Hartwig’s triple reverse order law revisited

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Abstract

We extend the classical Hartwig’s triple reverse order law for the Moore-Penrose inverse to closed-range bounded linear operators on infinite dimensional Hilbert spaces.

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1 Introduction

If $S$ is a semigroup with the unit 1, and if $a, b \in S$ are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses.

In this paper we specialize our investigations to the Moore-Penrose inverse of a triple product of closed range bounded linear operators on Hilbert spaces.

Let $X, Y, Z$ be Hilbert spaces, and let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from $X$ to $Y$. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $A \in \mathcal{L}(X,Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of $A$. An operator $B \in \mathcal{L}(Y,X)$ is an inner inverse of $A$, if $ABA = A$ holds. In this case $A$ is inner invertible, or relatively regular. It is well-known that $A$ is inner invertible if and only if $\mathcal{R}(A)$ is closed in $Y$. The Moore-Penrose inverse of $A \in \mathcal{L}(X,Y)$ is the operator $X \in \mathcal{L}(Y,X)$ which satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$  

The Moore-Penrose inverse of $A$ exists if and only if $\mathcal{R}(A)$ is closed in $Y$. If the Moore-Penrose inverse of $A$ exists, then it is unique, and it is denoted by $A^\dagger$.

The rule $(AB)^\dagger = B^\dagger A^\dagger$ (in the case when $A, B, AB$ have closed ranges) does not hold in general. The equivalence conditions can be found in [7] for complex matrices; see [8], [2] and [3] for closed range bounded linear

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operators on Hilbert spaces; see [9] for Moore-Penrose invertible elements in rings and $C^*$-algebras.

Notice that the reverse order rule attracts a significant attention (see [1], [4], [6], [10], [11] and [13]).

The classical result of Hartwig [12] deals with the triple reverse order law of the form

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger,$$  

where $A, B, C$ are matrices. Hartwig establishes several equivalent conditions such that (1) holds, offering a very general proof of the main result. However, one implication in [12] is not valid in infinite dimensional Hilbert spaces, and thus we find it interesting to extend Hartwig’s proof in this direction.

We start with some auxiliary results.

**Lemma 1.1.** Let $A \in \mathcal{L}(X, Y)$ have a closed range. Then $A$ has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $A_1$ is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof of the previous result is straightforward.

**Lemma 1.2.** [6] Let $A \in \mathcal{L}(X, Y)$ have a closed range. Let $X_1$ and $X_2$ be closed and mutually orthogonal subspaces of $X$, such that $X = X_1 \oplus X_2$. Let $Y_1$ and $Y_2$ be closed and mutually orthogonal subspaces of $Y$, such that $Y = Y_1 \oplus Y_2$. Then the operator $A$ has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$. Also,

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}.$$
(b) \[
A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},
\]
where \( D = A_1^*A_1 + A_2^*A_2 \) maps \( \mathcal{R}(A^*) \) into itself and \( D > 0 \). Also,
\[
A^1 = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.
\]

Here \( A_i \) denotes different operators in any of these two cases.

**Lemma 1.3.** Let \( A \in \mathcal{L}(X,Y) \) be closed range operator and let \( P_M \) be orthogonal projection from \( Y \) to closed subspace \( \mathcal{R}(M) \subset \mathcal{R}(A) \). Then \( A^*P_MA \) has a closed range.

**Proof.** According to Lemma 1.1 and Lemma 1.2, operators \( A \) and \( P_M \) have the following forms:
\[
A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},
\]
\[
P_M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix}.
\]

It is obvious that \( A^*P_MA = (P_MA)^*P_MA \), and by using well-known fact that for any bounded linear operator \( T \) holds: \( T^*T \) has closed range if and only if \( T \) has closed range, it is enough to prove that \( P_M A \) is closed range operator. From the form of \( P_M A \):
\[
P_M A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},
\]
we have \( \mathcal{R}(P_M A) = \mathcal{R}(A_1) = A_1(\mathcal{R}(A^*)) \), which is closed because \( A_1 \) is onto. Indeed, let us suppose \( A_1 \) is not onto; this means there is some \( y \in \mathcal{R}(M) \setminus \mathcal{R}(A_1) \). Because of \( \mathcal{R}(M) \subset \mathcal{R}(A) \), there is some \( x \in \mathcal{R}(A^*) \) such that \( y = A_1x + A_2x \), provided that \( A_2x \neq 0 \). Therefore, \( \mathcal{R}(M) \ni y - A_1x = A_2x \in \mathcal{N}(M) \), and sum \( \mathcal{R}(M) \oplus \mathcal{N}(M) \) is direct, so \( A_2x = 0 \), which is contradiction. Therefore, \( A_1 \) is onto. \( \Box \)
2 Main result

In this section we extend results due to Hartwig [12] concerning the triple reverse order law for the Moore-Penrose inverse from complex matrices to infinite dimensional settings.

In this section, let $X_i$, $i = 1, 2, 3, 4$, be arbitrary Hilbert spaces, and let $A \in \mathcal{L}(X_3, X_4)$, $B \in \mathcal{L}(X_2, X_3)$ and $C \in \mathcal{L}(X_1, X_2)$ be bounded linear operators with closed ranges. We use notations in the same way as in [12]:

$$
    M = ABC, \\
    E = A^\dagger A, \\
    P = EBF,
$$

$$
    X = C^\dagger B^\dagger A^\dagger, \\
    F = CC^\dagger, \\
    Q = FB^\dagger E.
$$

Recall that $K \in \mathcal{L}(X)$ is EP, if $K$ has a closed range, and $KK^\dagger = K^\dagger K$.

The main result is the following theorem.

**Theorem 2.1.** Let $A, B, C$ be closed-range operators such that $ABC$ also has a closed range. The following statements are equivalent:

(a) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$;

(b) $PQP = P$, $QPQ = Q$, and both of $A^*AP$, and $QPCC^*$ are Hermitian;

(c) $PQP = P$, $QPQ = Q$, and both of $A^*AP$, and $QPCC^*$ are EP;

(d) $PQP = P$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$;

(e) $(PQ)^2 = PQ$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

**Proof.** The proof given by Hartwig stays valid for $(a) \iff (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$. The only case which does not hold in general, is actually the implication $(e) \Rightarrow (b)$, which involves properties of the matrix rank. Thus, this part of the proof is not applicable to operators on infinite dimensional Hilbert space.

To complete the proof, we will prove $(e) \Rightarrow (a)$ in a different way, using properties of operator matrices.

Using Lemma 1.1 we conclude that the operator $C$ has the following matrix form:

$$
    C = \begin{bmatrix}
        C_1 & 0 \\
        0 & 0
    \end{bmatrix}, \quad \begin{bmatrix}
        \mathcal{R}(C^*) \\
        \mathcal{N}(C)
    \end{bmatrix} \to \begin{bmatrix}
        \mathcal{R}(C) \\
        \mathcal{N}(C^*)
    \end{bmatrix},
$$
where $C_1$ is invertible. Then

$$C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}.$$  

From Lemma 1.2 it follows that the operator $B$ has the following matrix form:

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $G = B_1B_1^* + B_2B_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(B))$. Then

$$B^\dagger = \begin{bmatrix} B_1^*G^{-1} & 0 \\ B_2^*G^{-1} & 0 \end{bmatrix}.$$  

From Lemma 1.2 it also follows that the operator $A$ has the following matrix form:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$  

Let us find the expressions for the operators $M$, $X$, $E$, $F$, $P$ and $Q$. It is easy to find that:

$$M = ABC = \begin{bmatrix} A_1B_1C_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1 = A_1B_1C_1;$$

$$X = C^\dagger B^\dagger A^\dagger = \begin{bmatrix} C_1^{-1}B_1^*G^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_1 = C_1^{-1}B_1^*G^{-1}A_1^*D^{-1};$$

$$E = A^\dagger A = \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2 \\ A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2 \end{bmatrix}; \quad F = CC^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix};$$

$$P = EBF = \begin{bmatrix} A_1^*D^{-1}A_1B_1 & 0 \\ A_2^*D^{-1}A_1B_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*D^{-1}M_1C_1^{-1} & 0 \\ A_2^*D^{-1}M_1C_1^{-1} & 0 \end{bmatrix};$$
\[ Q = FB^1E = \begin{bmatrix} B_1^*G^{-1}A_1^*D^{-1}A_1 & B_1^*G^{-1}A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1X_1A_1 & C_1X_1A_2 \end{bmatrix}. \]

It will be convinient to compute here matrix forms for some expressions appearing in the rest of the proof:

\[ PQ = \begin{bmatrix} A_1^*D^{-1}M_1X_1A_1 & A_1^*D^{-1}M_1X_1A_2 \\ A_2^*D^{-1}M_1X_1A_1 & A_2^*D^{-1}M_1X_1A_2 \end{bmatrix}; \]

\[ QP = \begin{bmatrix} C_1X_1M_1C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}; \]

\[ A^*AP = \begin{bmatrix} A_1^*M_1C_1^{-1} & 0 \\ A_2^*M_1C_1^{-1} & 0 \end{bmatrix}; \]

\[ CC^*P = \begin{bmatrix} C_1M_1^*D^{-1}A_1 & C_1M_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}; \]

\[ (PQ)^2 = \begin{bmatrix} A_1^*D^{-1}M_1X_1M_1X_1A_1 & A_1^*D^{-1}M_1X_1M_1X_1A_2 \\ A_2^*D^{-1}M_1X_1M_1X_1A_1 & A_2^*D^{-1}M_1X_1M_1X_1A_2 \end{bmatrix}. \]

Now, we will find equivalent expressions for the conditions \((a)\) and \((e)\) in the terms of the components of the operators \(A, B\) and \(C\).

\((a)\) : This is equivalent to \((A_1B_1C_1)^\dagger = C_1^{-1}B_1^*G^{-1}A_1^*D^{-1}\), or \(M_1^\dagger = X_1\).

\((e)\) : This is equivalent to the following three expressions:

\((e.1)\) \iff \(A_i^*D^{-1}(M_iX_i)^2A_j = A_i^*D^{-1}M_iX_1A_j\), for all \(i, j \in \{1, 2\}\);

\((e.2)\) \iff \(\mathcal{R}\left(\begin{bmatrix} A_1^*M_1C_1^{-1} & 0 \\ A_2^*M_1C_1^{-1} & 0 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} A_1^*X_1C_1^{-1} & 0 \\ A_2^*X_1C_1^{-1} & 0 \end{bmatrix}\right); \)

\((e.3)\) \iff \(\mathcal{R}\left(\begin{bmatrix} C_1M_1^*D^{-1}A_1 & C_1M_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} C_1X_1A_1 & C_1X_1A_2 \\ 0 & 0 \end{bmatrix}\right). \)

Recall that we prove the implication \((e) \implies (a)\).

Now, if we premultiply \((e.1)\) by \(A_i\), and use summation over \(i = 1, 2\) we yield \((M_1X_1)^2A_j = M_1X_1A_j\), for \(j = 1, 2\). If we now postmultiply last expression by \(A_j^\dagger\) and add them, we have \((M_1X_1)^2 = M_1X_1\). Therefore:

\((e.1) \implies (M_1X_1)^2 = M_1X_1\). \hspace{1cm} (2)
On the other hand, (e.2) is equivalent to:
\[ \mathcal{R}(A_i^* M_1 C_1^{-1}) = \mathcal{R}(A_i^* X_1^* C_1^*), \quad i = 1, 2. \]

Again, if \( A_i \) acts on both sides, and we add them, we obtain:
\[ \mathcal{R}(M_1 C_1^{-1}) = \mathcal{R}(X_1^* C_1^*). \]

Hence, we have
\[ \mathcal{R}(M_1) = \mathcal{R}(X_1^*), \]
which implies \( M_1 M_1^\dagger = X_1^\dagger X_1 \). Therefore,
\[ (e.2) \Rightarrow M_1 M_1^\dagger = X_1^\dagger X_1. \]

Let us now write (e.3) as:
\[ \mathcal{N}\left( \begin{bmatrix} A_1^* D_1^{-1} M_1 C_1^* & 0 \\ A_2^* D_1^{-1} M_1 C_1^* & 0 \end{bmatrix} \right) = \mathcal{N}\left( \begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix} \right). \]

Notice that
\[ \mathcal{N}\left( \begin{bmatrix} A_1^* D_1^{-1} M_1 C_1^* & 0 \\ A_2^* D_1^{-1} M_1 C_1^* & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : \begin{bmatrix} A_1^* D_1^{-1} M_1 C_1^* & 0 \\ A_2^* D_1^{-1} M_1 C_1^* & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \]
and we conclude:
\[ \mathcal{N}\left( \begin{bmatrix} A_1^* D_1^{-1} M_1 C_1^* & 0 \\ A_2^* D_1^{-1} M_1 C_1^* & 0 \end{bmatrix} \right) = \left( \mathcal{N}(A_1^* D_1^{-1} M_1 C_1^*) \cap \mathcal{N}(A_2^* D_1^{-1} M_1 C_1^*) \right) \oplus \mathcal{N}(C^*), \]
which is further equal (easy to see) to
\[ \mathcal{N}(M_1 C_1^*) \oplus \mathcal{N}(C^*). \]

With a little effort we find
\[ \mathcal{N}\left( \begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix} \right) = \left( \mathcal{N}(A_1^* X_1^* C_1^*) \cap \mathcal{N}(A_2^* X_1^* C_1^*) \right) \oplus \mathcal{N}(C^*) = \mathcal{N}(X_1^* C_1^*) \oplus \mathcal{N}(C^*). \]

Hence, the condition (e.3) implies:
\[ \mathcal{N}(M_1 C_1^*) = \mathcal{N}(X_1^* C_1^*), \]

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which is the same as $R(C_1M_1^*) = R(C_1X_1)$, or $R(M_1^*) = R(X_1)$, or even further: $M_1^*M_1 = X_1X_1^\dagger$.

Since we intend to prove $(e) \Rightarrow (a)$, it is enough to prove the following implication:

$$\left( (M_1X_1)^2 = M_1X_1, \quad M_1M_1^\dagger = X_1^\dagger X_1, \quad M_1^\dagger M_1 = X_1X_1^\dagger \right) \Rightarrow M_1^\dagger = X_1.$$

The following completes the proof:

\[
M_1 = M_1X_1X_1^\dagger = M_1X_1X_1^\dagger X_1X_1^\dagger = M_1X_1M_1^\dagger X_1^\dagger = \\
= M_1X_1M_1X_1^\dagger M_1^\dagger X_1^\dagger = M_1X_1X_1^\dagger M_1^\dagger X_1^\dagger = \\
= M_1M_1^\dagger X_1^\dagger = X_1^\dagger X_1X_1^\dagger = X_1^\dagger.
\]

For the sake of completeness, we remark that operators $A^*APQ$ and $QPCC^*$ from part (c) of our Theorem have closed ranges. It immediately follows from Lemma 1.3 because:

$$A^*APQ = A^*MM^\dagger A = A^*PR(M)A, \quad QPCC^* = CM^\dagger MC^* = CP_R(M)C^*.$$

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