Polynomially Riesz Perturbations

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Abstract

In this paper we investigate perturbation of left (right) Fredholm, Weyl and Browder operators by polynomially Riesz operators. We show how Baklouti's idea of "communication" enhances the perturbation properties of polynomially Riesz operators.

2010 *Mathematics subject classification*: 47A53, 46H05, 47A55.

Key words and phrases: homomorphism of Banach algebras; left and right Fredholm, Weyl and Browder operator; polynomially Riesz operator; perturbation.

1 Introduction

Let $\mathbb{C}$ be the set of all complex numbers and let X denote an infinite dimensional complex Banach space. We use $B(X)$ to denote the set of all linear bounded operators on X and $K(X)$ to denote the set of all compact operators on X. For $A \in B(X)$ we use $N(A)$ and $R(A)$, respectively, to denote the null-space and the range of A. Let $\alpha(A) = \dim N(A)$ if $N(A)$ is finite dimensional, and let $\alpha(A) = \infty$ if $N(A)$ is infinite dimensional. Similarly, let $\beta(A) = \dim X/R(A) = \mathrm{codim} R(A)$ if $X/R(A)$ is finite dimensional, and let $\beta(A) = \infty$ if $X/R(A)$ is infinite dimensional. Sets of upper and lower semi-Fredholm operators, respectively, are defined as $\Phi_+(X) = \{A \in B(X) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed}\}$, and $\Phi_-(X) = \{A \in B(X) : \beta(A) < \infty\}$. Operators in $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators. For such operators the index is defined by $i(A) = \alpha(A) - \beta(A)$. The set of Fredholm operators is defined as $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

[1] The authors are supported by the Ministry of Education and Science, Republic of Serbia, grant no. 174007.

The perturbation class of the set of Fredholm operators, denoted by $\text{Ptrb}\,(\Phi(X))$, is the set

$$\text{Ptrb}\,(\Phi(X)) = \{P \in B(X) : P + A \in \Phi(X) \ \text{ for every } A \in \Phi(X)\}.$$

An operator $A \in B(X)$ is relatively regular (or g-invertible) if there exists $B \in B(X)$ such that $ABA = A$. It is well-known that A is relatively regular if and only if $R(A)$ and $N(A)$ are closed and complemented subspaces of X. We say that an operator $A \in B(X)$ is *left Fredholm*, and write $A \in \Phi_l(X)$, if A is a relatively regular upper semi-Fredholm operator, while we say that A is *right Fredholm*, and write $A \in \Phi_r(X)$, if A is a relatively regular lower semi-Fredholm operator. In other words, A is left Fredholm if $R(A)$ is a closed and complemented subspace of X and $\alpha(A) < \infty$, while A is right Fredholm if $N(A)$ is a complemented subspace of X and $\beta(A) < \infty$. In [15] (p. 160) left (right) Fredholm operators are called left (right) essentially invertible. An operator $A \in B(X)$ is *left (right) Weyl* if A is left (right) Fredholm with $i(A) \leq 0$ ($i(A) \geq 0$). If $A \in \Phi(X)$ and $i(A) = 0$, then A is called Weyl.

The ascent of $A \in B(X)$, denoted by $\text{asc}(A)$, is the smallest $n \in \mathbb{N}$ such that $N(A^n) = N(A^{n+1})$. If such n does not exist, then $\text{asc}(A) = \infty$. The descent of A, denoted by $\text{dsc}(A)$, is the smallest $n \in \mathbb{N}$ such that $R(A^n) = R(A^{n+1})$. If such n does not exist, then $\text{dsc}(A) = \infty$. An operator $A \in B(X)$ is Browder if it is Fredholm, $\text{asc}(A) < \infty$ and $\text{dsc}(A) < \infty$. An operator $A \in B(X)$ is left Browder if it is left Fredholm of finite ascent, and A is right Browder if it is right Fredholm of finite descent [18]. It is well known that every Browder operator is Weyl ([12], Proposition 38.6(a)).

The left and right Fredholm spectrum of $A \in B(X)$, respectively, are defined by:

$$\begin{aligned} \sigma_f^{left}(A) &= \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_l(X)\}, \\ \sigma_f^{right}(A) &= \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_r(X)\}. \end{aligned}$$

The Fredholm spectrum of $A \in B(X)$ is given by:

$$\sigma_f(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi(X)\} = \sigma_f^{left}(A) \cup \sigma_f^{right}(A) \tag{1.1}$$

and the Browder spectrum of A is defined by:

$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Browder}\}.$$

An operator $A \in B(X)$ is Riesz, $A \in R(X)$, if $\{\lambda \in \mathbb{C} : A - \lambda \in \Phi(X)\} = \mathbb{C}\backslash\{0\}$, i.e. $\sigma_f(A) = \{0\}$. Immediately from definitions it follows (see also

[6], Theorem 5.5.9 and Lemma 5.6.1)

$$\mathrm{Ptrb}\,(\Phi(X)) \subset R(X). \tag{1.2}$$

In [3], Definition 2.1 (and earlier in [1], Definition 3.1) the concept of communicating operators is introduced. Precisely, let $T,\ S \in B(X)$, and let there exists a nonzero complex polynomial p such that $p(T) \in \mathrm{Ptrb}\,(\Phi(X))$; then there exists a unique nonzero complex polynomial m_T with leading coefficient 1 and of the minimal degree such that $m_T(T) \in \mathrm{Ptrb}(\Phi(X))$. It is said that T and S communicate if there exists a continuous map $\varphi : [0,1] \to \mathbb{C}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and for all λ zero of m_T, $\varphi(t)\lambda$ does not belong to the Fredholm spectrum of S. Baklouti proved ([3], Theorem 2.2) that if $T,\ S \in B(X)$ communicate and $TS - TS \in \mathrm{Ptrb}\,(\Phi(X))$, then $T - S$ is Fredholm with the same index as S. Moreover, if T and S commute and communicate, then $\mathrm{asc}(S) < \infty$ implies $\mathrm{asc}(T-S) < \infty$, as well $\mathrm{dsc}(S) < \infty$ implies $\mathrm{dsc}(T - S) < \infty$ ([3] Theorem 2.4).

In this paper we continue our discussion [19] of the perturbation of (one sided) Fredholm, Weyl and Browder elements by "polynomially Riesz" elements of a Banach algebra, but we focus on the Banach algebra $B(X)$. In Definition 3.1 we generalize Baklouti's concept of communicating operators, among others supposing that T belongs to the larger set of polynomially Riesz operators, i.e. that there exists a nonzero complex polynomial p such that $p(T)$ is a Riesz operator (then also there exists a unique nonzero complex polynomial π_T with leading coefficient 1 and of the minimal degree such that $\pi_T(T) \in R(X)$). Precisely, if T is polynomially Riesz and if there exists a continuous map $\varphi : [0,1] \to \mathbb{C}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and for all λ zero of π_T, $\varphi(t)\lambda$ does not belong to the left (right) Fredholm spectrum of S, then we shall say that S is in left (right) communication with T. We consider perturbations of left (right) Fredholm, Weyl and Browder operators by polynomially Riesz operators and pay particular attention to the advantage to be gained when perturbed operator is "in left (right) communication with" perturbing operator. Our main results are Theorem 3.2, for Fredholm and Weyl operators, and Theorem 3.4, for Browder operators, which extend (the second part of) Theorem 2.2 and Theorem 2.4 in [3], respectively. In Theorem 3.2 we show that if $A,\ B \in B(X)$, $AB - BA \in \mathrm{Ptrb}(\Phi(X))$, B is polynomially Riesz and A is in left (right) communication with B, then $A - B$ is left (right) Fredholm with the same index as A. In Theorem 3.4 we show that if A and B commute, B is polynomially Riesz, A is left (right) Browder and A is in left (right) communication with B, then $A - B$ is left (right) Browder.

This paper is divided into four sections. In the next section we recall some preliminary definitions and results from [19] concerning polynomially Riesz elements relative to a Banach algebra homomorphism and give Theorem 2.3 which generalizes the first part of Theorem 2.2 in [3]. Section 3 is devoted to perturbations of left (right) Fredholm, Weyl and Browder operators by polynomially Riesz operators and contains our main results already mentioned above. Throughout Section 4 we apply the results obtained in Section 3 in order to get results about perturbations of some shifts by polynomially Riesz operators.

2 Polynomially Riesz elements relative to a Banach algebra homomorphism

Let A and B be complex Banach algebras, with identities denoted in both cases with 1, and invertible groups A^{-1} and B^{-1}, respectively. The semigroup of left (right) invertible elements in A is denoted by A^{-1}_{left} (A^{-1}_{right}).

The *left* and *right spectrum* of $a \in A$, respectively, are defined by

$$\begin{aligned} \sigma^{left}(a) \equiv \sigma^{left}_A(a) &= \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}_{left}\} , \\ \sigma^{right}(a) \equiv \sigma^{right}_A(a) &= \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}_{right}\} . \end{aligned}$$

The *spectrum* of $a \in A$ is

$$\sigma(a) \equiv \sigma_A(a) = \sigma^{left}(a) \cup \sigma^{right}(a) .$$

The *radical* of A is the set

$$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subset A^{-1}\} = \{a \in A : 1 - aA \subset A^{-1}\}.$$

The set of *quasinilpotents* of A is

$$\mathrm{QN}(A) = \{a \in A : \sigma(a) = \{0\}\} = \{a \in A : 1 - \mathbb{C}a \subset A^{-1}\}.$$

It is well known that

$$a,\ b \in \mathrm{QN}(A),\ ab = ba \implies a + b \in \mathrm{QN}(A), \tag{2.1}$$

$$a \in A,\ b \in \mathrm{QN}(A),\ ab = ba \implies ab \in \mathrm{QN}(A). \tag{2.2}$$

A map $T : A \to B$ is a *homomorphism* if T is linear and satisfies $T(xy) = TxTy$, $x,\ y \in A$, and $T1 = 1$.

An element $a \in A$ is *left T Fredholm* if it has a left invertible image,

$$a \in T^{-1}(B^{-1}_{left}).$$

Right and two-sided T Fredholm elements are defined analogously.

An element $a \in A$ is *T Browder* if:

$$a = c + d \text{ with } c \in A^{-1},\ Td = 0,\ cd = dc.$$

The classes of two-sided T Fredholm and T Browder elements were introduced by R. Harte [10]. Fredholm theory relative to arbitrary Banach algebra homomorphism is recently discussed in [3], [14], [20], [19].

The induced left, right and two-sided T Fredholm spectra are given by

$$\sigma_T^{left}(a) = \sigma_B^{left}(Ta) \quad ; \quad \sigma_T^{right}(a) = \sigma_B^{right}(Ta);$$
$$\sigma_T(a) = \sigma_B(Ta).$$

The T Browder spectrum of a is given by

$$\beta_T(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not } T \text{ Browder}\}.$$

We shall describe $d \in A$ as *T Riesz* if

$$T(d) \in \mathrm{QN}(B)\ .$$

We shall write $\mathrm{Poly} = \mathbf{C}[z]$ for the algebra of complex polynomials.

We recall the following result ([19], Theorem 11.2):

Theorem 2.1. *If $a \in A$ and $d \in A$, if $H(B)$ is one of B^{-1}_{left}, B^{-1}_{right} and B^{-1}, and if $p \in \mathrm{Poly}$, then*

$$ad - da \in T^{-1}\mathrm{Rad}(B) \text{ and } p(d) \in T^{-1}\mathrm{QN}(B)\ ,$$

implies

$$p(a) \in T^{-1}H(B) \Longrightarrow a - d \in T^{-1}H(B)\ .$$

We shall say that $S \subseteq A$ is a *commutative ideal* if

$$S +_{comm} S \subseteq S\ ,\ A \cdot_{comm} S \subseteq S\ .$$

where we write

$$H +_{comm} K = \{c + d : (c, d) \in H \times K\ ,\ cd = dc\}$$

for the commuting sum and

$$H \cdot_{comm} K = \{c \cdot d : (c, d) \in H \times K\ ,\ cd = dc\}$$

for the commuting product of subsets $H, K \subseteq A$.

Clearly, every left or right ideal is a commutative ideal. From (2.1) and (2.2) it follows that the set $\mathrm{QN}(A)$ is a commutative ideal, as well the set $T^{-1}\mathrm{QN}(B)$.

If $S \subseteq A$ is an arbitrary set we shall write that $a \in \mathrm{Poly}^{-1}(S)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in S$. If $S \subseteq A$ is a commutative ideal, the set

$$\mathcal{P}_a^S = \{p \in \mathrm{Poly} : p(a) \in S\}$$

of polynomials p for which $p(a) \in S$ is an ideal of the algebra Poly. Since the natural numbers are well ordered there exists a unique polynomial p of minimal degree with leading coefficient 1 contained in $\mathcal{P}_a^S$ which we call the minimal polynomial of a; we shall write $p = \pi_a \equiv \pi_a^S$. Then $\mathcal{P}_a^S$ is generated by $p = \pi_a$, i.e. $\mathcal{P}_a^S = \pi_a \cdot \mathrm{Poly}$. Hence, for every $q \in \mathcal{P}_a^S$, $\pi_a^{-1}(0) \subset q^{-1}(0)$.

We say that the homomorphism T has the *strong Riesz property* if

$$\forall\ a \in A\ : \partial\sigma(a) \subset \sigma(Ta) \cup \mathrm{iso}\,\sigma(a),$$

where $\mathrm{iso}\,\sigma(a)$ denotes the set of the isolated points of $\sigma(a)$.

We recall the following result proved in [19] (see [19], Theorem 11.1):

Theorem 2.2. *Let $T : A \to B$ be a homomorphism and let $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN(B)}$. Then*

$$\sigma_T(a) = \pi_a^{-1}(0),$$

where π_a is the minimal polynomial of a.

If in particular T has the strong Riesz property, then

$$\sigma_T(a) = \beta_T(a) = \pi_a^{-1}(0).$$

From [19], Theorem 10.1 it follows that if a is left (right) T Fredholm, $d \in T^{-1}\mathrm{QN(B)}$ and $ad - da \in T^{-1}\mathrm{Rad(B)}$, then $a - d$ is left (right) T Fredholm.

The following theorem shows that it holds more generally.

Theorem 2.3. *Let $T : A \to B$ be a homomorphism and let $a,\ d \in A$ such that $ad - da \in T^{-1}\mathrm{Rad(B)}$, $d \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN(B)}$ with the minimal polynomial π_d. Then*

$$\pi_d^{-1}(0) \cap \sigma_T^{left}(a) = \emptyset \Longrightarrow a - d \text{ is left Fredholm,}$$

and

$$\pi_d^{-1}(0) \cap \sigma_T^{right}(a) = \emptyset \Longrightarrow a - d \text{ is right Fredholm.}$$

Proof. Suppose that $ad - da \in T^{-1}\mathrm{Rad(B)}$, $d \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN(B)}$ and suppose that any zero of the polynomial π_d does not belong to $\sigma_T^{left}(a)$. Then $0 \notin \pi_d(\sigma_T^{left}(a)) = \sigma_T^{left}(\pi_d(a))$ and hence $\pi_d(a)$ is left T Fredholm. From Theorem 2.1 it follows that $a - d$ is left T Fredholm. □

3 Polynomially Riesz operators

The Calkin algebra over X is the quotient algebra $C(X) = B(X)/K(X)$, and let $\Pi : B(X) \to C(X)$ denote the natural homomorphism. Let us remark that the Fredholm operators are the Fredholm elements relative to the homomorphism Π ([6], Theorem 3.2.8), the left (right) Fredholm operators are the left (right) Fredholm elements relative to Π ([5], Chapter 5.1, Theorem 5). Also, the Browder operators are the Browder elements relative to the homomorphism Π ([6], Theorem 1.4.5; [15], Theorem 20.21; [2], Theorem 3.48).

Since Π is an onto homomorphism, then $\Pi^{-1}(\mathrm{Rad}(C(X))) = \mathrm{Ptrb}\,(\Phi(X))$ ([6], Theorem 5.5.9). Recall that $A \in R(X)$ if and only if $\Pi(A)$ is quasinilpotent in $C(X)$ ([2], p. 179 and 180). In other words the Riesz operators are the Riesz elements relative to the homomorphism Π, i.e. $R(X) = \Pi^{-1}(\mathrm{QN}(C(X)))$.

From the punctured neighbourhood theorem ([15], Theorem 18.7, [11], Theorem 7.8.5) for $A \in B(X)$ it follows that

$$\partial\sigma(A) \subset \sigma_f(A) \cup \mathrm{iso}\,\sigma(A), \tag{3.1}$$

i.e. the homomorphism Π has the strong Riesz property.

Riesz operator perturbations were studied by Rakočević in [16] and recently in [18]. Baklouti [3] investigated perturbation by operators that belong to $\mathrm{Poly}^{-1}\mathrm{Ptrb}\,(\Phi(X))$. In this section we study polynomially Riesz operator perturbations.

Theorem 3.1. *Let $A,\ B \in B(X)$ such that $AB - BA \in \operatorname{Ptrb}(\Phi(X))$ and $B \in \operatorname{Poly}^{-1}R(X)$ with the minimal polynomial π_B. Then*

$$\pi_B^{-1}(0) \cap \sigma_f^{left}(A) = \emptyset \Longrightarrow A - B \text{ is left Fredholm},$$

and

$$\pi_B^{-1}(0) \cap \sigma_f^{right}(A) = \emptyset \Longrightarrow A - B \text{ is right Fredholm}.$$

Proof. Theorem 2.3, applied to the Calkin homomorphism $\Pi : B(X) \to C(X)$. □

Because of (1.2) and (1.1), we remark that Theorem 3.1 is an improvement of the first part of Theorem 2.2 in [3].

Now we define when an operator $A \in B(X)$ is in (left, right) communication with another operator $B \in B(X)$. This is a generalization of the concept of communicating operators introduced in [1] (Definition 3.1) and considered also in [3] (Definition 2.1).

Definition 3.1. Let $A,\ B \in B(X)$ and $B \in \operatorname{Poly}^{-1}R(X)$ with the minimal polynomial π_B. If there is a continuous map $\varphi : [0,1] \to \mathbb{C}$ for which

$$\varphi(0) = 0,\ \varphi(1) = 1$$

and

$$\varphi([0,1])\pi_B^{-1}(0) \cap \sigma_f(A) = \emptyset, \tag{3.2}$$

we shall say that A is *in communication with* B.

If (3.2) holds for $\sigma_f^{left}(A)$ ($\sigma_f^{right}(A)$) instead of $\sigma_f(A)$, then we shall say that A is *in left (right) communication with* B.

Clearly, if A is in communication with B, then A is in left and right communication with B.

We remark that this is in some sense a curious notation: we are insisting that B is special, namely polynomially Riesz, and also that A is (left, right) Fredholm.

Notice that if $B \in R(X)$ and A is left Fredholm, then $\pi_B(z) = z$, $0 \notin \sigma_f^{left}(A)$ and hence A is in left communication with B .

We remark that if $B \in \operatorname{Poly}^{-1}R(X)$ with the minimal polynomial π_B and $\sigma_f^{left}(A)$ is a finite set such that $\sigma_f^{left}(A) \cap (\{0\} \cup \pi_B^{-1}(0)) = \emptyset$, then A is in left communication with B.

Therefore, if $p(B) \in R(X)$ for some polynomial p, $A = \lambda I$ with $\lambda \neq 0$ and $p(\lambda) \neq 0$, then $\sigma_f(A) = \{\lambda\}$, and so A is in communication with B.

The following theorem is an improvement of the second part of Theorem 2.2 in [3].

Theorem 3.2. *For $A,\ B \in B(X)$, if $AB - BA \in \operatorname{Ptrb}(\Phi(X))$, $B \in \operatorname{Poly}^{-1}R(X)$ and if A is in left (right) communication with B, then $A - B$ is left (right) Fredholm and $i(A - B) = i(A)$.*

Proof. Suppose that $AB - BA \in \operatorname{Ptrb}(\Phi(X))$, $B \in \operatorname{Poly}^{-1}R(X)$ and A is in left communication with B. Let $\pi_B(z) = \Pi_{i=1}^{n}(z - \lambda_i)$ be the minimal polynomial of B. Then there exists a continuous map $\varphi : [0, 1] \to \mathbb{C}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and for all $i = 1, \dots, n$, $\varphi(t)\lambda_i \notin \sigma_f^{left}(A)$ for every $t \in [0, 1]$. For $t \in [0, 1]$ and $p_t(z) = \Pi_{i=1}^{n}(z - \varphi(t)\lambda_i)$ we have $p_t(\varphi(t)B) = (\varphi(t))^n \pi_B(B) \in R(X)$ and hence $\varphi(t)B \in \operatorname{Poly}^{-1}R(X)$. Since any zero of the polynomial p_t does not belong to $\sigma_f^{left}(A)$ and since $A(\varphi(t)B) - (\varphi(t)B)A \in \operatorname{Ptrb}(\Phi(X))$, by Theorem 3.1 we obtain that $A - \varphi(t)B$ is left Fredholm for every $t \in [0, 1]$. Thus $A - B$ is left Fredholm and from the local constancy of the index we get $i(A - B) = i(A)$. □

From Theorem 8 in [18] it follows that for $A,\ B \in B(X)$, if A is left (right) Weyl, B is Riesz and $AB - BA \in \operatorname{Ptrb}(\Phi(X))$, then $A - B$ is left (right) Weyl. The following theorem is an extension of this result.

Theorem 3.3. *For $A,\ B \in B(X)$, let $AB - BA \in \operatorname{Ptrb}(\Phi(X))$, $B \in \operatorname{Poly}^{-1}R(X)$ and let A be in left (right) communication with B. If A is left (right) Weyl, then $A - B$ is left (right) Weyl and $i(A - B) = i(A)$.*

Proof. From Theorem 3.2. □

For $A \in B(X)$ set $N^\infty(A) = \cup_n N(A^n)$ for the hyper-kernel of A and $R^\infty(A) = \cap_n R(A^n)$ for the hyper-range of A. We shall write $\overline{N}^\infty(A)$ for the closure of the hyper-kernel of A.

Theorem 3.4. *Let $A,\ B \in B(X)$ such that $AB = BA$ and $B \in \operatorname{Poly}^{-1}R(X)$.*

(3.4.1) *If A is in left communication with B and $asc(A) < \infty$, then A and $A - B$ are left Browder and $i(A - B) = i(A)$.*

(3.4.2) *If A is in right communication with B and $dsc(A) < \infty$, then A and $A - B$ are right Browder and $i(A - B) = i(A)$.*

Proof. (3.4.1): Let $AB = BA$, $\operatorname{asc}(A) < \infty$, $B \in \operatorname{Poly}^{-1}R(X)$ and let A be in left communication with B. Then there exists a continuous map $\varphi : [0, 1] \to \mathbb{C}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and for all λ zero of π_B, $\varphi(t)\lambda \notin \sigma_f^{left}(A)$ for every $t \in [0, 1]$. By the proof of Theorem 3.2, $A - \varphi(t)B$ is left Fredholm for every $t \in [0, 1]$ and $i(A - B) = i(A)$. Since A and B commute and $\varphi : [0, 1] \to \mathbb{C}$ is a continuous map, from [8], Theorem 3 it follows that the function $t \to \overline{N}^\infty(A - \varphi(t)B) \cap R^\infty(A - \varphi(t)B)$ is a locally constant function on the set $[0,1]$ and therefore, this function is constant on $[0,1]$. As $\mathrm{asc}(A) < \infty$, from [17], Proposition 1.6(i) it follows $\overline{N^\infty}(A) \cap R^\infty(A) = N^\infty(A) \cap R^\infty(A) = \{0\}$ and hence $\overline{N^\infty}(A-B) \cap R^\infty(A-B) = \{0\}$. It implies $N^\infty(A-B) \cap R^\infty(A-B) = \{0\}$, and again by [17], Proposition 1.6(i), we get $\mathrm{asc}(A-B) < \infty$. Thus, A and $A-B$ are left Browder.

(3.4.2): Suppose that $AB = BA$, $\mathrm{dsc}(A) < \infty$, $B \in \mathrm{Poly}^{-1}R(X)$ and A is in right communication with B. Then A and $A-B$ are right Fredholm by Theorem 3.2, $A'B' = B'A'$, $\mathrm{asc}(A') = \mathrm{dsc}(A) < \infty$ and $B' \in \mathrm{Poly}^{-1}R(X')$ by [2], Corollary 3.114. Since $\sigma_f^{left}(A') \subset \sigma_f^{right}(A)$, A' is in left communication with B'. From (3.4.1) we get $\mathrm{dsc}(A-B) = \mathrm{asc}(A'-B') < \infty$. Therefore, A and $A-B$ are right Browder. □

Baklouti proved ([3] Theorem 2.4) that if $T,\ S \in B(X)$ commute, $T \in \mathrm{Poly}^{-1}\mathrm{Ptrb}\,(\Phi(X))$, and if S is in communication with T, then $\mathrm{asc}(S) < \infty$ implies $\mathrm{asc}(T-S) < \infty$, as well $\mathrm{dsc}(S) < \infty$ implies $\mathrm{dsc}(T-S) < \infty$. We remark that Theorem 3.4 extend the result of Baklouti.

The well known result about the stability of the Browder operators under commuting Riesz operator perturbations ([16], Corollary 2) is extended in the following theorem.

Theorem 3.5. *Let $A,\ B \in B(X)$ such that $AB = BA$ and $B \in \mathrm{Poly}^{-1}R(X)$. If A is Browder and A is in communication with B, then $A-B$ is Browder.*

Proof. Follows from Theorem 3.4. □

We give a new proof for the stability of the left (right) Browder operators under commuting Riesz operator perturbations. This assertion is also proved in [18]. However, the proof of this result in [18] is based on the punctured neighbourhood theorem ([15], Theorem 18.7).

Theorem 3.6. *Let $A,\ B \in B(X)$. If A is left (right) Browder and B Riesz such that $AB = BA$, then $A+B$ is left (right) Browder and $i(A+B) = i(A)$.*

Proof. Suppose that A is left Browder, $B \in R(X)$ and $AB = BA$. Then $P = -B \in R(X)$ and hence $P \in \mathrm{Poly}^{-1}R(X)$ with the minimal polynomial $\pi_P(z) = z$. Since $0 \notin \sigma_f^{left}(A)$, we conclude that A is in left communication with P. Now from Theorem 3.4 it follows that $A-P = A+B$ is left Browder and $i(A+B) = i(B)$. □

The approximate point spectrum and the surjectivity spectrum of $A \in B(X)$, respectively, are given by

$$\begin{aligned} \sigma_a(A) &= \{\lambda \in \mathbb{C} : A-\lambda \text{ is not bounded below}\}, \\ \sigma_s(A) &= \{\lambda \in \mathbb{C} : A-\lambda \text{ is not surjective}\}. \end{aligned}$$

We shall say that an operator $A \in B(X)$ is *almost bounded below* (*almost surjective*) if there exists $\epsilon > 0$ such that $A - \lambda$ is bounded below (surjective) for every $\lambda \in \mathbb{C}$, $0 < |\lambda| < \epsilon$. In other words, A is almost bounded below (surjective) if and only if 0 is not an accumulation point of the approximate point (surjectivity) spectrum of A.

Corollary 3.1. *Let $A,\ B \in B(X)$ such that $AB = BA$ and $B \in \text{Poly}^{-1}R(X)$.*
(3.1.1) *If A is in left communication with B and if A is almost bounded below, then A and $A - B$ are left Browder and $i(A - B) = i(A)$.*
(3.1.2) *If A is in right communication with B and if A is almost surjective, then A and $A - B$ are right Browder and $i(A - B) = i(A)$.*

Proof. (3.1.1): Suppose that $AB = BA$, $B \in \text{Poly}^{-1}R(X)$ and A is in left communication with B. From Definition 3.1 it follows that $0 \notin \sigma_f^{left}(A)$, i.e. A is left Fredholm. Since A is almost bounded below, from [15], Corollary 20.20, or also from [18], Theorem 5, we have that $\text{asc}A < \infty$. Now the conclusion follows from Theorem 3.4.

(3.1.2) can be proved similarly. □

Theorem 3.7. *Let $A,\ B \in B(X)$. Then*

$$A,\ B \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_A^{-1}(0) \cap (\pi_B^{-1}(0) \cup \{0\}) = \emptyset$$

implies

$$AB - BA \in \text{Ptrb}\,(\Phi(X)) \Longrightarrow A \text{ is Browder and } A - B \text{ is Weyl},$$

and

$$AB = BA \Longrightarrow A,\ A - B \text{ are Browder}.$$

Proof. Let $A,\ B \in \text{Poly}^{-1}R(X)$, and let $\pi_A^{-1}(0) \cap (\pi_B^{-1}(0) \cup \{0\}) = \emptyset$. From $A \in \text{Poly}^{-1}R(X)$, according to Theorem 2.2, it follows that

$$\sigma_f(A) = \sigma_b(A) = \pi_A^{-1}(0). \tag{3.3}$$

Since $0 \notin \pi_A^{-1}(0)$, from (3.3) it follows that A is Browder. Now, since $\sigma_f(A)$ is finite and does not contain 0 and also does not contain any zero of the minimal polynomial of B, we conclude that A is in communication with B.

Suppose now that $AB - BA \in \text{Ptrb}\,(\Phi(X))$. From Theorem 3.3 we get that $A - B$ is Weyl.

If in particular A and B commute, from Theorem 3.5 it follows that $A - B$ is Browder. □

Theorem 3.8. *Let X be a Hilbert space and let $A \in B(X)$ be semi-Fredholm and self-adjoint. Then for $B \in B(X)$,*

$$B \in \mathrm{Poly}^{-1} R(X) \quad \text{and} \quad \pi_B^{-1}(0) \cap \mathbb{R} = \emptyset$$

implies

$$AB - BA \in \mathrm{Ptrb}\,(\Phi(X)) \Longrightarrow A - B \text{ is Weyl},$$

and

$$AB = BA \Longrightarrow A - B \text{ is Browder}.$$

Proof. Since A is self-adjoint it follows that $\alpha(A) = \beta(A)$ and so, A is Fredholm and $0 \notin \sigma_f(A)$. Because of $N(A) = N(A^*A) = N(A^2)$ we have asc$A < \infty$ and since $R(A)$ is closed, then $X = N(A) \oplus R(A)$ and hence, $R(A) = R(A^2)$, i.e. dsc$A < \infty$. Therefore, A is Browder. From selfadjointness of A it follows that $\sigma(A) \subset \mathbb{R}$ and hence, $\sigma_f(A) \subset \mathbb{R}\backslash\{0\}$. If $B \in \mathrm{Poly}^{-1}R(X)$ and $\pi_B^{-1}(0) \cap \mathbb{R} = \emptyset$, then defining $\varphi : [0,1] \to \mathbb{C}$ with $\varphi(t) = t$, $t \in [0,1]$, we have that $\varphi(t)\lambda \notin \sigma_f(A)$ for every $\lambda \in \pi_B^{-1}(0)$ and every $t \in [0,1]$. Thus A is in communication with B and the assertions follow from Theorem 3.3 and Theorem 3.5. □

4 Shifts

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathbb{C}^{\mathbb{N}_0}$ be the linear space of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ℓ_∞, c and c_0 denote the set of bounded, convergent and convergent sequences with null limit. We write $\ell_p = \{x \in \mathbb{C}^{\mathbb{N}_0} : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \le p < \infty$. For $n = 0, 1, 2, \ldots$, let $e^{(n)}$ denote the sequences such that $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. The forward and the backward unilateral shifts U and V are linear operators on $\mathbb{C}^{\mathbb{N}_0}$ defined by

$$Ue^{(n)} = e^{(n+1)} \quad \text{and} \quad Ve^{(n+1)} = e^{(n)}, \quad n = 0, 1, 2, \ldots .$$

Invariant subspaces for U and V include c_0, c, ℓ_∞ and ℓ_p, $p \ge 1$. Recall that for every $1 \le p < \infty$,

$$\ell_1 \subset \ell_p \subset c_0 \subset c \subset \ell_\infty,$$

and for each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, U, $V \in B(X)$ and $\|U\| = \|V\| = 1$. On the Hilbert space ℓ_2 we also have that V is the Hilbert conjugate operator of U, $V = U^*$.

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ and $\mathbb{S} = \partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. It is well known that, for $X = \ell_2$, $\sigma(U) = \sigma(V) = \mathbb{D}$ ([9], Solution 67; [7], Proposition 27.7 (b)), while from [13], Proposition 1.6.15, it follows that $\sigma(U) = \mathbb{D}$ for $X = \ell_p$, $p \ge 1$.

Theorem 4.1. *For each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and the forward and backward unilateral shifts U, $V \in B(X)$ there are equalities*

$$\sigma(U) = \sigma(V) = \mathbb{D}. \tag{4.1}$$

Proof. Since

$$\|U\| = \|V\| = 1 \tag{4.2}$$

it is clear that

$$\sigma(U) \cup \sigma(V) \subset \mathbb{D}. \tag{4.3}$$

Observe

$$V(1 - UV) = 0 \neq I - UV;$$

also

$$N(V) = (I - UV)X \neq \{0\}. \tag{4.4}$$

From (4.2) it is clear that

$$|\lambda| < 1 \Longrightarrow I - \lambda U \in B(X)^{-1}. \tag{4.5}$$

Since $V - \lambda = V(I - \lambda U)$, from (4.4) and (4.5) it follows

$$N(V - \lambda) = (V - \lambda)^{-1}(0) = (I - \lambda U)^{-1}V^{-1}(0) \neq \{0\}, \tag{4.6}$$

and hence, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma(V)$. Now, since $\sigma(V)$ is closed we obtain

$$\mathbb{D} \subset \sigma(V). \tag{4.7}$$

Since U is not surjective, it follows that $0 \in \sigma(U)$. Suppose that $\lambda \in \mathbb{C}$ and $0 < |\lambda| < 1$. We show that $e_0 \notin R(\lambda - U)$. If there exists $x = (x_k)_{k=0}^\infty$ such that $(\lambda - U)x = e_0$, then

$$(\lambda x_0, \lambda x_1 - x_0, \lambda x_2 - x_1, \dots) = (1, 0, 0, \dots)$$

and hence

$$x = (\frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}, \dots),$$

which is not a bounded sequence and so, it is not in X. Therefore, $\lambda \in \sigma(U)$ and hence, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma(U)$. Consequently,

$$\mathbb{D} \subset \sigma(U). \tag{4.8}$$

From (4.3), (4.7) and (4.8) it follows (4.1). $\square$

For the case $X = \ell_2$, it is known that $\sigma_f(U) = \mathbb{S}$ ([4], Example 1.2; [7], Proposition 27.7(b)). In [1], Remark 2.9 it was shown that $\sigma_f(V) = \mathbb{S}$ for $X = \ell_p$, $p \geq 1$.

Theorem 4.2. *For each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and the forward and backward unilateral shifts U, $V \in B(X)$ there are equalities*

$$\sigma_f(U) = \sigma_f(V) = \mathbb{S}. \tag{4.9}$$

Proof. From (4.1) we have

$$\sigma_f(U) \cup \sigma_f(V) \subset \mathbb{D}. \tag{4.10}$$

Observe

$$VU = I \quad \text{and} \quad UV = I - P_0,$$

where P_0 is the projector defined by $P_0(x_0, x_1, x_2, \dots) = (x_0, 0, 0, \dots)$. Clearly, $P_0 \in K(X)$ and we have that $\Pi(U)$ is invertible in the Calkin algebra $C(X)$ and $\Pi(V)$ is its inverse. Hence U and V are Fredholm. Let $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$. Since $\dfrac{1}{|\lambda|} > 1 \geq \|\Pi(V)\|$, it follows that $\dfrac{1}{\lambda} \notin \sigma(\Pi(V))$ and therefore, $\lambda \notin \sigma((\Pi(V))^{-1}) = \sigma(\Pi(U))$. Analogously we conclude that $\lambda \notin \sigma(\Pi(V))$. Hence $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \cap \sigma_f(U) = \emptyset$ and $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \cap \sigma_f(V) = \emptyset$, which togeather with (4.10) gives

$$\sigma_f(U) \cup \sigma_f(V) \subset \mathbb{S}. \tag{4.11}$$

From the strong Riesz property (3.1) and (4.1) we obtain

$$\mathbb{S} \subset \sigma_f(U) \text{ and } \mathbb{S} \subset \sigma_f(V). \tag{4.12}$$

From (4.11) and (4.12) we get (4.9). □

We remark that U and V are Fredholm with $i(U) = -1$ and $i(V) = 1$. Since $\operatorname{asc} U = 0$, $\operatorname{dsc} U = +\infty$, $\operatorname{asc} V = \infty$ and $\operatorname{dsc} V = 0$ it follows that U is left Browder but not right Browder, while V is right Browder but not left Browder.

Theorem 4.3. *Let $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and let $U \in B(X)$ be the forward unilateral shift. Then for $T \in B(X)$,*

$$T \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

implies

$$UT - TU \in \operatorname{Ptrb}(\Phi(X)) \Longrightarrow U - T \text{ is Fredholm and } i(U-T) = -1,$$

and

$$UT = TU \Longrightarrow U - T \text{ is Fredholm, left Browder and } i(U-T) = -1.$$

Proof. Suppose that $T \in \mathrm{Poly}^{-1}R(X)$ and $\pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Then defining $\varphi : [0,1] \to \mathbb{C}$ with $\varphi(t) = t$, $t \in [0,1]$, we have that $\varphi(t)\lambda \notin \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \sigma_f(U)$ for every $\lambda \in \pi_T^{-1}(0)$ and every $t \in [0,1]$ and so, we conclude that the unilateral shift U is in communication with T.

If $UT - TU \in \operatorname{Ptrb}(\Phi(X))$, from Theorem 3.2 it follows that $U - T$ is Fredholm and $i(U-T) = i(U) = -1$.

Moreover, if $UT = TU$, then, since U is left Browder, from Theorem 3.4 it follows that $U - T$ is left Browder. □

Theorem 4.4. *Let $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and let $V \in B(X)$ be the backward unilateral shift. Then for $T \in B(X)$,*

$$T \in \mathrm{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

implies

$$VT - TV \in \operatorname{Ptrb}(\Phi(X)) \Longrightarrow V - T \text{ is Fredholm and } i(V-T) = 1,$$

and

$$VT = TV \Longrightarrow V - T \text{ is Fredholm, right Browder and } i(V-T) = 1.$$

Proof. Analogously to the proof of Theorem 4.3. □

Let $\mathbb{C}^{\mathbb{Z}}$ be the linear space of all complex sequences $x = (x_k)_{k=-\infty}^{\infty}$. Let $c_0(\mathbb{Z})$ be the set of all sequences $x = (x_k)_{k=-\infty}^{\infty}$ such that $\lim_{k\to\infty} x_k = \lim_{k\to\infty} x_{-k} = 0$, i.e. $x_k \to 0$ when $|k| \to \infty$. For $x = (x_k)_{k=-\infty}^{\infty} \in c_0(\mathbb{Z})$ set $\|x\| = \sup_k |x_k|$. We write $\ell_p(\mathbb{Z}) = \{x \in \mathbb{C}^{\mathbb{Z}} : \sum_{k=-\infty}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$, and for $x = (x_k)_{k=-\infty}^{\infty} \in \ell_p(\mathbb{Z})$, $\|x\| = \left(\sum_{k=-\infty}^{\infty} |x_k|^p\right)^{1/p}$. Remark that $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$ are Banach spaces.

For $k = \dots, -2, -1, 0, 1, 2, \dots$, let $\delta^{(k)}$ denote the sequences such that $\delta_k^{(k)} = 1$ and $\delta_i^{(k)} = 0$ for $i \neq k$. The forward and the backward bilateral shifts W_1 and W_2 are linear operators on $\mathbb{C}^{\mathbb{Z}}$ defined by

$$W_1\delta^{(k)} = \delta^{(k+1)} \quad \text{and} \quad W_2\delta^{(k+1)} = \delta^{(k)}, \quad k = \dots, -2, -1, 0, 1, 2, \dots .$$

Obviously, $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$ are invariant subspaces for W_1 and W_2, and $W_1^{-1} = W_2$. For each $X \in \{c_0(\mathbb{Z}), \ell_p(\mathbb{Z})\}$, W_1 and W_2 are isometries. On the Hilbert space $\ell_2(\mathbb{Z})$ we have that W_2 is the Hilbert conjugate operator of W_1, that is W_1 and W_2 are unitary.

For $X = \ell_2(\mathbb{Z})$ it is known that $\sigma(W_1) = \sigma(W_2) = \mathbb{S}$ and $\sigma_f(W_1) = \mathbb{S}$ ([9], Solution 68; [7], Proposition 27.7 (c)).

Theorem 4.5. *If X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, then for the forward and backward bilateral shifts $W_1,\ W_2 \in B(X)$ there are equalities*

$$\sigma(W_1) = \sigma(W_2) = \mathbb{S}. \tag{4.13}$$

Proof. Since

$$\|W_1\| = \|W_2\| = 1 \tag{4.14}$$

it follows that

$$\sigma(W_1) \cup \sigma(W_2) \subset \mathbb{D}. \tag{4.15}$$

Let $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$. Then $1/|\lambda| > 1$ and from (4.14) it follows that $1/\lambda \notin \sigma(W_2)$ and hence $\lambda \notin \sigma(W_2^{-1}) = \sigma(W_1)$. From (4.15) it follows that

$$\sigma(W_1) \subset \mathbb{S}.$$

Suppose that $\lambda \in \mathbb{S}$. We prove that $R(\lambda - W_1)$ does not contain δ_0. For $x = (x_k)_{k=-\infty}^{\infty} \in X$, from $(\lambda - W_1)x = \delta_0$ we get

$$\ldots, \lambda x_{-2} - x_{-3} = 0, \lambda x_{-1} - x_{-2} = 0, \lambda x_0 - x_{-1} = 1, \lambda x_1 - x_0 = 0, \lambda x_2 - x_1 = 0, \ldots,$$

and hence

$$\begin{gathered} x_1 = \tfrac{1}{\lambda} x_0,\ x_2 = \tfrac{1}{\lambda^2} x_0,\ x_3 = \tfrac{1}{\lambda^3} x_0, \ldots \\ x_{-2} = \lambda x_{-1},\ x_{-3} = \lambda^2 x_{-1},\ \ldots\ . \end{gathered}$$

From $\lim_{k\to\infty} x_k = 0$, $\lim_{k\to\infty} x_{-k} = 0$ and $|\lambda| = 1$ we conclude that $x_0 = 0$ and $x_{-1} = 0$ which contradict the fact that $\lambda x_0 - x_{-1} = 1$.

Consequently, $\sigma(W_1) = \mathbb{S}$. Hence $\sigma(W_2) = \sigma(W_1^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(W_1)\} = \mathbb{S}$. ☐

Theorem 4.6. *If X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, then for the forward and backward bilateral shifts $W_1,\ W_2 \in B(X)$ there are equalities*

$$\sigma_f(W_1) = \sigma_f(W_2) = \mathbb{S}. \tag{4.16}$$

Proof. From (4.13) it follows that

$$\partial\sigma(W_1) = \operatorname{acc}\sigma(W_1) = \mathbb{S}, \tag{4.17}$$

where $\operatorname{acc}\sigma(W_1)$ denotes the set of the accumulation points of $\sigma(W_1)$. From (3.1) and (4.17) it follows that

$$\mathbb{S} \subset \sigma_f(W_1). \tag{4.18}$$

Since $\sigma_f(W_1) \subset \sigma(W_1)$, from (4.13) and (4.18) we get $\sigma_f(W_1) = \mathbb{S}$. Analogously, $\sigma_f(W_2) = \mathbb{S}$. ☐

Theorem 4.7. *Let X be one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$ and let $W_1 \in B(X)$ be the forward bilateral shift. Then for $T \in B(X)$,*

$$T \in \mathrm{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

implies

$$W_1T - TW_1 \in \mathrm{Ptrb}\,(\Phi(X)) \Longrightarrow W_1 - T \text{ is Weyl},$$

and

$$W_1T = TW_1 \Longrightarrow W_1 - T \text{ is Browder}.$$

Proof. Suppose that $T \in \mathrm{Poly}^{-1}R(X)$ and $\pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. From (4.16), as in the proof of Theorem 4.3, we conclude that the bilateral shift W_1 is in communication with T.

If $W_1T - TW_1 \in \mathrm{Ptrb}\,(\Phi(X))$, from Theorem 3.3 we get that $W_1 - T$ is Weyl.

Suppose that $W_1T = TW_1$. Since W_1 is invertible, it is Browder and from Theorem 3.5 it follows that $W_1 - T$ is Browder. ☐

Theorem 4.8. *Let X be one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$ and let $W_2 \in B(X)$ be the backward bilateral shift. Then for $T \in B(X)$,*

$$T \in \mathrm{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

implies

$$W_2T - TW_2 \in \mathrm{Ptrb}\,(\Phi(X)) \Longrightarrow W_2 - T \text{ is Weyl},$$

and

$$W_2T = TW_2 \Longrightarrow W_2 - T \text{ is Browder}.$$

Proof. Analogously to the proof of Theorem 4.7. ☐

References

[1] B. Abdelmoumen and H. Baklouti, *Perturbation results on semi-Fredholm operators and applications*, J. Inequal. Appl. (2009), doi:10.1155/2009/284526, Article ID 284526, 13 pages.

[2] P. Aiena, Fredholm and local spectral theory with applications to multipliers, Kluwer (2004).

[3] H. Baklouti, *T-Fredholm analysis and applications to operator theory*, Jour. Math. Anal. Appl. **369** (2010) 283-289.

[4] S. K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. J., **20**(1970), 529-544.

[5] S.R. Caradus, *Generalized inverses and operator theory*, Queens Papers Pure Appl. Math. **38**, Queens University Kingston Ontario 1978.

[6] S.R. Caradus, W.E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Dekker 1974.

[7] J. B. Conway, *A Course in Operator Theory*, American Mathematical Society, Providence, RI, USA, 1999.

[8] M. A. Goldman and S. N. Kračkovskii, *Behaviour of the space of zero elements with finite-dimensional salient on the Riesz kernel under perturbations of the operator*, Dokl. Akad. Nauk SSSR **221**(1975), 532-534 (in Russian); English transl.: Soviet Math. Dokl. **16**(1975), 370-373.

[9] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, 1967.

[10] R.E. Harte, *Fredholm theory relative to a Banach algebra homomorphism*, Math. Zeit. **179** (1982) 431-436.

[11] R.E. Harte, *Invertibility and singularity*, Dekker 1988.

[12] H. G. Heuser, *Functional Analysis*, Wiley Interscience, Chichester, 1982.

[13] K. B. Laursen and M. Neumann, *An Introduction to Local Spectral Theory*, London, UK, 2000.

[14] H. du T. Mouton, S. Mouton and H. Raubenheimer, *Ruston elements and Fredholm theory relative to arbitrary homomorphisms*, Quaestiones Mathematicae 34(3) (2011), 341-359.

[15] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Birkhäuser 2007.

[16] V. Rakočević, *Semi-Browder operators and perturbations*, Studia Math. **122**(1996), 131-137.

[17] T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. Roy. Irish Acad. Sect. A **87**(1987), 137-146.

[18] S.Č. Živković-Zlatanović, D.S. Djordjević and R.E. Harte, *On left and right Browder operators*, Jour. Korean Math. Soc. **48**(2011), 1053-1063.

[19] S.Č. Živković-Zlatanović, D.S. Djordjević and R.E. Harte, *Ruston, Riesz and perturbation classes*, J. Math. Anal. Appl. **389**(2012), 871-886.

[20] S.Č. Živković-Zlatanović and R.E. Harte, *On almost essentially Ruston elements of a Banach algebra*, Filomat **24(3)** (2010) 149-155.

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