IDENTITIES CONCERNING THE REVERSE ORDER LAW FOR THE MOORE-PENROSE INVERSE

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Abstract

We prove some identities related to the reverse order law for the Moore-Penrose inverse of operators on Hilbert spaces, extending some results from (Y. Tian and S. Cheng, Linear Multilinear Algebra 52 (2004)) and (R. E. Cline, SIAM Review, Vol. 6, No. 1 (1964)) to infinite dimensional settings.

2010 *Mathematics Subject Classification*: 47A05, 15A09.

Keywords and phrases: Moore-Penrose inverse, weighted Moore-Penrose inverse, reverse order law.

1 Introduction

If S is a semigroup with the unit 1, and if $a, b \in S$ are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses. Notice that the classical result for the reverse order law for the Moore-Penrose inverse is proved in [8] for complex matrices, and in [1], [2] and [9] for bounded linear operators on Hilbert spaces (meaning $(AB)^\dagger = B^\dagger A^\dagger$ if and only if: $A^\dagger A$ commutes with BB^*, and A^*A commutes with $BB^\dagger$). Latter, the corresponding result for elements in a ring with involution is proved in [10]. A significant number of papers treat the sufficient or equivalent conditions such that the reverse order law holds in some sense (see [4], [5], [6], [7], [11]).

The reverse order law has applications in the numerical computation of the Moore-Penrose inverse of a product of operators. In this paper we specialize our investigation to certain identities related to the Moore-Penrose inverse of closed-range bounded linear operators on Hilbert spaces.

Let X and Y be Hilbert spaces, and let $\mathcal{L}(X, Y)$ denote the set of all linear bounded operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. For $A \in \mathcal{L}(X, Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A. An operator $B \in \mathcal{L}(Y, X)$ is an inner inverse of A, if

[1] The authors are supported by the Ministry of Science, Republic of Serbia, grant no. 174007.

$ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in Y. The Moore-Penrose inverse of $A \in \mathcal{L}(X,Y)$ is the operator $A^\dagger \in \mathcal{L}(Y,X)$ which satisfies the Penrose equations

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in Y, and in this case $A^\dagger$ is unique.

Let $M \in \mathcal{L}(Y)$ and $N \in \mathcal{L}(X)$ be positive and invertible operators, and let $A \in \mathcal{L}(X,Y)$ have a closed range. Then there exists the unique operator $B \in \mathcal{L}(Y,X)$ such that the following equations are satisfied:

$$ABA = A,\ BAB = B,\ (MAB)^* = MAB,\ (NBA)^* = NBA.$$

Such B is denoted by $A^\dagger_{M,N}$ and it is known as the weighted Moore-Penrose inverse of A with respect to the weights M and N.

There is one very useful result linking the ordinary and the weighted Moore-Penrose inverse ([10], Theorem 5):

$$A^\dagger_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}AN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}.$$

The paper is organized as follows. In the rest of Introduction we formulate some auxiliary results. In Section 2 we prove some identities related to various mixed-type reverse order rules for the Moore-Penrose inverse of products of Hilbert space operators with closed ranges. We also consider one classical identity proved by Cline in [3]. Present paper is an extension of results from [11] and [3] to infinite dimensional settings. Recall that the results in [11] are obtained mostly using the finite dimensional methods. In this paper we use operator matrices on arbitrary Hilbert spaces.

Lemma 1.1. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then A has a matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof is straightforward.

Lemma 1.2. *Let $A \in \mathcal{L}(X,Y)$ be a closed range operator and let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ be orthogonal decompositions with closed subspaces. Then A has the following matrix representations:*
(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (D is positive and invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^* A_1 + A_2^* A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$ (D is positive and invertible). Also,

$$A^\dagger = \begin{bmatrix} D^{-1} A_1^* & D^{-1} A_2^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Here A_1, A_2 may be different operators in (a) *and* (b)*.*

The previous lemma is proved in [6]. The following result is well-known, and it can be found in [9].

Lemma 1.3. *Let $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$ have closed ranges. Then AB has a closed range if and only if $A^\dagger A B B^\dagger$ has a closed range.*

The following lemma appears to be useful later, when we deal with the weighted Moore-Penrose inverses.

Lemma 1.4 ([6])**.** *Let X and Y be Hilbert spaces, let $C \in \mathcal{L}(X,Y)$ have a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $DCC^\dagger = CC^\dagger D$.*

Lemma 1.5. *Let W, X, Y, Z be Hilbert spaces, let $P \in \mathcal{L}(X,Y)$, $Q \in \mathcal{L}(Y,Z)$ and $R \in \mathcal{L}(W,X)$ be operators such that P, Q, QP, PR have closed ranges. If Q and R are invertible, then:*

(a) $(P(QP)^\dagger)^\dagger = QPP^\dagger$;

(b) $((PR)^\dagger P)^\dagger = P^\dagger PR$.

Proof. For (a) we verify that the operators $A = P(QP)^\dagger$ and $B = QPP^\dagger$ satisfy the Penrose equations. As a sample we check that $ABA = A$:

$$\begin{aligned} ABA &= P(QP)^\dagger QPP^\dagger P(QP)^\dagger = Q^{-1}QP(QP)^\dagger QP(QP)^\dagger = \\ &= Q^{-1}QP(QP)^\dagger = P(QP)^\dagger = A. \end{aligned}$$

The remaining equations are checked analogously. Equation (b) is verified in a similar manner. Notice that from (a) and (b) we conclude that $P(QP)^\dagger$ and $(PR)^\dagger P$ have closed ranges. $\square$

2 Main results

In this section we prove the results concerning the mixed-type reverse-order law for the Moore-Penrose inverse of a product of two and multiple Hilbert space operators with close ranges.

Theorem 2.1. *Let X, Y, Z, W be Hilbert spaces, and let $A \in \mathcal{L}(Z, W)$, $B \in \mathcal{L}(Y, Z)$ and $C \in \mathcal{L}(X, Y)$ be the operators, such that A, B, C, AB, ABC have closed ranges. Then the following hold:*

(a) $(AB)^\dagger = (A^\dagger AB)^\dagger (ABB^\dagger)^\dagger$;

(b) $(AB)^\dagger = [(A^\dagger)^* B]^\dagger (B^\dagger A^\dagger)^* [A(B^\dagger)^*]^\dagger$;

(c) $(ABC)^\dagger = (A^\dagger ABC)^\dagger B(ABCC^\dagger)^\dagger$;

(d) $(ABC)^\dagger = [(AB)^\dagger ABC]^\dagger B^\dagger [ABC(BC)^\dagger]^\dagger$;

(e) $(ABC)^\dagger = [(ABB^\dagger)^\dagger ABC]^\dagger B[ABC(B^\dagger BC)^\dagger]^\dagger$;

(f) $(ABC)^\dagger = [(A^\dagger)^* BC]^\dagger (A^\dagger)^* B(C^\dagger)^* [AB(C^\dagger)^*]^\dagger$;

(g) $(ABC)^\dagger = \{[A(B^\dagger)^*]^\dagger ABC\}^\dagger B^* BB^* \{ABC[(B^\dagger)^* C]^\dagger\}^\dagger$;

(h) $(ABC)^\dagger = \{[(AB)^\dagger]^* C\}^\dagger [(AB)^\dagger]^* B^\dagger [(BC)^\dagger]^* \{A[(BC)^\dagger]^*\}^\dagger$.

Proof. According to lemmas from the previous section, it is easy to conclude that operators A, B and C have the following matrix representations with the respect to the appropriate decompositions of spaces:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

Moreover,

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

Finally,

$$C = \begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

where $E = C_1^*C_1 + C_2^*C_2$ is invertible and positive in $\mathcal{L}(\mathcal{R}(C^*))$. Then

$$C^\dagger = \begin{bmatrix} E^{-1}C_1^* & E^{-1}C_2^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}.$$

(a): Notice that $\mathcal{R}(A^\dagger AB) = A^\dagger A(\mathcal{R}(B)) = A^\dagger A(\mathcal{R}(BB^\dagger)) = \mathcal{R}(A^\dagger ABB^\dagger)$ is closed according to Lemma 1.3. Also, $\mathcal{R}(B^*A^*)$ is closed. Again, from Lemma 1.3 and $\mathcal{R}((ABB^\dagger)^*) = \mathcal{R}((B^*)^\dagger B^*A^*) = \mathcal{R}((B^*)^\dagger B^*A^*(A^*)^\dagger) = \mathcal{R}((A^\dagger ABB^\dagger)^*)$, it follows that $\mathcal{R}(ABB^\dagger)$ is closed. Now, using matrix forms of A and B, we have:

$$ABB^\dagger = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \; (ABB^\dagger)^\dagger = \begin{bmatrix} A_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}; \; A^\dagger AB = \begin{bmatrix} A_1^*D^{-1}A_1B_1 & 0 \\ A_2^*D^{-1}A_1B_1 & 0 \end{bmatrix},$$

$$(A^\dagger AB)^\dagger = ((A^\dagger AB)^*(A^\dagger AB))^\dagger (A^\dagger AB)^* =$$
$$= \begin{bmatrix} (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_1 & (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $(AB)^\dagger = (A^\dagger AB)^\dagger (ABB^\dagger)^\dagger$ is equivalent to

$$(A_1B_1)^\dagger = (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_1A_1^\dagger,$$

which is further equivalent to

$$(A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger.$$

The last equality follows by checking Penrose equations; as a sample we check the second one:

$$\begin{aligned}
&(A_1B_1)^\dagger A_1B_1(A_1B_1)^\dagger = \\
&= (D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger = \\
&= (D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger = (A_1B_1)^\dagger.
\end{aligned}$$

(b): Notice that $\mathcal{R}(((A^\dagger)^*B)^*) = \mathcal{R}(B^*A^\dagger) = \mathcal{R}(B^*A^*) = \mathcal{R}((AB)^*)$ is closed, so $\mathcal{R}((A^\dagger)^*B)$ is closed. Also, $\mathcal{R}(A(B^\dagger)^*) = \mathcal{R}(A((B^*B)^\dagger B^*)^*) = \mathcal{R}(AB(B^*B)^\dagger) = AB(\mathcal{R}((B^*B)^\dagger)) = AB(\mathcal{R}(B^*)) = A(\mathcal{R}(B)) = \mathcal{R}(AB)$ is closed. Again, using matrix forms of A and B, we have that $(AB)^\dagger = [(A^\dagger)^*B]^\dagger(B^\dagger A^\dagger)^*[A(B^\dagger)^*]^\dagger$ is equivalent to the following:

$$(A_1B_1)^\dagger = (D^{-1}A_1B_1)^\dagger D^{-1}A_1(B_1^{-1})^*(A_1(B_1^{-1})^*)^\dagger.$$

The last equality can easily be proved by checking the Penrose equations.
(c): Notice that $\mathcal{R}((A^\dagger ABC)^*) = (BC)^*(\mathcal{R}(A^\dagger A)) = (BC)^*(\mathcal{R}(A^*)) = \mathcal{R}((ABC)^*)$ is closed. Also, $\mathcal{R}(ABCC^\dagger) = AB(\mathcal{R}(CC^\dagger) = AB(\mathcal{R}(C)) = \mathcal{R}(ABC)$ is closed. Now we show that $(ABC)^\dagger = (A^\dagger ABC)^\dagger B(ABCC^\dagger)^\dagger$. First we compute factors appearing on the right side. Denote:

$$T = A^\dagger ABC = \begin{pmatrix} A_1^*D^{-1}A_1B_1C_1 & 0 \\ A_2^*D^{-1}A_1B_1C_1 & 0 \end{pmatrix}.$$

Now,

$$T^\dagger = (T^*T)^\dagger T^* = \begin{pmatrix} XA_1 & XA_2 \\ 0 & 0 \end{pmatrix},$$

where

$$X = (C_1^*B_1^*A_1^*D^{-1}A_1B_1C_1)^\dagger C_1^*B_1^*A_1^*D^{-1}.$$

Let S be the following:

$$S = ABCC^\dagger = \begin{pmatrix} A_1B_1C_1E^{-1}C_1^* & A_1B_1C_1E^{-1}C_2^* \\ 0 & 0 \end{pmatrix}.$$

It is easy to find:

$$S^\dagger = S^*(SS^*)^\dagger = \begin{pmatrix} C_1E^{-1}C_1^*B_1^*A_1^*(A_1B_1C_1E^{-1}C_1^*B_1^*A_1^*)^\dagger & 0 \\ C_2E^{-1}C_1^*B_1^*A_1^*(A_1B_1C_1E^{-1}C_1^*B_1^*A_1^*)^\dagger & 0 \end{pmatrix}.$$

Therefore, the statement (c) is equivalent to:

$$\begin{aligned}(A_1B_1C_1)^\dagger &= (C_1^*B_1^*A_1^*D^{-1}A_1B_1C_1)^\dagger C_1^*B_1^*A_1^*D^{-1}A_1 \times \\ &\times \; B_1C_1E^{-1}C_1^*B_1^*A_1^*(A_1B_1C_1E^{-1}C_1^*B_1^*A_1^*)^\dagger,\end{aligned}$$

i.e.

$$(A_1B_1C_1)^\dagger = (D^{-1/2}A_1B_1C_1)^\dagger D^{-1/2}A_1B_1C_1E^{-1/2}(A_1B_1C_1E^{-1/2})^\dagger.$$

This formula can be proved on analogous way as in (a).
(f): Notice that $\mathcal{R}(((A^\dagger)^*BC)^*) = (BC)^*(\mathcal{R}(A^\dagger)) = (BC)^*(\mathcal{R}(A^*)) = \mathcal{R}((ABC)^*)$ is closed, so $\mathcal{R}((A^\dagger)^*BC)$ is closed. Also, $\mathcal{R}(AB(C^\dagger)^*) = AB(\mathcal{R}((C^\dagger)^*)) = AB(\mathcal{R}(C)) = \mathcal{R}(ABC)$ is closed. An easy computation shows that

$$(ABC)^\dagger = [(A^\dagger)^*BC]^\dagger(A^\dagger)^*B(C^\dagger)^*[AB(C^\dagger)^*]^\dagger$$

is equivalent to:

$$(A_1B_1C_1)^\dagger = (D^{-1}A_1B_1C_1)^\dagger D^{-1}A_1B_1C_1E^{-1}(A_1B_1C_1E^{-1})^\dagger.$$

This equality follows a standard argument.

So far we have proved four identities. Now we use (c), to show that (d), (e) and (g) are satisfied. Also, we use (f) to prove that (h) holds.

(d): An easy computation shows that (d) is equivalent to the following:

$$(A_1B_1C_1)^\dagger = [(A_1B_1)^\dagger A_1B_1C_1]^\dagger B_1^{-1}[A_1B_1C_1(B_1C_1)^\dagger]^\dagger.$$

If we put: $A' = A_1B_1,\ B' = B_1^{-1},\ C' = B_1C_1$, then (d) becomes already proven identity (c) for operators A', B', C'. For the completeness, notice that the following operator ranges are closed:

$$\begin{aligned}&\mathcal{R}(A') = \mathcal{R}(AB),\ \mathcal{R}(B') = \mathcal{R}(B^*),\ \mathcal{R}(C') = \mathcal{R}(BC),\\ &\mathcal{R}(A'B') = \mathcal{R}(A),\ \mathcal{R}(B'C') = \mathcal{R}(C),\ \mathcal{R}(A'B'C') = \mathcal{R}(ABC).\end{aligned}$$

(e): An easy computation shows that (e) is equivalent to the following:

$$(A_1B_1C_1)^\dagger = [A_1^\dagger A_1B_1C_1]^\dagger B_1[A_1B_1C_1C_1^\dagger]^\dagger.$$

The last identity is proved in (c).
(g): An easy computation shows that

$$(ABC)^\dagger = \{[A(B^\dagger)^*]^\dagger ABC\}^\dagger B^*BB^*\{ABC[(B^\dagger)^*C]^\dagger\}^\dagger$$

is equivalent to the following:

$$(A_1B_1C_1)^\dagger = \{[A_1(B_1^*)^{-1}]^\dagger A_1B_1C_1\}^\dagger B_1^*B_1B_1^*\{A_1B_1C_1[(B_1^*)^{-1}C_1]^\dagger\}^\dagger.$$

We put: $A'' := A_1(B_1^*)^{-1}$, $B'' := B_1^*B_1B_1^*$, $C'' := (B_1^*)^{-1}C_1$. Now we have that the following operator ranges are closed:

$$\begin{aligned}
&\mathcal{R}(A'') = A_1(\mathcal{R}((B_1^*)^{-1})) = \mathcal{R}(AB),\ \mathcal{R}(B'') = \mathcal{R}(B^*),\\
&\mathcal{R}(C'') = \mathcal{R}(BC),\ \mathcal{R}(A''B'') = \mathcal{R}(A_1B_1B_1^*) = \mathcal{R}(AB),\\
&\mathcal{R}((B''C'')^*) = \mathcal{R}((B^*BC)^*) = C^*(\mathcal{R}(B^*B)) = \mathcal{R}((BC)^*),\\
&\mathcal{R}(A''B''C'') = \mathcal{R}(ABC).
\end{aligned}$$

So, conditions of the identity (c) are satisfied. Hence, (g) follows from (c).
(h): An easy computation shows that (h) is equivalent to the following:

$$(A_1B_1C_1)^\dagger = \{[(A_1B_1)^\dagger]^*C_1\}^\dagger[(A_1B_1)^\dagger]^*B_1^{-1}[(B_1C_1)^\dagger]^*\{A_1[(B_1C_1)^\dagger]^*\}^\dagger.$$

If we put: $A''' := A_1B_1$, $B''' := B_1^{-1}$, $C''' := B_1C_1$, then (h) becomes already proven identity (f). For the completeness, notice that the following operator ranges are closed:

$$\begin{aligned}
&\mathcal{R}(A''') = \mathcal{R}(AB),\ \mathcal{R}(B''') = \mathcal{R}(B^*),\ \mathcal{R}(C''') = \mathcal{R}(BC),\\
&\mathcal{R}(A'''B''') = \mathcal{R}(A),\ \mathcal{R}(B'''C''') = \mathcal{R}(C),\ \mathcal{R}(A'''B'''C''') = \mathcal{R}(ABC).
\end{aligned}$$

□

Remark 2.1. *The existence of the Moore-Penrose inverses of various operators in the previous theorem, follows from the closedness of operator ranges* $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(C), \mathcal{R}(AB), \mathcal{R}(BC), \mathcal{R}(ABC)$. *This fact is important, and it is explained in details. The same is true for the rest of theorems.*

The next two corollaries are immediate consequences of Theorem 2.1.(a).

Corollary 2.1. *Let* X, Y, Z *be Hilbert spaces, and let* $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ *be operators, such that* A, B, AB *have closed ranges. If* $A^\dagger AB = B$ *and* $ABB^\dagger = A$, *then* $(AB)^\dagger = B^\dagger A^\dagger$.

Corollary 2.2. *Let* P *and* Q *be two orthogonal projectors, i.e.* $P^2 = P = P^*$ *and* $Q^2 = Q = Q^*$. *Then* $(PQ)^\dagger$ *is an idempotent.*

Moreover, all other corollaries from [11] are also true with some slight changes in their formulations.

If U, V are operators acting on the same space, then $[U, V] = UV - VU$ is the usual notation for their commutator.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be operators, such that A, B, AB have closed ranges. Let $M \in \mathcal{L}(Z)$ and $N \in \mathcal{L}(X)$ be positive and invertible operators. Then the weighted Moore-Penrose inverse of AB with respect to M and N satisfies the following two identities:*

(a) $(AB)^{\dagger}_{M,N} = (A^{\dagger}AB)^{\dagger}_{I,N}(ABB^{\dagger})^{\dagger}_{M,I}$;

(b) $(AB)^{\dagger}_{M,N} = [(A^{\dagger}_{M,I})^{*}B]^{\dagger}_{M^{-1},N}(B^{\dagger}_{I,N}A^{\dagger}_{M,I})^{*}[A(B^{\dagger}_{I,N})^{*}]^{\dagger}_{M,N^{-1}}$.

Proof. By using well-known relation $A^{\dagger}_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}AN^{-\frac{1}{2}})^{\dagger}M^{\frac{1}{2}}$, it is easy to obtain that (a) is equivalent to:

$$(M^{\frac{1}{2}}ABN^{-\frac{1}{2}})^{\dagger} = (A^{\dagger}ABN^{-\frac{1}{2}})^{\dagger}(M^{\frac{1}{2}}ABB^{\dagger})^{\dagger}. \tag{1}$$

Let us denote: $\tilde{A} = M^{\frac{1}{2}}A$, $\tilde{B} = BN^{-\frac{1}{2}}$. We prove the following:

$$(M^{-\frac{1}{2}}\tilde{A})^{\dagger} = \tilde{A}^{\dagger}M^{\frac{1}{2}}.$$

The last statement holds if and only if $M^{-\frac{1}{2}}\tilde{A}\tilde{A}^{\dagger}M^{\frac{1}{2}}$ is Hermitian, which is equivalent to $[M, \tilde{A}\tilde{A}^{\dagger}] = 0$. Using Lemma 1.4, the last expression is equivalent to $\mathcal{R}(M\tilde{A}) = \mathcal{R}(\tilde{A})$, which is valid, because of the invertibility of the Hermitian operator M. Analogously we prove that:

$$(\tilde{B}N^{\frac{1}{2}})^{\dagger} = N^{-\frac{1}{2}}\tilde{B}^{\dagger}.$$

Now, (1) becomes:

$$(\tilde{A}\tilde{B})^{\dagger} = (\tilde{A}^{\dagger}\tilde{A}\tilde{B})^{\dagger}(\tilde{A}\tilde{B}\tilde{B}^{\dagger})^{\dagger},$$

which is already proven identity in Theorem 2.1.(a).

Analogously, we prove the statement (b). □

Theorem 2.3. *Let X, Y, Z, W be Hilbert spaces, and let $A \in \mathcal{L}(Z,W)$, $B \in \mathcal{L}(Y,Z)$, $C \in \mathcal{L}(X,Y)$ be operators, such that A, B, C, AB, BC, ABC have closed ranges. Let $M \in \mathcal{L}(W)$ and $N \in \mathcal{L}(X)$ be positive and invertible operators. Then the weighted Moore-Penrose inverse of ABC with respect to M and N satisfies the following identities:*

(a) $(ABC)^{\dagger}_{M,N} = (A^{\dagger}ABC)^{\dagger}_{I,N}B(ABCC^{\dagger})^{\dagger}_{M,I}$;

(b) $(ABC)^{\dagger}_{M,N} = ((AB)^{\dagger}ABC)^{\dagger}_{I,N}B^{\dagger}(ABC(BC)^{\dagger})^{\dagger}_{M,I}$;

(c) $(ABC)^{\dagger}_{M,N} = ((ABB^{\dagger})^{\dagger}ABC)^{\dagger}_{I,N}B(ABC(B^{\dagger}BC)^{\dagger})^{\dagger}_{M,I}$;

(d) $(ABC)^{\dagger}_{M,N} = [(A^{\dagger}_{M,I})^* BC]^{\dagger}_{M^{-1},N} (A^{\dagger}_{M,I})^* B (C^{\dagger}_{I,N})^* [AB(C^{\dagger}_{I,N})^*]^{\dagger}_{M,N^{-1}}$;

(e) $(ABC)^{\dagger}_{M,N} = \{[A(B^{\dagger})^*]^{\dagger} ABC\}^{\dagger}_{I,N} B^* B B^* \{ABC[(B^{\dagger})^* C]^{\dagger}\}^{\dagger}_{M,I}$;

(f) $(ABC)^{\dagger}_{M,N} = \{[(AB)^{\dagger}_{M,I}]^* C\}^{\dagger}_{M^{-1},N} [(AB)^{\dagger}_{M,I}]^* B^{\dagger} [(BC)^{\dagger}_{I,N}]^* \times$
$\times \{A[(BC)^{\dagger}_{I,N}]^*\}^{\dagger}_{M,N^{-1}}$.

Proof. The proof in all cases is similar to the proof of Theorem 2.2. First, we transform all weighted Moore-Penrose inverses to the ordinary ones, then we put: $\tilde{A} = M^{\frac{1}{2}} A$, $\tilde{B} = B$, $\tilde{C} = C N^{-\frac{1}{2}}$, and apply Lemma 1.4. After that, all cases reduce to already-proven identities from Theorem 2.1. $\square$

Some more general identities can also be derived from previous theorems.

Theorem 2.4. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be operators, such that A, B, AB have closed ranges. Let $M \in \mathcal{L}(Z), N \in \mathcal{L}(X)$ and $P \in \mathcal{L}(Y)$ be positive and invertible operators. Then the weighted Moore-Penrose inverse of AB with respect to M and N satisfies the following identity:*

$$(AB)^{\dagger}_{M,N} = (A^{\dagger}_{I,P} AB)^{\dagger}_{P,N} (ABB^{\dagger}_{P,I})^{\dagger}_{M,P}.$$

Proof. The proof is similar to the proof of Theorem 2.2. First, we transform all weighted Moore-Penrose inverses to the ordinary ones, which gives:

$$(M^{\frac{1}{2}} ABN^{-\frac{1}{2}})^{\dagger} = [(AP^{-\frac{1}{2}})^{\dagger} ABN^{-\frac{1}{2}}]^{\dagger} [M^{\frac{1}{2}} AB (P^{\frac{1}{2}} B)^{\dagger}]^{\dagger}.$$

If we put: $\tilde{A} = M^{\frac{1}{2}} AP^{-\frac{1}{2}}$, $\tilde{B} = P^{\frac{1}{2}} BN^{-\frac{1}{2}}$, and then apply Lemma 1.4, this statement reduces to the already-proven identity from Theorem 2.1.(a). $\square$

The following theorem can be proven similarly.

Theorem 2.5. *Let X, Y, Z, W be Hilbert spaces, and let $A \in \mathcal{L}(Z, W), B \in \mathcal{L}(Y, Z)$, $C \in \mathcal{L}(X, Y)$ be operators, such that A, B, C, AB, BC, ABC have closed ranges. Let $M \in \mathcal{L}(W), N \in \mathcal{L}(X), P \in \mathcal{L}(Y)$, $Q \in \mathcal{L}(Z)$ be positive and invertible operators. Then the weighted Moore-Penrose inverse of ABC with respect to M and N satisfies the following identities:*

(a) $(ABC)^{\dagger}_{M,N} = (A^{\dagger}_{I,P} ABC)^{\dagger}_{P,N} B (ABCC^{\dagger}_{Q,I})^{\dagger}_{M,Q}$;

(b) $(ABC)^{\dagger}_{M,N} = ((AB)^{\dagger}_{I,Q} ABC)^{\dagger}_{Q,N} B^{\dagger}_{P,Q} (ABC(BC)^{\dagger}_{P,I})^{\dagger}_{M,P}$;

(c) $(ABC)^{\dagger}_{M,N} = ((ABB^{\dagger}_{P,I})^{\dagger}_{M,P} ABC)^{\dagger}_{P,N} B (ABC(B^{\dagger}_{I,Q} BC)^{\dagger}_{Q,N})^{\dagger}_{M,Q}$.

Now, we return to one classical matrix identity from [3]. Our next theorem shows that the result from [3] holds for bounded linear Hilbert space operators.

Theorem 2.6. *Let X, Y, Z be Hilbert spaces, let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be operators such that A, B, AB have closed ranges. Then:*

$$(AB)^\dagger = (A^\dagger AB)^\dagger (AB(A^\dagger AB)^\dagger)^\dagger.$$

Proof. Using a method described in Theorem 2.1 (and decompositions and matrix forms for A and B) we conclude that $(AB)^\dagger = (A^\dagger AB)^\dagger (AB(A^\dagger AB)^\dagger)^\dagger$ is equivalent to the following (D is positive and invertible as in Lemma 1.2):

$$(A_1B_1)^\dagger = (D^{-\frac{1}{2}} A_1B_1)^\dagger (A_1B_1(D^{-\frac{1}{2}} A_1B_1)^\dagger)^\dagger,$$

which is, by Lemma 1.5, further equivalent to:

$$(A_1B_1)^\dagger = (D^{-\frac{1}{2}} A_1B_1)^\dagger D^{-\frac{1}{2}} A_1B_1(A_1B_1)^\dagger.$$

We check directly all four Penrose equations, so we have the proof. $\square$

3 Conclusion remarks

We proved several identities related to the reverse order law for the Moore-Penrose inverse of bounded linear operators on Hilbert spaces. The assumption of closed-range operator is essential for the existence of the Moore-Penrose inverse. We used operator matrices which follow naturally from orthogonal decompositions of Hilbert spaces. This method seems to be effective in investigating the infinite dimensional case. On the other hand, corresponding results for complex matrices were proved in [3] and [11] using finite dimensional methods, mostly properties of the rank of a complex matrix. It will be challenging to examine whether similar/analogous statements hold for some $\{i, j, k\}-$inverses ($\{i, j, k\} \subset \{1, 2, 3, 4\}$) instead of the Moore-Penrose inverses.

Acknowledgement: We are grateful to the referee for valuable comments and suggestions which improve readability of the paper.

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