Hartwig's triple reverse order law revisited

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Abstract

We extend the classical Hartwig's triple reverse order law for the Moore-Penrose inverse to closed-range bounded linear operators on infinite dimensional Hilbert spaces.

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1 Introduction

If S is a semigroup with the unit 1, and if $a, b \in S$ are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses.

In this paper we specialize our investigations to the Moore-Penrose inverse of a triple product of closed range bounded linear operators on Hilbert spaces.

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $A \in \mathcal{L}(X,Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A. An operator $B \in \mathcal{L}(Y,X)$ is an inner inverse of A, if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in Y. The Moore-Penrose inverse of $A \in \mathcal{L}(X,Y)$ is the operator $X \in \mathcal{L}(Y,X)$ which satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

The Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in Y. If the Moore-Penrose inverse of A exists, then it is unique, and it is denoted by $A^\dagger$.

The rule $(AB)^\dagger = B^\dagger A^\dagger$ (in the case when A, B, AB have closed ranges) does not hold in general. The equivalence conditions can be found in [7] for complex matrices; see [8], [2] and [3] for closed range bounded linear

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operators on Hilbert spaces; see [9] for Moore-Penrose invertible elements in rings and C^*-algebras.

Notice that the reverse order rule attracts a significant attention (see [1], [4], [6], [10], [11] and [13]).

The classical result of Hartwig [12] deals with the triple reverse order law of the form

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger, \tag{1}$$

where A, B, C are matrices. Hartwig establishes several equivalent conditions such that (1) holds, offering a very general proof of the main result. However, one implication in [12] is not valid in infinite dimensional Hilbert spaces, and thus we find it interesting to extend Hartwig's proof in this direction.

We start with some auxiliary results.

Lemma 1.1. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof of the previous result is straightforward.

Lemma 1.2. [6] *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$. Also,

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^* A_1 + A_2^* A_2$ *maps* $\mathcal{R}(A^*)$ *into itself and* $D > 0$*. Also,*

$$A^\dagger = \begin{bmatrix} D^{-1} A_1^* & D^{-1} A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i *denotes different operators in any of these two cases.*

Lemma 1.3. *Let* $A \in \mathcal{L}(X, Y)$ *be closed range operator and let* P_M *be orthogonal projection from* Y *to closed subspace* $\mathcal{R}(M) \subset \mathcal{R}(A)$*. Then* $A^* P_M A$ *has a closed range.*

Proof. According to Lemma 1.1 and Lemma 1.2, operators A and P_M have the following forms:

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},$$

$$P_M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix}.$$

It is obvious that $A^* P_M A = (P_M A)^* P_M A$, and by using well-known fact that for any bounded linear operator T holds: $T^* T$ has closed range if and only if T has closed range, it is enough to prove that $P_M A$ is closed range operator. From the form of $P_M A$:

$$P_M A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},$$

we have $\mathcal{R}(P_M A) = \mathcal{R}(A_1) = A_1(\mathcal{R}(A^*))$, which is closed because A_1 is onto. Indeed, let us suppose A_1 is not onto; this means there is some $y \in \mathcal{R}(M) \setminus \mathcal{R}(A_1)$. Because of $\mathcal{R}(M) \subset \mathcal{R}(A)$, there is some $x \in \mathcal{R}(A^*)$ such that $y = A_1 x + A_2 x$, provided that $A_2 x \neq 0$. Therefore, $\mathcal{R}(M) \ni y - A_1 x = A_2 x \in \mathcal{N}(M)$, and sum $\mathcal{R}(M) \oplus \mathcal{N}(M)$ is direct, so $A_2 x = 0$, which is contradiction. Therefore, A_1 is onto. □

2 Main result

In this section we extend results due to Hartwig [12] concerning the triple reverse order law for the Moore-Penrose inverse from complex matrices to infinite dimensional settings.

In this section, let X_i, $i = 1, 2, 3, 4$, be arbitrary Hilbert spaces, and let $A \in \mathcal{L}(X_3, X_4)$, $B \in \mathcal{L}(X_2, X_3)$ and $C \in \mathcal{L}(X_1, X_2)$ be bounded linear operators with closed ranges. We use notations in the same way as in [12]:

$$\begin{aligned} M &= ABC, & X &= C^\dagger B^\dagger A^\dagger, \\ E &= A^\dagger A, & F &= CC^\dagger, \\ P &= EBF, & Q &= FB^\dagger E. \end{aligned}$$

Recall that $K \in L(X)$ is EP, if K has a closed range, and $KK^\dagger = K^\dagger K$.

The main result is the following theorem.

Theorem 2.1. *Let A, B, C be closed-range operators such that ABC also has a closed range. The following statements are equivalent:*

(a) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$;

(b) $PQP = P$, $QPQ = Q$, *and both of* A^*APQ, *and* $QPCC^*$ *are Hermitian;*

(c) $PQP = P$, $QPQ = Q$, *and both of* A^*APQ, *and* $QPCC^*$ *are EP;*

(d) $PQP = P$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$;

(e) $(PQ)^2 = PQ$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

Proof. The proof given by Hartwig stays valid for $(a) \Leftrightarrow (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$. The only case which does not hold in general, is actually the implication $(e) \Rightarrow (b)$, which involves properties of the matrix rank. Thus, this part of the proof is not applicable to operators on infinite dimensional Hilbert space.

To complete the proof, we will prove $(e) \Rightarrow (a)$ in a different way, using properties of operator matrices.

Using Lemma 1.1 we conclude that the operator C has the following matrix form:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where C_1 is invertible. Then

$$C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}.$$

From Lemma 1.2 it follows that the operator B has the following matrix form:

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $G = B_1B_1^* + B_2B_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(B))$. Then

$$B^\dagger = \begin{bmatrix} B_1^*G^{-1} & 0 \\ B_2^*G^{-1} & 0 \end{bmatrix}.$$

From Lemma 1.2 it also follows that the operator A has the following matrix form:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

Let us find the expressions for the operators M, X, E, F, P and Q. It is easy to find that:

$$M = ABC = \begin{bmatrix} A_1B_1C_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1 = A_1B_1C_1;$$

$$X = C^\dagger B^\dagger A^\dagger = \begin{bmatrix} C_1^{-1}B_1^*G^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$X_1 = C_1^{-1}B_1^*G^{-1}A_1^*D^{-1};$$

$$E = A^\dagger A = \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2 \\ A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2 \end{bmatrix}; \quad F = CC^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix};$$

$$P = EBF = \begin{bmatrix} A_1^*D^{-1}A_1B_1 & 0 \\ A_2^*D^{-1}A_1B_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*D^{-1}M_1C_1^{-1} & 0 \\ A_2^*D^{-1}M_1C_1^{-1} & 0 \end{bmatrix};$$

$$Q = FB^\dagger E = \begin{bmatrix} B_1^* G^{-1} A_1^* D^{-1} A_1 & B_1^* G^{-1} A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1 X_1 A_1 & C_1 X_1 A_2 \\ 0 & 0 \end{bmatrix}.$$

It will be convinient to compute here matrix forms for some expressions appearing in the rest of the proof:

$$PQ = \begin{bmatrix} A_1^* D^{-1} M_1 X_1 A_1 & A_1^* D^{-1} M_1 X_1 A_2 \\ A_2^* D^{-1} M_1 X_1 A_1 & A_2^* D^{-1} M_1 X_1 A_2 \end{bmatrix};$$

$$QP = \begin{bmatrix} C_1 X_1 M_1 C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix};$$

$$A^* A P = \begin{bmatrix} A_1^* M_1 C_1^{-1} & 0 \\ A_2^* M_1 C_1^{-1} & 0 \end{bmatrix};$$

$$CC^* P^* = \begin{bmatrix} C_1 M_1^* D^{-1} A_1 & C_1 M_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix};$$

$$(PQ)^2 = \begin{bmatrix} A_1^* D^{-1} M_1 X_1 M_1 X_1 A_1 & A_1^* D^{-1} M_1 X_1 M_1 X_1 A_2 \\ A_2^* D^{-1} M_1 X_1 M_1 X_1 A_1 & A_2^* D^{-1} M_1 X_1 M_1 X_1 A_2 \end{bmatrix}.$$

Now, we will find equivalent expressions for the conditions (a) and (e) in the terms of the components of the operators A, B and C.

(a) : This is equivalent to $(A_1 B_1 C_1)^\dagger = C_1^{-1} B_1^* G^{-1} A_1^* D^{-1}$, or $M_1^\dagger = X_1$.

(e) : This is equivalent to the following three expressions:

$$\begin{aligned}
(e.1) \quad &\Leftrightarrow \quad A_i^* D^{-1} (M_1 X_1)^2 A_j = A_i^* D^{-1} M_1 X_1 A_j, \qquad \text{for all } i, j \in \{1, 2\}; \\
(e.2) \quad &\Leftrightarrow \quad \mathcal{R}\left(\begin{bmatrix} A_1^* M_1 C_1^{-1} & 0 \\ A_2^* M_1 C_1^{-1} & 0 \end{bmatrix} \right) = \mathcal{R}\left(\begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix} \right); \\
(e.3) \quad &\Leftrightarrow \quad \mathcal{R}\left(\begin{bmatrix} C_1 M_1^* D^{-1} A_1 & C_1 M_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} \right) = \mathcal{R}\left(\begin{bmatrix} C_1 X_1 A_1 & C_1 X_1 A_2 \\ 0 & 0 \end{bmatrix} \right).
\end{aligned}$$

Recall that we prove the implication (e) $\Longrightarrow$ (a).

Now, if we premultiply (e.1) by A_i, and use summation over $i = 1, 2$ we yield $(M_1 X_1)^2 A_j = M_1 X_1 A_j$, for $j = 1, 2$. If we now postmultiply last expression by A_j^* and add them, we have $(M_1 X_1)^2 = M_1 X_1$. Therefore:

$$(e.1) \implies (M_1 X_1)^2 = M_1 X_1. \tag{2}$$

On the other hand, (e.2) is equivalent to:

$$\mathcal{R}(A_i^* M_1 C_1^{-1}) = \mathcal{R}(A_i^* X_1^* C_1^*), \quad i = 1, 2.$$

Again, if A_i acts on both sides, and we add them, we obtain:

$$\mathcal{R}(M_1 C_1^{-1}) = \mathcal{R}(X_1^* C_1^*).$$

Hence, we have

$$\mathcal{R}(M_1) = \mathcal{R}(X_1^*),$$

which implies $M_1 M_1^\dagger = X_1^\dagger X_1$. Therefore,

$$(e.2) \Rightarrow M_1 M_1^\dagger = X_1^\dagger X_1.$$

Let us now write (e.3) as:

$$\mathcal{N}\left(\begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix}\right).$$

Notice that

$$\mathcal{N}\left(\begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : \begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\},$$

and we conclude:

$$\mathcal{N}\left(\begin{bmatrix} A_1^* D^{-1} M_1 C_1^* & 0 \\ A_2^* D^{-1} M_1 C_1^* & 0 \end{bmatrix}\right) = \Big(\mathcal{N}(A_1^* D^{-1} M_1 C_1^*) \cap \mathcal{N}(A_2^* D^{-1} M_1 C_1^*)\Big) \oplus \mathcal{N}(C^*),$$

which is further equal (easy to see) to

$$\mathcal{N}(M_1 C_1^*) \oplus \mathcal{N}(C^*).$$

With a little effort we find

$$\begin{aligned} \mathcal{N}\left(\begin{bmatrix} A_1^* X_1^* C_1^* & 0 \\ A_2^* X_1^* C_1^* & 0 \end{bmatrix}\right) &= \Big(\mathcal{N}(A_1^* X_1^* C_1^*) \cap \mathcal{N}(A_2^* X_1^* C_1^*)\Big) \oplus \mathcal{N}(C^*) = \\ &= \mathcal{N}(X_1^* C_1^*) \oplus \mathcal{N}(C^*). \end{aligned}$$

Hence, the condition (e.3) implies:

$$\mathcal{N}(M_1 C_1^*) = \mathcal{N}(X_1^* C_1^*),$$

which is the same as $\mathcal{R}(C_1M_1^*) = \mathcal{R}(C_1X_1)$, or $\mathcal{R}(M_1^*) = \mathcal{R}(X_1)$, or even further: $M_1^\dagger M_1 = X_1X_1^\dagger$.

Since we intend to prove $(e) \Rightarrow (a)$, it is enough to prove the following implication:

$$\Big((M_1X_1)^2 = M_1X_1, \quad M_1M_1^\dagger = X_1^\dagger X_1, \quad M_1^\dagger M_1 = X_1X_1^\dagger\Big) \Rightarrow M_1^\dagger = X_1.$$

The following completes the proof:

$$\begin{aligned} M_1 &= M_1X_1X_1^\dagger = M_1X_1X_1^\dagger X_1X_1^\dagger = M_1X_1M_1M_1^\dagger X_1^\dagger = \\ &= M_1X_1M_1X_1X_1^\dagger M_1^\dagger X_1^\dagger = M_1X_1X_1^\dagger M_1^\dagger X_1^\dagger = \\ &= M_1M_1^\dagger X_1^\dagger = X_1^\dagger X_1X_1^\dagger = X_1^\dagger. \end{aligned}$$

For the sake of completeness, we remark that operators A^*APQ and $QPCC^*$ from part (c) of our Theorem have closed ranges. It immediately follows from Lemma 1.3 because:

$$A^*APQ = A^*MM^\dagger A = A^*P_{\mathcal{R}(M)}A, \; QPCC^* = CM^\dagger MC^* = CP_{\mathcal{R}(M)}C^*.$$

□

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