Quasihyponormal and Strongly Quasihyponormal Matrices in Inner Product Spaces

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Abstract

We introduce the notion of quasihyponormal and strongly quasihyponormal matrices in spaces equipped with possibly degenerate indefinite inner product, based on the works that studied hyponormal and strongly hyponormal matrices in these spaces. Also, we generalize some results which are already known for normal and hyponormal matrices.

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1 Introduction

Let C^n be the space equipped with an indefinite inner product induced by possibly singular Hermitian matrix $H \in C^{n\times n}$ via the formula

$$[x, y] = \langle Hx, y\rangle$$

where $\langle ., .\rangle$ denotes the standard inner product on C^n. If the Hermitian matrix H is invertible, then the indefinite inner product is nondegenerate. In that case, for every matrix $T \in C^{n\times n}$ there is the unique matrix $T^{[*]}$ satisfying

$$[T^{[*]}x, y] = [x, Ty], \quad \text{for all } x, y \in C^n,$$

and it is given by $T^{[*]} = H^{-1}T^*H$. In these spaces the notion of H-quasihyponormal matrix can be introduced by analogy with the quasihyponormal operators in Hilbert space, i.e. with

$$HT^{[*]}(T^{[*]}T - TT^{[*]})T \geq 0.$$

Spaces with a degenerate inner product (when Gram matrix H is singular) often appear in applications, e.g. in the theory of operator pencils [5].

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That kind of spaces is less researched. One of problems that arises here is that the H-adjoint of the matrix $T \in C^{n\times n}$ need not exist. The examples can be found in ([4, 11]).

A matrix $T \in C^{n\times n}$ can always be interpreted as a linear relation via its graph $\Gamma(T)$, where: $\Gamma(T) := \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix} : x \in C^n \right\} \subseteq C^{2n}$. As in ([4, 10, 11]), we will consider H-adjoint $T^{[*]}$ not as a matrix, but as a linear relation in C^n, i.e. a subspace of C^{2n}. The H-adjoint of T is the linear relation $T^{[*]} = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in C^{2n} : [y,\omega] = [z,x] \text{ for all } \begin{pmatrix} x \\ \omega \end{pmatrix} \in T \right\}$. We just mention that we can always find the basis of C^n such that the matrices H and T have the forms:

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}. \tag{1}$$

where $H_1, T_1 \in \mathcal{C}^{m\times m}, m \leq n$, and H_1 is invertible.

Here H_1 is invertible Hermitian matrix and the inner product induced by it is nondegenerate. From the ([10], Proposition 2.6) we have

$$T^{[*]_H} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ {T_1}^{[*]_{H_1}} y_1 \\ z_2 \end{pmatrix} : {T_2}^* H_1 y_1 = 0 \right\}.$$

Here we will suppress the subscripts H and H_1 whenever it is clear from the context what is meant. Also, $e_k =< 0,..,0,1,0,...0 >^\top \in C^n$ will denote the k^{th} standard unit vector. Of course, $R(T)$ and $KerT$ will denote the range and kernel of a matrix T, respectively. About indefinite inner product spaces see ([3, 1, 2]).

This paper is organized as follows. In the second section we give some basic definitions and properties concerning subspaces, linear relations and notion of H-hyponormality. In section 3. we give the definition of H-quasihyponormal matrices and linear relation. In the fourth section we introduce strongly H-quasihyponormal matrices and linear relations and investigate their connection with Moore-Penrose H-normal matrices. In section 5, we conclude by assertion that for H-quasihyponormal matrices $KerH$ is contained in an invariant H-neutral subspace.

2 Preliminaries

Let H be possibly singular Hermitian $n \times n$ matrix that defines indefinite inner product. If $L \subset C^n$ is a subspace, then its orthogonal companion in C^n is defined by

$$L^{[\perp]} = \{x \in C^n : [x, y] = 0 \text{ for all } y \in L\}.$$

The orthogonal complement of some subspace L is not necessarily the direct complement. It is true if and only if L is nondegenerate. If L and M are subspaces in C^n, then by $L[\dot{+}]M$ we denote the direct H-orthogonal sum of L and M. A vector $x \in C^n$ is called H-positive (H-negative, H-neutral) if $[x, x] > 0$ (resp. $[x, x] < 0$, $[x, x] = 0$), and H-nonnegative (H-nonpositive) if x is not H-negative (not H-positive). A subspace $L \subset C^n$ is called positive with respect to $[.,.]$ (or H-positive) if $[x, x] > 0$ for all nonzero x in L. Similarly H-negative, H-neutral, H-nonpositive, H-nonnegative subspaces are defined. The subspace L is called maximal H-nonnegative if it is not properly contained in any larger H-nonnegative subspace. In [3] it was proved that H-nonnegative subspace is maximal if and only if its dimension is equal to the number of positive eigenvalues of H counted with multiplications. A subspace $L \subset C^n$ is T-invariant if $TL \subseteq L$.

A linear relation $T \subseteq C^{2n}$ is H-symmetric if $T \subseteq T^{[*]}$ and H-normal if $TT^{[*]} \subseteq T^{[*]}T$. A linear relation $T \subseteq C^{2n}$ is called H-nonnegative if T is H-symmetric and if $[y, x] \geq 0$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in T$. In [4] the definition of the H-hyponormal linear relation is given.

Definition 2.1. *The linear relation $T \subseteq C^{2n}$ is H-hyponormal if $T^{[*]}T$ has full domain and if $T^{[*]}T - TT^{[*]}$ is H-nonnegative.*

Also, it is important to mention the result given in [4], Proposition 2.6, that if $T \in C^{n \times n}$ is a matrix and T and H are in the form (1), then the linear relation $T^{[*]}T$ has full domain if and only if $T_2^* H_1 T_1 = 0$ and $T_2^* H_1 T_2 = 0$.

In this paper we introduce definition of H-quasihyponormal linear relation and matrices. Also, we give the connection with H-hyponormal matrices and check how some of their properties can be extended to H-quasihyponormal case.

3 H-quasihyponormal matrices

Let H be a Hilbert space. The operator $T \in B(H)$ is quasihyponormal if $||T^*Tx|| \leq ||T^2x||$, for every $x \in H$. It can be written as $\langle T^*Tx|T^*Tx\rangle \leq \langle T^2x|T^2x\rangle$, i.e. $(T^*T)^2 \leq (T^*)^2T^2$.

By analogy with this, we could define the H-quasihyponormal matrices in indefinite inner product spaces. For an invertible matrix H, the matrix T is H-quasihyponormal if it satisfies the condition:

$$[T^{[*]}Tx, T^{[*]}Tx] \leq [T^2x, T^2x].$$

This condition can be written in the form $[(T^{[*]}T)^2x, x] \leq [(T^{[*]})^2T^2x, x]$, i.e. $H((T^{[*]})^2T^2 - (T^{[*]}T)^2) \geq 0$.

It is convenient to write it as $HT^{[*]}(T^{[*]}T - TT^{[*]})T \geq 0$.

If H is invertible, then we can write the last inequality as: $T^*H(T^{[*]}T - TT^{[*]})T \geq 0$.

As it is known, if the Hermitian matrix $H \in \mathcal{C}^{n\times n}$ is invertible, then an H-hyponormal matrix T by definition satisfies $H(T^{[*]}T - TT^{[*]}) \geq 0$, i.e. $T^{[*]}T - TT^{[*]}$ is H-nonnegative.
It is easy to see that in the case of invertible matrix H, every H-hyponormal matrix is H-quasihiponormal matrix, as well.
Our aim is to extend the notion of H-quasihyponormality to the case of singular matrix H.

Theorem 3.1. *Let $T \subseteq \mathcal{C}^{2n}$ be a linear relation. Then $(T^{[*]})^2T^2 - (T^{[*]}T)^2$ is H-symmetric, i.e.,*

$$(T^{[*]})^2T^2 - (T^{[*]}T)^2 \subseteq ((T^{[*]})^2T^2 - (T^{[*]}T)^2)^{[*]}.$$

Proof. From the proof of the Proposition 4.4. [4] it follows that

$$T^2 \subseteq ((T^{[*]})^2)^{[*]} \text{ and } (T^{[*]})^2 \subseteq (T^2)^{[*]}$$

and from Proposition 2.3(iii) [4] we have

$$(T^{[*]})^2T^2 \subseteq (T^2)^{[*]}((T^{[*]})^2)^{[*]} \subseteq ((T^{[*]})^2T^2)^{[*]}. \tag{2}$$

In [4] it is already shown that $T^{[*]}T$ and $TT^{[*]}$ are H-symmetric linear relations, so

$$(T^{[*]}T)^2 = T^{[*]}TT^{[*]}T \subseteq (T^{[*]}T)^{[*]}(T^{[*]}T)^{[*]} \subseteq \left(T^{[*]}TT^{[*]}T\right)^{[*]} = \left((T^{[*]}T)^2\right)^{[*]}. \tag{3}$$

Now, (2), (3) and Proposition 2.3(ii) [4] imply

$$(T^{[*]})^2T^2 - (T^{[*]}T)^2 \subseteq \Big((T^2)^{[*]}T^2 - (T^{[*]}T)^2\Big)^{[*]},$$

i.e. $(T^{[*]})^2T^2 - (T^{[*]}T)^2$ is H-symmetric.

□

Let T and H be in the form (1). Then we have

$$(T^{[*]}T)^2 = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1^{[*]}T_1T_1^{[*]}(T_1y_1 + T_2y_2) + T_1^{[*]}T_2z_2 \\ \omega_2 \end{pmatrix} : \begin{array}{c} T_2^*H_1(T_1y_1 + T_2y_2) = 0 \\ T_2^*H_1T_1T_1^{[*]}(T_1y_1 + T_2y_2) + T_2^*H_1T_2z_2 = 0 \end{array} \right\}.$$

Here, z_2 and ω_2 are arbitrary numbers. To avoid the emptiness of domain, we will assume that ${T_2}^*H_1T_2 = 0$. Under this assumption we have:

$$(T^{[*]}T)^2 = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1^{[*]}T_1T_1^{[*]}(T_1y_1 + T_2y_2) + T_1^{[*]}T_2z_2 \\ \omega_2 \end{pmatrix} : \begin{array}{c} T_2^*H_1T_1y_1 = 0 \\ T_2^*H_1T_1T_1^{[*]}(T_1y_1 + T_2y_2) = 0 \end{array} \right\}.$$

Similarly, using $T_2^*H_1T_2 = 0$, we obtain:

$$(T^{[*]})^2T^2 = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ ({T_1}^{[*]})^2({T_1}^2 + T_2T_3)y_1 + ({T_1}^{[*]})^2(T_1T_2 + T_2T_4)y_2 \\ z_2 \end{pmatrix} : \begin{array}{c} {T_2}^*H_1T_1(T_1y_1 + T_2y_2) = 0 \\ {T_2}^*H_1{T_1}^{[*]}T_1(T_1y_1 + T_2y_2) + {T_2}^*H_1{T_1}^{[*]}T_2(T_3y_1 + T_4y_2) = 0 \end{array} \right\}.$$

Finally, $(T^{[*]})^2T^2 - (T^{[*]}T)^2 =$

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ T_1{}^{[*]}(T_1{}^{[*]}T_1 - T_1T_1{}^{[*]})(T_1y_1 + T_2y_2) + (T_1{}^{[*]})^2T_2(T_3y_1 + T_4y_2) - T_1{}^{[*]}T_2z_2 \\ \omega_2 \end{pmatrix} : \begin{array}{c} T_2{}^*H_1T_1y_1 = 0 \\ T_2{}^*H_1T_1(T_1y_1 + T_2y_2) = 0 \\ T_2{}^*H_1T_1T_1{}^{[*]}(T_1y_1 + T_2y_2) = 0 \\ T_2{}^*H_1T_1{}^{[*]}T_1(T_1y_1 + T_2y_2) + T_2{}^*H_1T_1{}^{[*]}T_2(T_3y_1 + T_4y_2) = 0 \end{array} \right\}.$$

Theorem 3.2. *Let $T \in \mathcal{C}^{n\times n}$ be a matrix, T and H be in the form (1) and let $T_2{}^*H_1T_2 = 0$. Then $(T^{[*]})^2T^2 - (T^{[*]}T)^2$ is H-nonnegative if and only if*

$$(T_1y_1 + T_2y_2)^*H_1(T_1^{[*]}T_1 - T_1T_1^{[*]})(T_1y_1 + T_2y_2) \geq 0,$$

for all y_1, y_2 satisfying
(1) $T_2^*H_1T_1y_1 = 0$,
(2) $T_2^*H_1T_1(T_1y_1 + T_2y_2) = 0$,
(3) $T_2^*H_1T_1T_1{}^{[*]}(T_1y_1 + T_2y_2) = 0$,
(4) $T_2^*H_1T_1{}^{[*]}T_1(T_1y_1 + T_2y_2) + T_2^*H_1T_1{}^{[*]}T_2(T_3y_1 + T_4y_2) = 0$.

Proof. The linear relation $(T^{[*]})^2T^2 - (T^{[*]}T)^2$ is H-symmetric by Theorem 3.1. Thus, from the previous paragraph one could see that $(T^{[*]})^2T^2 - (T^{[*]}T)^2$ is H-nonnegative if and only if

$$y_1{}^*H_1T_1{}^{[*]}(T_1{}^{[*]}T_1 - T_1T_1{}^{[*]})(T_1y_1 + T_2y_2) + y_1{}^*H_1(T_1{}^{[*]})^2T_2(T_3y_1 + T_4y_2) - y_1{}^*H_1T_1{}^{[*]}T_2z_2 \geq 0,$$

under conditions (1)..(4).
From (1) we have $y_1{}^*H_1T_1{}^{[*]}T_2z_2 = 0$, and from (2) $y_1{}^*T_1{}^*T_1{}^*H_1T_2 = -y_2{}^*T_2{}^*T_1{}^*H_1T_2$, and so $y_1{}^*H_1(T_1{}^{[*]})^2T_2(T_3y_1+T_4y_2) = -y_2{}^*T_2{}^*T_1{}^*H_1T_2(T_3y_1+T_4y_2)$.

Now we get

$$y_1{}^*T_1{}^*H_1(T_1{}^{[*]}T_1 - T_1T_1{}^{[*]})(T_1y_1 + T_2y_2) - y_2{}^*T_2{}^*T_1{}^*H_1T_2(T_3y_1 + T_4y_2) \geq 0.$$

The condition (4) implies

$y_1{}^*T_1{}^*H_1(T_1{}^{[*]}T_1 - T_1T_1{}^{[*]})(T_1y_1+T_2y_2)+y_2{}^*T_2{}^*T_1{}^*H_1T_1(T_1y_1+T_2y_2) \geq 0.$

After some calculations we get

$(T_1y_1+T_2y_2)^*H_1T_1{}^{[*]}T_1(T_1y_1+T_2y_2) - (T_1y_1+T_2y_2)^*H_1T_1T_1{}^{[*]}(T_1y_1 + T_2y_2) + y_2{}^*T_2{}^*H_1T_1T_1{}^{[*]}(T_1y_1+T_2y_2) \geq 0.$

Because of (3) we finally get:

$$(T_1y_1+T_2y_2)^*H_1(T_1{}^{[*]}T_1 - T_1T_1{}^{[*]})(T_1y_1+T_2y_2) \geq 0.$$

□

For an invertible matrix $H \in C^{n\times n}$ it is well known that H-quasihyponormality of a matrix T implies H-hyponormality on $R(T)$.

Similarly to [4] (Definition 3.5. and Definition 3.1) we give the notion of H-hyponormality on a subspace.

Definition 3.1. *A linear relation $T \subseteq C^{2n}$ is called H-hyponormal on a subspace $M \subseteq C^n$ if $T^{[*]}T$ has full domain and if $T^{[*]}T - TT^{[*]}$ is H-nonnegative on M.*

Definition 3.2. *A linear relation $T \subseteq C^{2n}$ is called H-nonnegative on a subspace $M \subseteq C^n$ if T is H-symmetric and if*

$$[y,x] \geq 0 \text{ for all } \begin{pmatrix} x \\ y \end{pmatrix} \in T, \text{ where } x \in M.$$

According to Theorem 3.2. we could define H-quasihyponormal matrices in indefinite inner product spaces in the following way: Let $T \in C^{n\times n}$ and $H \in C^{n\times n}$ be matrices given in the form (1). Then the matrix T is H-quasihyponormal if $T_2^*H_1T_2 = 0$ and $(T^{[*]})^2T^2 - (T^{[*]}T)^2$ is H-nonnegative. But, without the condition $T_2^*H_1T_1 = 0$, H-quasihyponormality will never imply H-hyponormality on any subspace of C^n. Thus, this definition would not be satisfactory as the next example shows.

Example 3.1. *Let* $T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -1 \\ \hline 0 & 0 & 0 \end{array}\right]$ *and* $H = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$. *Then* $T_1^{[*]} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ *and* $T_2^*H_1T_2 = 0$.

Let $y = \left(\frac{y_1}{y_2}\right) = \begin{pmatrix} y_{11} \\ y_{12} \\ \hline y_2 \end{pmatrix}$ *be partitioned conformably with* T. *Then we have*

$T_2^* H_1 T_1 y_1 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = y_{11} + y_{12} = 0$ *if and only if* $y_{12} = -y_{11}$.

$T_2^* H_1 T_1 (T_1 y_1 + T_2 y_2) = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y_2 \\ -y_2 \end{pmatrix} \right) = 0$, *for all* y_2.

$T_2^* H_1 T_1 T_1^{[*]} (T_1 y_1 + T_2 y_2) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y_2 \\ -y_2 \end{pmatrix} \right) = 0$, *for all* y_2.

$T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{pmatrix} y_2 \\ y_2 \end{pmatrix} \right) = 0$, *for all* y_2,

so y *is in domain of* $T^{[*]}(T^{[*]}T - TT^{[*]})T$ *if and only if* $y = \begin{pmatrix} y_{11} \\ -y_{11} \\ y_2 \end{pmatrix}$.

In this case we have: $(T_1 y_1 + T_2 y_2)^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]})(T_1 y_1 + T_2 y_2) = 0$. *Thus, the matrix* T *is* H*-quasihyponormal.*

Is this matrix T H*-hyponormal on some subspace of* C^n*? Of course, the answer is negative because the condition* $T_2^* H_1 T_1 = 0$, *which is in definition of* H*-hyponormal matrices is not satisfied, ([4], Proposition 3.6.).*

In previous example the domain of $T^{[*]}(T^{[*]}T - TT^{[*]})T$ is too small so we will require that, as in H-hyponormal case, $T^{[*]}T$ has full domain, i.e. that $T_2^* H_1 T_2 = 0$ and $T_2^* H_1 T_1 = 0$ are satisfied ([4], Proposition 2.6.).

Now, we can give the definition for the H-quasihyponormal linear relations.

Definition 3.3. *A linear relation* $T \subseteq C^{2n}$ *is called* H*-quasihyponormal if* $T^{[*]}T$ *has full domain and if* $(T^{[*]})^2 T^2 - (T^{[*]}T)^2$ *is* H*-nonnegative.*

In the next theorem we give characterization of H-quasihyponormal matrices.

Theorem 3.3. *Let* $T \in \mathcal{C}^{n \times n}$ *be a matrix and* T *and* H *be in the form (1). Then* T *is* H*- quasihyponormal if and only if* $T^{[*]}T$ *has full domain and*

$$y_1^* T_1^* H_1 (T_1{}^{[*]} T_1 - T_1 T_1{}^{[*]}) T_1 y_1 \geq y_2^* T_2^* T_1^* H_1 T_1 T_2 y_2$$

for all y_1, y_2 *satisfying* $T_2^* T_1^* H_1 T_1 (T_1 y_1 + T_2 y_2) = 0$.

Proof. Let the linear relation $T^{[*]}T$ have full domain. That means that $T_2^* H_1 T_1 = 0$ and $T_2^* H_1 T_2 = 0$. Now, according to Theorem 3.2. (under the additional assumption of $T_2^* H_1 T_1 = 0$), we have: $(T^{[*]})^2 T^2 - (T^{[*]}T)^2$ is H-nonnegative if and only if

$$(T_1 y_1 + T_2 y_2)^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]})(T_1 y_1 + T_2 y_2) \geq 0 \tag{4}$$

for all y_1, y_2 satisfying $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$. We can write (4) as

$$y_1^* T_1^* H_1 T_1^{[*]} T_1 T_1 y_1 + y_1^* T_1^* H_1 T_1^{[*]} T_1 T_2 y_2 - y_1^* T_1^* H_1 T_1 T_1^{[*]} T_1 y_1 -$$

$$y_1^* T_1^* H_1 T_1 T_1^{[*]} T_2 y_2 + y_2^* T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) - y_2^* T_2^* H_1 T_1 T_1^{[*]} (T_1 y_1 + T_2 y_2) \geq 0.$$

Now $T_2^* H_1 T_1 = 0$ (and so $T_1^{[*]} T_2 = 0$) implies $y_1^* T_1^* H_1 T_1 T_1^{[*]} T_2 y_2 = 0$ and $y_2^* T_2^* H_1 T_1 T_1^{[*]} (T_1 y_1 + T_2 y_2) = 0$. Also, from the condition $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$ we have $y_2^* T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$ and $y_1^* T_1^* H_1 T_1^{[*]} T_1 T_2 y_2 = -y_2^* T_2^* H_1 T_1^{[*]} T_1 T_2 y_2$, so (4) reduces to

$$y_1^* T_1^* H_1 (T_1{}^{[*]} T_1 - T_1 T_1{}^{[*]}) T_1 y_1 \geq y_2^* T_2^* T_1^* H_1 T_1 T_2 y_2.$$

□

It is easy to see that if matrices T and H are given in the form (1) and $T^{[*]}T$ has full domain, then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in dom (T^{[*]})^2 T^2 - (T^{[*]}T)^2$$

if and only if $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$.

Our class of H-quasihyponormal matrices should contain all H- hyponormal matrices, i.e. we are going to prove that the class of all H-hyponormal matrices is a proper subclass of H-quasihyponormal matrices. So we have the following result.

Theorem 3.4. *Let* $T \in C^{n \times n}$ *be a matrix,* T *and* H *be in the form (1). Then if* T *is* H*-hyponormal matrix then* T *is* H*-quasihyponormal matrix.*

Proof. Let T be an H-hyponormal matrix. By Proposition 3.6. in [4] it means that ${T_2}^* H_1 T_2 = 0$, ${T_2}^* H_1 T_1 = 0$ and ${y_1}^* H_1({T_1}^{[*]} T_1 - T_1 {T_1}^{[*]}) y_1 \geq 0$, for all y_1 satisfying ${T_2}^* H_1 y_1 = 0$.
We have $(T_1 y_1 + T_2 y_2)^* H_1({T_1}^{[*]} T_1 - T_1 {T_1}^{[*]})(T_1 y_1 + T_2 y_2) \geq 0$ for all y_1 and y_2 as ${T_2}^* H_1(T_1 y_1 + T_2 y_2) = 0$ is obviously satisfied, so by Theorem 3.3. we get that T is H-quasihyponormal matrix. □

To show that the class of H-quasihyponormal matrices does not coincide with H-hyponormal matrices, we give the next example.

Example 3.2. *Let* $T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$ *and*
$H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$. *Then we show* $T_2^* H_1 T_2 =$
$\begin{pmatrix} 3 & 1 & 2 & 2 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix} = 0$ *and*
$T_2^* H_1 T_1 = \begin{pmatrix} 3 & 1 & 2 & 2 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix},$
so $T_2^* H_1 T_2 = 0$ *and* $T_2^* H_1 T_1 = 0$.
Further on, we have ${T_1}^{[*]} = {H_1}^{-1} T_1^* H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and*
$H_1(T_1^{[*]} T_1 - T_1 T_1^{[*]}) = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

The vector $y = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ y_2 \end{pmatrix}$ *is in the domain of* $T^{[*]}T - TT^{[*]}$, *because of* $T_2^* H_1 y_1 = 0$, *but for that* y_1 *we have* $y_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) y_1 = -5 < 0$, *so we conclude that* T *is not* H*-hyponormal matrix ([4], Proposition 3.6.).*

Now we check if T *is* H*-quasihyponormal matrix. Let* $y_1 = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \end{pmatrix}$.

Then $T_1 y_1 + T_2 y_2 = \begin{pmatrix} y_{12} + 3y_2 \\ y_{12} + y_2 \\ y_{12} + 2y_2 \\ 2y_2 \end{pmatrix}$. $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$ *just for* $y_{12} = -y_2$, *i.e.* y *is in the domain of* $T^{[*]}(T^{[*]}T - TT^{[*]})T$ *if and only if it has the form* $y = \begin{pmatrix} y_{11} \\ -y_2 \\ y_{13} \\ y_{14} \\ y_2 \end{pmatrix}$. *Hence we have* $T_1 y_1 + T_2 y_2 = \begin{pmatrix} 2y_2 \\ 0 \\ y_2 \\ 2y_2 \end{pmatrix}$.
Finally, we get

$$(T_1 y_1 + T_2 y_2)^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]})(T_1 y_1 + T_2 y_2) = \overline{y_2} y_2 \geq 0.$$

Thus T *is* H*-quasihyponormal matrix.*

Now, the H-quasihyponormal matrices defined like this have the desired property given in the next result.

Corollary 3.1. *Let* $T \in C^{n \times n}$ *be a matrix,* T *and* H *be in the form (1). Then if* T *is* H*-quasihyponormal matrix then* T *is* H*-hyponormal on* $R(T) \cap dom(T^{[*]})^2 T$.

Proof. Let T be H-quasyhyponormal matrix, where T and H are given in the form (1). That means that $T_2^* H_1 T_2 = 0$, $T_2^* H_1 T_1 = 0$ and $(T_1 y_1 + T_2 y_2)^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]})(T_1 y_1 + T_2 y_2) \geq 0$ for all y_1 and y_2 that satisfy $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$.
As $T^{[*]}T$ has full domain, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in dom(T^{[*]})^2 T$ if and only if $T_2^* H_1 T_1^{[*]} T_1 z_1 =$

0. If $z \in R(T) \cap (T^{[*]})^2 T$ then $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} T_1 y_1 + T_2 y_2 \\ T_3 y_1 + T_4 y_2 \end{pmatrix}$ for some y_1 and y_2 and $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$. We have $T_2^* H_1 z_1 = T_2^* H_1 (T_1 y_1 + T_2 y_2) = 0$, because of $T_2^* H_1 T_2 = T_2^* H_1 T_1 = 0$. For this z we get $z_1^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) z_1 = (T_1 y_1 + T_2 y_2)^* H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]})(T_1 y_1 + T_2 y_2) \geq 0$. Thus, T is H-hyponormal on $R(T) \cap dom(T^{[*]})^2 T$ by Proposition 3.6. in [4] and Definition 3.1. □

We are familiar with the fact that in the case of H being negative semidefinite H-hyponormality implies H-normality. It is not the case on the relation H-quasihyponormality - H-hyponormality, i.e. for negative semi-definite matrix H, H-quasyhyponormality does not imply H-hyponormality as the next example shows.

Example 3.3. $T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$ *and*

$H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$. *We have that* $T_2 = 0$ *so* $T_2^* H_1 y_1 = 0$ *and* $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$ *for all* y_1 *and* y_2 *of appropriate sizes.* $T_1^{[*]} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$. $H_1 T_1^{[*]} (T_1^{[*]} T_1 - T_1 T_1^{[*]}) T_1 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \geq 0$, *so* T *is* H*-quasihyponormal matrix by Theorem 3.3.*
Also, $H_1 (T_1^{[*]} T_1 - T_1 T_1^{[*]}) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ *which is not nonnegative. This proves that* T *is not* H*-hyponormal matrix ([4]Proposition 3.6).*

4 Strongly H-quasihyponormal matrices

In [11] H-normal matrices are defined by the inclusion $TT^{[*]} \subseteq T^{[*]}T$. The H-normal matrix T has the property that $KerH$ is always T-invariant. Also, it was shown that if T and H are in forms (1), then T is H-normal if and only if $T_2 = 0$ and T_1 is H_1-normal.

A matrix T is called Moore-Penrose H-normal if $HTH^\dagger T^* H = T^* HT$, where $H^\dagger$ denotes Moore-Penrose generalized inverse of H. Recall that if T and H are in the form (1), then the Moore-Penrose generalized inverse of H is given by $H^\dagger = \begin{bmatrix} H_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ and the matrix T is Moore-Penrose H-normal if and only if $T_2^* H_1 T_2 = 0$, $T_2^* H_1 T_1 = 0$ and T_1 is H_1-normal.

In [9] the authors presented result that if matrix T is Moore-Penrose H-normal then $kerH$ is always contained in a T-invariant H-neutral subspace. In [4] it was shown that the class of H-hyponormal matrices does not have this property because it is too general, so the authors in [4] defined a new class of matrices - strongly H-hyponormal matrices. This class is the proper subclass of H-hyponormal matrices, and small enough to ensure that the kernel of H is always contained in an invariant H-neutral subspace.

As we saw, the class of H-quasihyponormal matrices is larger than the class of H-hyponormal matrices and, of course, it is not the case that $kerH$ is contained in a T-invariant H-neutral subspace, when T is H-quasihyponormal matrix, neither.

Now we will find the class of matrices which is larger than the strongly H-hyponormal matrices, but still has the property that kernel of H is contained in an invariant H-neutral subspace. This new class will be proper subclass of H-quasihyponormal matrices.

Definition 4.1. *Let $T \subseteq C^{2n}$ be a linear relation.*

T is called strongly H-quasihyponormal of degree $k \in N$ if T is H-quasihyponormal and $(T^{[*]})^i T^i$ has full domain for all $i = 1, ..., k$.

T is called strongly H-quasihyponormal if T is strongly H-quasihyponormal of degree k for all $k \in N$.

Here, we will use the result of Proposition 4.4, [4], that for the matrices T and H, given in the form (1), the assertions
(1) $(T^{[*]})^i T^i$ has full domain for $1 \leq i \leq k$, and
(2) $T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_1 = 0$ and $T_2^* H_1 (T_1^{[*]})^{i-1} T_1^{i-1} T_2 = 0$ for $1 \leq i \leq k$
are equivalent. As in [4], Proposition 4.5, we can deduce the next result.

Theorem 4.1. *Let $T \in C^{n \times n}$ be a matrix. If T is strongly H-quasihyponormal degree $k = rankH$, then T is strongly H-quasihyponormal.*

Now, we give the characterization of strongly H-quasihyponormal matrices.

Theorem 4.2. *A matrix T is strongly H-quasihyponormal if and only if*

$$y_1{}^* T_1{}^* H_1 (T_1{}^{[*]} T_1 - T_1 T_1{}^{[*]}) T_1 y_1 \geq 0$$

for all y_1, when ${T_2}^* H_1 (T_1^{[*]})^{i-1} {T_1}^{i-1} T_1 = 0$, ${T_2}^* H_1 (T_1^{[*]})^{i-1} {T_1}^{i-1} T_2 = 0$, for all $1 \leq i \leq k$, where $k = rankH$.

It is clear that the class of strongly H-hyponormal matrices is a subclass of strongly H-quasihyponormal matrices. These two classes does not coincide, as it is shown in the following example.

Example 4.1. *Let* $T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$ *and*

$H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$. *As $T_2 = 0$ it is clear that ${T_2}^* H_1 (T_1^{[*]})^{i-1} {T_1}^{i-1} T_1 = 0$, and ${T_2}^* H_1 (T_1^{[*]})^{i-1} {T_1}^{i-1} T_2 = 0$, for all $i = 1, 2$, so $(T^{[*]})^2 T^2$ has full domain.*

Also, $T_2^* H_1 T_1^{[*]} T_1 (T_1 y_1 + T_2 y_2) = 0$ are satisfied for all y_1 and y_2 of appropriate sizes. We have $T_1^{[*]} = \begin{bmatrix} -2 & 0 \\ -1 & 0 \end{bmatrix}$ *and* $H_1(T_1^{[*]} T_1 - T_1 T_1^{[*]}) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$.

$(T_1y_1+T_2y_2)^* H_1(T_1^{[*]}T_1 - T_1T_1^{[*]})(T_1y_1+T_2y_2) = y_1^* \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} y_1 = y_1^* \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} y_1 = \begin{pmatrix} y_{11}^* & y_{12}^* \end{pmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = (2y_{11} - y_{12})^*(2y_{11} - y_{12}) \geq 0$, *thus T is strongly H-quasihyponormal matrix by Theorem 4.2.*

On the other hand, $T_2^* H_1 y_1 = 0$ for all y_1, but $H_1(T_1^{[*]}T_1 - T_1T_1^{[*]}) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ *which is not nonnegative, so by Proposition 3.6. in [4], T is not strongly H-hyponormal matrix.*

The class of all strongly H-quasihyponormal matrices does not coincide with the class of H-quasihyponormal matrices, neither. This fact is illustrated by the Example 3.2.. In that example we saw that T is H-quasihyponormal matrix, but it is easy to verify that $T_2^* H_1 T_1^{[*]} T_1 T_1 \neq 0$, so T is not strongly H-quasihyponormal matrix.

The Moore-Penrose H-normal matrices were investigated in [6, 9, 11], and their connection with H-hyponormal and strongly H-hyponormal matrices is given in [4]. We give the relation between H-quasihyponormal and strongly H-quasihyponormal matrices and the Moore-Penrose H-normal matrices.

Theorem 4.3. *Let $T \in C^{n\times n}$ be a matrix and let T and H be in the forms as in (1). Then the following assertions are equivalent:*
(i) T is Moore-Penrose H-normal matrix;
(ii) T is strongly H-quasihyponormal matrix and T_1 is H_1-normal;
(iii) T is H-quasihyponormal matrix and T_1 is H_1-normal.

Proof. In [4], Theorem 5.5. it was shown that if T is Moore-Penrose H-normal matrix then T is strongly H-hyponormal matrix and T_1 is H_1-normal. It is clear that T is strongly H-quasihyponormal matrix, too, so (1) implies (2).
If T is strongly H-quasihyponormal matrix, then we have by definition that (2) implies (3).
Let T be H-quasihyponormal matrix. Then we have ${T_2}^* H_1 T_2 = 0$ and $T_2^* H_1 T_1 = 0$ and together with T_1 is being H_1-normal and Lemma 5.1. in [4], we get (1). □

As we see, in the special case when T is a matrix and T_1 is H_1-normal, the properties of Moore-Penrose H-normal, strongly H-hyponormal, H-hyponormal, strongly H-quasihyponormal and H-quasihyponormal matrices are equivalent. We remark that in [4] the equivalence of the first tree classes is shown.

5 Invariant semidefinite subspaces of H-quasihyponormal matrices

The next theorem shows that for a strongly H-quasihyponormal matrix T, given in the form (1), $KerH$ is always contained in T-invariant H-neutral subspace. In [4], Theorem 6.1. it is shown that it is true for H-hyponormal matrices. Herein we do not give the proof of our theorem because it is completely identical to the proof of Theorem 6.1. in [4]. It is not unexpected at all because the main ingredient of the proof is the "domain condition", which is identical for strongly H-hyponormal and strongly H-quasihyponormal matrices.

Theorem 5.1. *Let $T \in C^{n\times n}$ be a strongly H-quasihyponormal matrix. Let M be the smallest T-invariant subspace containing the kernel of H. Then M is H-neutral. In particular, if T and H are in the forms (1), then $M = M_0[\dot{+}]kerH$, where M_0 (canonically identified with a subspace of C^m)*

is H_1-neutral and the smallest T_1-invariant subspace that contains the range of T_2.

The main question is if it is possible to extend the subspace M from previous theorem to maximal H-nonpositive subspace, as it is done for H-hyponormal matrices; or we should find additional hypotheses that will make it possible. To obtain that we have to give the answer for the quasihyponormal matrices in *nondegenerate* inner product spaces. Here the Hermitian matrix H that determines indefinite inner product $[.,.]$ is *invertible.*

Unfortunately, some of the theorems important for this extension do not hold for H-quasihyponormal matrices, as it is the case with the next result, taken from [8]. The Example 5.1. proves it.

Theorem 5.2. *Let X be H-hyponormal and let $A = 1/2(X + X^{[*]})$ and $S = 1/2(X - X^{[*]})$ denote its H-selfadjoint and H-skew-adjoint parts, respectively.*
1. If the spectral subspace of A associated with the real spectrum of A is not H-negative (not H-positive, respectively), then there exists a common eigenvector of A and S that corresponds to a real eigenvalue of A and is H-nonnegative (H-nonpositive, respectively).
2. If the spectral subspace of S associated with the purely imaginary (possibly including zero) spectrum of S is not H-negative (not H-positive, respectively), then there exists a common eigenvector of A and S that corresponds to a purely imaginary eigenvalue of S and is H-nonnegative (H-nonpositive, respectively).

Example 5.1. *Let* $X = \begin{bmatrix} 0 & 1-ib & 0 \\ -ib & 0 & 1-ib \\ 0 & -ib & 0 \end{bmatrix}$, *where b is an arbitrary real number and* $H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. *Then* $X^{[*]} = \begin{bmatrix} 0 & 1+ib & 0 \\ ib & 0 & 1+ib \\ 0 & ib & 0 \end{bmatrix}$ *and* $HX^{[*]}(X^{[*]}X - XX^{[*]})X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4b^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *so X is H-quasihyponormal matrix. Its H-selfadjoint and H-skew-adjoint parts are* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and $S = \begin{bmatrix} 0 & -ib & 0 \\ -ib & 0 & -ib \\ 0 & -ib & 0 \end{bmatrix}$*, respectively. The spectral subspace of* A *associated with the real axis is* $U = Span\{e_1\}$*, which is not* H*-nonnegative. The only eigenvector of* A *is* e_1*, which is obviously an eigenvector of* S *just in the case of* $b = 0$*. So for* $b \neq 0$*,* A *and* S *do not have a common eigenvector. For* $b = 0$*, the matrix* X *is* H*-hyponormal and in that case* A *and* S *really have a common eigenvector.*

In [8] it was shown that for H-normal matrix T, invariant maximal H-semidefinite subspaces are also invariant for the adjoint $T^{[*]}$. In [7] that result was generalized for H-hyponormal matrices if the subspace under consideration is assumed to be H-nonpositive. We will show that it is not true for H-quasihyponormal matrices.

Example 5.2. *Let* $X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$*,* $H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$*. We have* $X^{[*]} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $HX^{[*]}(X^{[*]}X - XX^{[*]})X = 0$*, so* X *is* H*-quasihyponormal matrix. Clearly, the subspace* $U := Span\{e_2, e_3, e_4\}$ *is* H*-nonpositive* X*-invariant subspace of maximal dimension. But* $X^{[*]}e_2 = -e_1 \notin U$*, proving that* U *is not* $X^{[*]}$*-invariant.*

Thus, the solution of the problem of finding additional assumptions for which the extension on maximal invariant H-nonpositive subspace would be possible for strongly H-quasihyponormal matrices demands appropriate results for H-quasihyponormal matrices in nondegenerate indefinite inner product spaces, which will be the subject of a later research.

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