (AA^*)^m AA^\dagger((A^*)^\dagger A^\dagger)^m &= (AA^*)^{m-1}((A^*)^\dagger A^\dagger)^{m-1}(A^*)^\dagger A^\dagger = \\
&= AA^\dagger(A^*)^\dagger A^\dagger = AA^\dagger,
\end{aligned}
$$

which is Hermitian;

$$(IV)\ A^\dagger((A^*)^\dagger A^\dagger)^m(AA^*)^m A = A^\dagger AA^\dagger A = A^\dagger A,$$

which is Hermitian. □

2 Main results

The following theorem is a generalization of the main result from [13] to the infinite dimensional settings.

Theorem 2.1. *Let X, Y and Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be bounded linear operators, such that A, B and AB have closed ranges. The following statements are equivalent:*

(a) $(AB)^\dagger = B^\dagger A^\dagger$;

(b) $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$;

(c) $((A^\dagger)^* B)^\dagger = B^\dagger A^*$;

(d) $(A(B^\dagger)^*)^\dagger = B^* A^\dagger$;

(e) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(f) $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

(g) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(h) $(AB)^\dagger = B^\dagger(ABB^\dagger)^\dagger$ *and* $(AAB^\dagger)^\dagger = BB^\dagger A^\dagger$;

(i) $(AB)^\dagger = (A^*AB)^\dagger A^*$ *and* $(A^*AB)^\dagger = B^\dagger(A^*A)^\dagger$;

(j) $(AB)^\dagger = B^*(ABB^*)^\dagger$ *and* $(ABB^*)^\dagger = (BB^*)^\dagger A^\dagger$;

(k) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$ *and* $(A^*ABB^*)^\dagger = (BB^*)^\dagger(A^*A)^\dagger$;

(l) $(AB)^\dagger = B^*B(AA^*ABB^*B)^\dagger AA^*$ *and*
$(AA^*ABB^*B)^\dagger = (BB^*B)^\dagger(AA^*A)^\dagger$;

(m) $(AB)^\dagger = B^*BB^*((A^*A)^2(BB^*)^2)^\dagger A^*AA^*$ *and*
$((A^*A)^2(BB^*)^2)^\dagger = ((BB^*)^2)^\dagger((A^*A)^2)^\dagger$;

(n) $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}$ *and* $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}$.

Proof. Let we say something about the existence of the Moore-Penrose inverse of various terms appearing in the formulas above. The existence of $(A^\dagger ABB^\dagger)^\dagger$ follows immediately from Lemma 1.3. It is easy to see the existence of $((A^\dagger)^*B)^\dagger$ and $(A(B^\dagger)^*)^\dagger$. We have

$$\mathcal{R}(B^*A^*A) = B^*(\mathcal{R}(A^*A)) = B^*(\mathcal{R}(A^*)) = \mathcal{R}((AB)^*)$$

is closed, which implies the existence of $(A^*AB)^\dagger$, $(A^\dagger AB)^\dagger$ and also of $(A^*ABB^*)^\dagger$, because of:

$$\mathcal{R}(A^*ABB^*) = A^*A(\mathcal{R}(BB^*)) = A^*A(\mathcal{R}(B)) = \mathcal{R}(A^*AB) = \mathcal{R}((B^*A^*A)^*).$$

On completely analogous way one can prove the existence of $(ABB^\dagger)^\dagger$ and $(ABB^*)^\dagger$.

First, we enlist some parts of the proof regardless of the decomposition we will use later.
$(a) \Leftrightarrow (n)$: This is already proven as Corollary 6.2.4 in [8].
$(a) \Leftrightarrow (e)$: Already proven as Theorem 2.4.c) in [7].
$(f) - (m) \Rightarrow (a)$: Those implications are proven on the same way: the second part of the statement is replaced onto the first one, and common identities (see Proposition 1.1 and Lemma 1.5) are applied if necessary. As a result, we yield statement (a). For illustration, we will present two specific cases:
$(j) \Rightarrow (a)$: $(AB)^\dagger = B^*(ABB^*)^\dagger = B^*(BB^*)^\dagger A^\dagger = B^\dagger A^\dagger$.
$(m) \Rightarrow (a)$: Here we will use Lemma 1.5 for $n = 2$.

$$\begin{aligned} (AB)^\dagger &= B^*BB^*((A^*A)^2(BB^*)^2)^\dagger = B^*BB^*((BB^*)^2)^\dagger((A^*A)^2)^\dagger A^*AA^* = \\ &= B^*BB^*(BB^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(A^*A)^\dagger A^*AA^* = \\ &= B^*BB^\dagger(BB^*)^\dagger(A^*A)^\dagger A^\dagger AA^* = B^*(BB^*)^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger. \end{aligned}$$

$(a) \Rightarrow (b)$: $B^\dagger A^\dagger = (AB)^\dagger = (AB)^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$.
$(a) \Rightarrow (g)$: $(AB)^\dagger = B^\dagger A^\dagger = B^\dagger A^\dagger AA^\dagger = (A^\dagger AB)^\dagger A^\dagger$, according to the already proven statement (e).
$(a) \Rightarrow (h)$: $(AB)^\dagger = B^\dagger A^\dagger = B^\dagger BB^\dagger A^\dagger = B^\dagger(A^\dagger BB^\dagger)^\dagger$, according to the

already proven statement (e).
For the rest of the proof, we will use the following operator decompositions. Using Lemma 1.1, we conclude that the operator B has the following matrix form:

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From Lemma 1.2 also follows that the operator A has the following matrix form:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

$(a) \Leftrightarrow (c) \Leftrightarrow (d)$: Easy computations shows that statements (a), (c) and (d) are equivalent to: $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$, $(D^{-1}A_1B_1)^\dagger = B_1^{-1}A_1^*$ and $(A_1(B_1^*)^{-1})^\dagger = B_1^*A_1^*D^{-1}$, respectively. Each of them is further equivalent to the following:

$$A_1A_1^*D^{-1}A_1 = A_1, \ \ [A_1A_1^*, D^{-1}] = 0, \ \ [B_1B_1^*, A_1^*D^{-1}A_1] = 0. \tag{1}$$

Proving the statements: $(a) \Rightarrow (f)$ and $(a) \Rightarrow (i)-(j)$ are very similar, so we will show it only on the case $(a) \Rightarrow (i)$. Using the decomposition described above, it is easy to conclude that (i) becomes:

$$\begin{aligned} (A_1B_1)^\dagger &= (D^{1/2}A_1B_1)^\dagger D^{1/2}, \\ (D^{1/2}A_1B_1)^\dagger D^{-1/2}A_i &= B_1^{-1}A_1^*D^{-2}A_i, \ \ i = 1, 2. \end{aligned}$$

Now we will show that (a) implies the first statement, by checking all four Penrose equations. For the first and the second equation, it is clear. Let we check the third and forth.

$$\begin{aligned} (III)\ D^{1/2}A_1B_1(A_1B_1)^\dagger D^{-1/2} &= D^{1/2}A_1B_1B_1^{-1}A_1^*D^{-1}D^{-1/2} = \\ &= D^{1/2}A_1A_1^*D^{-1}D^{-1/2}, \end{aligned}$$

which is, under the premise (a), Hermitian.

$$\begin{aligned}(IV)\ (A_1B_1)^\dagger D^{-1/2}D^{1/2}A_1B_1 &= B_1^{-1}A_1^*D^{-1}D^{-1/2}D^{1/2}A_1B_1 = \\ &= B_1^{-1}A_1^*D^{-1}A_1B_1,\end{aligned}$$

For the sake of completeness, we enlist the equivalent forms for (f) and (j) :

$$(f):\ (A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2},$$
$$(D^{-1/2}A_1)^\dagger D^{-1/2}A_i = A_1^*D^{-1}A_i,\ \ i=1,2;$$

and

$$(j):\ (A_1B_1)^\dagger = B_1^*(A_1B_1B_1^*)^\dagger,$$
$$(D^{1/2}A_1B_1)^\dagger D^{-1/2}A_i = (B_1B_1^*)^{-1}A_1^*D^{-1}.$$

The proof that $(a) \Rightarrow (k) - (m)$ will be omitted here, because it will be found later, in Theorem 2.2, for more general case.
Now, it remains only part:
$(b) \Rightarrow (a)$: If we use usual matrix forms for the operators A and B, it actually remains to be proven that:

$$(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}A_1A_1^*D^{-1} \Rightarrow (A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}.$$

Let us denote $W = A_1^*D^{-1}A_1$. For the expression $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}A_1A_1^*D^{-1}$, proper Penrose equations are the following:

1. $A_1 = A_1W^2$;
2. $W^3A_1^* = WA_1^*$;
3. $[A_1WA_1^*, D^{-1}] = 0$;
4. $[B_1B_1^*, W^2] = 0$.

On the other side, Penrose equations for $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$ are the following:

1. $A_1 = A_1W$;

2. $WA_1^* = A_1^*$;

3. $[A_1A_1^*, D^{-1}] = 0$;

4. $[B_1B_1^*, W] = 0$.

The operator W is Hermitian, moreover - it is positive ($W = T^*T$, where $T = D^{-1/2}A_1$). This is the reason $I + W$ is invertible, so we have:

$$\begin{aligned} A_1 = A_1W^2 &\Leftrightarrow A_1(I - W^2) = 0 \Leftrightarrow A_1(I - W)(I + W) = 0 \\ &\Rightarrow A_1(I - W) = 0, \end{aligned}$$

which means

$$A_1 = A_1W.$$

By using this fact, we have the proof immediately. □

The next theorem presents one possible way for generalization of some statements from the previous theorem.

Theorem 2.2. *Let X, Y and Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be bounded linear operators, such that A, B and AB have closed ranges. Let m and n be arbitrary nonnegative integers. The following statements are equivalent:*

(a) $(AB)^\dagger = B^\dagger A^\dagger$;

(l') $(AB)^\dagger = (B^*B)^n((AA^*)^m AB(B^*B)^n)^\dagger (AA^*)^m$ *and* $((AA^*)^m AB(B^*B)^n)^\dagger = (B(B^*B)^n)^\dagger((AA^*)^m A^*)^\dagger$;

(m') $(AB)^\dagger = B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger (A^*A)^m A^*$ *and* $((A^*A)^{m+1}(BB^*)^{n+1})^\dagger = ((BB^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}$.

Proof. First, we show the existence of the operators $((A^*A)^{m+1}(BB^*)^{n+1})^\dagger$ and $((AA^*)^m AB(B^*B)^n)^\dagger$. By Lemma 1.3, if P and Q are closed-range operators, then PQ is closed-range if and only if $P^\dagger PQQ^\dagger$ is closed range. Let we put

$$P = (A^*A)^m, \;\; Q = (BB^*)^n.$$

They are closed-range as a powers of Hermitian closed-range operators A^*A and BB^*. (The Hermitian operator $H \in \mathcal{L}(X)$ is closed-range if and only if $0 \notin acc(\sigma(H))$. According to the spectral mapping theorem, if Hermitian operator H is closed-range, then H^n is also closed-range for arbitrary positive integer n.) Let us compute $P^\dagger PQQ^\dagger$:

$$\begin{aligned} P^\dagger PQQ^\dagger &= ((A^*A)^m)^\dagger (A^*A)^m (BB^*)^n ((BB^*)^n)^\dagger = \\ &= ((A^*A)^\dagger)^m (A^*A)^m (BB^*)^n ((BB^*)^\dagger)^n = A^\dagger ABB^\dagger, \end{aligned}$$

which is closed-range operator because of Lemma 1.3. Thus, we proved that $(A^*A)^{m+1}(BB^*)^{n+1}$ has closed range, which implies the existence of its Moore-Penrose inverse.

Let us now put

$$P = (AA^*)^m A, \;\; Q = B(B^*B)^n.$$

By computing $P^\dagger PQQ^\dagger$:

$$\begin{aligned} P^\dagger PQQ^\dagger &= ((AA^*)^m A)^\dagger (AA^*)^m AB(B^*B)^n (B(B^*B)^n)^\dagger = \\ &= A^\dagger((A^*)^\dagger A^\dagger)^m (AA^*)^m AB(B^*B)^n (B^\dagger(B^*)^\dagger)^n B^\dagger = A^\dagger ABB^\dagger, \end{aligned}$$

we conclude using Lemma 1.3 that it is closed range operator, which implies $(AA^*)^m AB(B^*B)^n$ has closed range, and because of that the Moore-Penrose inverse.

Now, the proof starts.

$$\begin{aligned} (l') \Rightarrow (a): \;\; (AB)^\dagger &= (B^*B)^n((AA^*)^m AB(B^*B)^n)^\dagger (AA^*)^m = \\ &= (B^*B)^n (B(B^*B)^n)^\dagger ((AA^*)^m A)^\dagger (AA^*)^m = \\ &= (B^*B)^n ((B^*B)^\dagger)^n B^\dagger A^\dagger ((AA^*)^\dagger)^m (AA^*)^m = B^\dagger A^\dagger, \end{aligned}$$

where we used the following fact: if H is hermitian, then $H^2H^\dagger = H = H^\dagger H^2$.

$(a) \Rightarrow (l')$: Using the decompositions:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix},$$

the implication becomes:

$$(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1} \Rightarrow \begin{cases} (A_1B_1)^\dagger = (B_1^*B_1)^n (D^m A_1B_1(B_1^*B_1)^n)^\dagger D^m, \\ (D^m A_1 B_1 (B_1^*B_1)^n)^\dagger = (B_1^*B_1)^{-n} B_1^{-1} A_1^* D^{-(m+1)}. \end{cases}$$

We can easily prove that $(D^m A_1B_1(B_1^*B_1)^n)^\dagger = (B_1^*B_1)^{-n}(A_1B_1)^\dagger D^{-m}$, by immediately checking all four Penrose equations under the premise $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$.

The second part is now clear:

$$(D^m A_1B_1(B_1^*B_1)^n)^\dagger = (B_1B_1^*)^{-n}(A_1B_1)^\dagger D^{-m} = (B_1^*B_1)^{-l}B_1^{-1}A_1^*D^{-(m+1)},$$

so we completed this part of the proof.

$(m') \Rightarrow (a)$:

$$\begin{aligned}
(AB)^\dagger &= B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger(A^*A)^m A^* = \\
&= (B^*B)^n B^*((BB^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}A^*(AA^*)^m = \\
&= (B^*B)^n B^*(BB^*)^\dagger((BB^*)^\dagger)^n((A^*A)^\dagger)^m(A^*A)^\dagger A^*(AA^*)^m = \\
&= (B^*B)^{n-1}B^*BB^\dagger((BB^*)^\dagger)^n((A^*A)^\dagger)^m A^\dagger AA^*(AA^*)^{m-1} = \\
&= (B^*B)^{n-1}B^*((BB^*)^\dagger)^n((A^*A)^\dagger)^m A^*(AA^*)^{m-1} = \\
&= \dots \\
&= B^*(BB^*)^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger.
\end{aligned}$$

$(a) \Rightarrow (m')$: Here we also use the following decompositions:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix},$$

but in the calculation there are some steps which should be explained. Let us denote $T = (A^*A)^{m+1}(BB^*)^{n+1}$. It is easier to compute on the following way:

$$T = A^*(AA^*)^m AB(B^*B)^n B^* = \begin{pmatrix} A_1^*D^m A_1(B_1B_1^*)^{n+1} & 0 \\ A_2^*D^m A_1(B_1B_1^*)^{n+1} & 0 \end{pmatrix};$$

now,

$$T^\dagger = \begin{pmatrix} (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_1 & (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_2 \\ 0 & 0 \end{pmatrix}.$$

Remains to find $((A^*A)^\dagger)^{m+1}$. It can be computed on this way:

$$(A^*A)^\dagger = A^\dagger(A^\dagger)^* = \begin{pmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2 \\ A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2 \end{pmatrix}.$$

It is easy to prove by induction that for arbitrary nonnegative integer k:

$$((A^*A)^\dagger)^k = \begin{pmatrix} A_1^*D^{-(k+1)}A_1 & A_1^*D^{-(k+1)}A_2 \\ A_2^*D^{-(k+1)}A_1 & A_2^*D^{-(k+1)}A_2 \end{pmatrix}.$$

Also, it is clear:

$$(A^*A)^{k+1} = A^*(AA^*)^k A = \begin{pmatrix} A_1^*D^k A_1 & A_1^*D^k A_2 \\ A_2^*D^k A_1 & A_2^*D^k A_2 \end{pmatrix}.$$

Now, we have all necessary terms for computing (m') in the terms of $A_1,\ A_2$ and B_1. Thus, we should prove that $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$ implies:

$$\begin{cases} (A_1B_1)^\dagger = B_1^*(B_1B_1^*)^n(D^{m+1/2}A_1B_1B_1^*(B_1B_1^*)^n)^\dagger D^{m+1/2}, \\ (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_i = (B_1^*B_1)^{-(n+1)}A_1^*D^{-(m+2)}A_i,\ i = 1, 2. \end{cases}$$

We can prove the first part is true by checking all four Penrose equations for

$$(D^{m+1/2}A_1B_1B_1^*(B_1B_1^*)^n)^\dagger = (B_1B_1^*)^{-n}(B_1^*)^{-1}(A_1B_1)^\dagger D^{-(m+1/2)},$$

under the premise $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$.

Now, the second part:

$$\begin{aligned} & (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_i \\ = \ & (B_1B_1^*)^{-n}(B_1^*)^{-1}(A_1B_1)^\dagger D^{-(m+1/2)}D^{-1/2}A_i = \\ = \ & (B_1B_1^*)^{-n}(B_1^*)^{-1}B_1^{-1}A_1^*D^{-1}D^{-(m+1/2)}D^{-1/2}A_i = \\ = \ & (B_1B_1^*)^{-(n+1)}A_1^*D^{-(m+2)}A_i. \end{aligned}$$

□

Remark 2.1. *If we put $m = 0,\ n = 0$ in statement (l'), it becomes (k) from the Theorem 2.1, if $m = 1,\ n = 1$ it becomes (m). Also if we put $m = 1,\ n = 1$ in (m'), it becomes (l). Suppose other and further generalizations are possible.*

The next result is immediate corollary of the Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let X and Y be Hilbert spaces, and let $A \in \mathcal{L}(X, Y)$ be bounded linear closed-range operator. The following statements are equivalent:*

(a) $(A^2)^\dagger = (A^\dagger)^2$, *namely, A is a bi-dagger;*

(b) $((A^\dagger)^*A)^\dagger = A^\dagger A^*$;

(c) $(A(A^\dagger)^*)^\dagger = A^*A^\dagger$;

(d) $(A^2A^\dagger)^\dagger = A(A^\dagger)^2$ *and* $(A^\dagger A^2)^\dagger = (A^\dagger)^2A$;

(e) $(A^2)^\dagger = A^\dagger(A^\dagger A^2A^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger A^2A^\dagger)^\dagger = A(A^\dagger)^2A$;

(f) $(A^2)^\dagger = (A^\dagger A^2)^\dagger A^\dagger$ *and* $(A^\dagger A^2)^\dagger = (A^\dagger)^2A$;

(g) $(A^2)^\dagger = A^\dagger(A^2A^\dagger)^\dagger$ *and* $(A^2A^\dagger)^\dagger = A(A^\dagger)^2$;

(h) $(A^2)^\dagger = (A^*A^2)^\dagger A^*$ *and* $(A^*A^2)^\dagger = A^\dagger(A^*A)^\dagger$;

(i) $(A^2)^\dagger = A^*(A^2A^*)^\dagger$ *and* $(A^2A^*)^\dagger = (AA^*)^\dagger A^\dagger$;

(j) $(A^2)^\dagger = A^*(A^*A^2A^*)^\dagger A^*$ *and* $(A^*A^2A^*)^\dagger = (AA^*)^\dagger(A^*A)^\dagger$;

(k) $(A^2)^\dagger = A^*A(AA^*A^2A^*A)^\dagger AA^*$ *and* $(AA^*A^2A^*A)^\dagger = (AA^*A)^\dagger(AA^*A)^\dagger$;

(l) $(A^2)^\dagger = A^*AA^*((A^*A)^2(AA^*)^2)^\dagger A^*AA^*$ *and* $((A^*A)^2(AA^*)^2)^\dagger = ((AA^*)^2)^\dagger((A^*A)^2)^\dagger$;

(m) $(A^2)^\dagger = (A^*A)^n((AA^*)^mA^2(A^*A)^n)^\dagger(AA^*)^m$ *and* $((AA^*)^mA^2(A^*A)^n)^\dagger = (A(A^*A)^n)^\dagger((AA^*)^mA^*)^\dagger$;

(n) $(A^2)^\dagger = A^*(AA^*)^n((A^*A)^{m+1}(AA^*)^{n+1})^\dagger(A^*A)^mA^*$ *and* $((A^*A)^{m+1}(AA^*)^{n+1})^\dagger = ((AA^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}$.

For the sake of completeness, we shall repeat some results already proven in [7] as the $(c)-$parts of the Theorems 2.2, 2.3 and 2.4.

Theorem 2.3. *Let X, Y and Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$ be bounded linear operators, such that A, B and AB have closed ranges. The following statements are equivalent:*

(a) $(AB)^\dagger = B^\dagger A^\dagger$;

(b1) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;

(b2) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;

(b3) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$;

(c1) $AB(AB)^\dagger A = ABB^\dagger$ *and* $A^\dagger AB = B(AB)^\dagger AB$;

(c2) $[A^*A, BB^\dagger] = 0$ *and* $[A^\dagger A, BB^*] = 0$;

(d1) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(d2) $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger$.

The following theorem establishes the connection between the basic reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ and mixed-type reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$. This mixed-type reverse order law is deeply considered in the paper [5].

Theorem 2.4. *Let X, Y and Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be bounded linear operators, such that A, B and AB have closed ranges. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$, and AB satisfies any one of the following conditions:*

(a) $ABB^\dagger A^\dagger AB = AB$;

(b) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;

(c) $[A^\dagger A, BB^\dagger] = 0$;

(d) $A^\dagger ABB^\dagger$ *is an idempotent;*

(e) $BB^\dagger A^\dagger A$ *is an idempotent;*

(f) $B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$;

(g) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$.

Proof. Statements $(a) - (g)$ are mutually equivalent, as it is proved in the Theorem 2.1. from [7]. From this result and from the statement (f), Theorem 2.1, the conclusion is easy to obtain. □

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd Edition, Springer, New York, 2003.

[2] R. H. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math.25 (1973), 489–495.

[3] R. H. Bouldin, *Generalized inverses and factorizations*, Recent applications of generalized inverses, Pitman Ser. Res. Notes in Math. No. 66 (1982), 233–248.

[4] S. R. Caradus, *Generalized inverses and operator theory*, Queen's paper in pure and applied mathematics, Queen's University, Kingston, Ontario, 1978.

[5] N. Č. Dinčić, D. S. Djordjević and D. Mosić, *Mixed-type reverse order law and its equivalencies*, Studia Math. 204 (2011), 123–136.

[6] D. S. Djordjević, *Unified approach to the reverse order rule for generalized inverses*, Acta Sci. Math. (Szeged) l67 (2001), 761–776.

[7] D. S. Djordjević and N. Č. Dinčić, *Reverse order law for the Moore-Penrose inverse*, J. Math. Anal. Appl. 361 (1) (2010), 252–261.

[8] D. S. Djordjević and V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

[9] T. N. E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8 (1966), 518–521.

[10] R. E. Hartwig, *The reverse order law revisited*, Linear Algebra Appl. 76 (1986), 241–246.

[11] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34 (1982), 43–52.

[12] J. J. Koliha, D. S. Djordjević and D. Cvetković Ilić, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl. 426 (2007), 371–381.

[13] Y. Tian, *The equivalence between* $(AB)^\dagger = B^\dagger A^\dagger$ *and other mixed-type reverse-order laws*, International Journal of Mathematical Education in Science and Technology, Vol. 37, No. 3, (2007), 331–339.

[14] Y. Tian, *Using rank formulas to characterize equalities for Moore-Penrose inverses of matrix products*, Appl. Math. Comput. 147 (2004), 581–600.

Nebojša Č. Dinčić
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
E-mail: ndincic@hotmail.com

Dragan S. Djordjević
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
E-mail: ganedj@eunet.rs dragan@pmf.ni.ac.rs