Several expressions for the generalized Drazin inverse of a block matrix in a Banach algebra

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Abstract

We present some new representations for the generalized Drazin inverse of a block matrix with generalized Schur complement being generalized Drazin invertible in a Banach algebra under conditions weaker than those used in recent papers on the subject.

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1 Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. For $a \in \mathcal{A}$, the symbols $\sigma(a)$ and $\rho(a)$ will denote the spectrum and the resolvent set of a, respectively. We use $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively, to denote the sets of all nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$.

The concept of the generalized Drazin inverse in Banach algebras was introduced by Koliha (see [16]). In [13], Harte presented an alternative definition of a generalized Drazin inverse in a ring. For $a \in \mathcal{A}$, if there exists an element $b \in \mathcal{A}$ which satisfies

$$bab = b, \qquad ab = ba, \qquad a - a^2b \in \mathcal{A}^{qnil},$$

then b is called the generalized Drazin inverse of a (or Koliha–Drazin inverse of a), and a is generalized Drazin invertible. If the generalized Drazin inverse of a exists, it is unique and denoted by a^d. The set of all generalized Drazin invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^d$. If $a \in \mathcal{A}^d$, the spectral idempotent a^π of a corresponding to the set $\{0\}$ is given by $a^\pi = 1 - aa^d$. The Drazin inverse is a special case of the generalized Drazin inverse for which $a - a^2b \in \mathcal{A}^{nil}$. Obviously, if a is Drazin invertible, then it is generalized Drazin invertible. The group inverse is the Drazin inverse for which the condition $a - a^2b \in \mathcal{A}^{nil}$ is replaced with $a = aba$. We use $a^\#$ to denote the group inverse of a, and we use $\mathcal{A}^\#$ to denote the set of all group invertible elements of $\mathcal{A}$.

We need the following important result from [3].

Lemma 1.1. [3, Lemma 2.4] *Let $b, q \in \mathcal{A}^{qnil}$ and let $qb = 0$. Then $q + b \in \mathcal{A}^{qnil}$.*

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The next result is proved for matrices [15, Theorem 2.1], for bounded linear operators [12, Theorem 2.3] and for elements of Banach algebra [3].

Lemma 1.2. [3, Example 4.5] *Let $a, b \in \mathcal{A}^d$ and let $ab = 0$. Then*

$$(a+b)^d = \sum_{n=0}^{\infty}(b^d)^{n+1}a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

If $a \in \mathcal{A}^{qnil}$, then a^d exists and $a^d = 0$. Consequently, by Lemma 1.2, the following lemma, which the part (i) is proved by Castro–González and Koliha [3] and part (ii) for bounded linear operators in [12, Theorem 2.2], holds.

Lemma 1.3. *Let $b \in \mathcal{A}^d$ and $a \in \mathcal{A}^{qnil}$.*

(i) [3, Corollary 3.4] *If $ab = 0$, then $a + b \in \mathcal{A}^d$ and $(a+b)^d = \sum\limits_{n=0}^{\infty}(b^d)^{n+1}a^n$.*

(ii) *If $ba = 0$, then $a + b \in \mathcal{A}^d$ and $(a+b)^d = \sum\limits_{n=0}^{\infty} a^n (b^d)^{n+1}$.*

Recall that, if $p = p^2 \in \mathcal{A}$ is an idempotent, we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{1}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ and let the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$.

The Drazin inverse is very useful, and has various applications in singular differential or difference equations, Markov chains, cryptography, iterative method and numerical analysis (see [6, 7]).

The problem of finding an explicit representation for the Drazin inverse of a complex block matrix in terms of its blocks was first proposed by Campbell and Meyer [5]. This problem is quite complicated, and there was no explicit formula for the Drazin inverse of a block matrix. Some special cases have been considered in [4, 9, 14, 18, 22, 23, 24].

The generalized Schur complement plays an important role in the representations for the Drazin inverse of a block matrix. Miao [18] and Wei [23] have been studied the Drazin inverse of a block matrix with the generalized Schur complement being nonsingular or it is equal to zero.

The following result is well-known for complex matrices (see [18]) and it is proved for elements of Banach algebra [19].

Lemma 1.4. [19, Lemma 2.2] *Let* $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ *and let* $w = aa^d + a^d bca^d$ *be such that* $aw \in (p\mathcal{A}p)^d$. *If* $ca^\pi = 0$, $a^\pi b = 0$ *and the generalized Schur complement* $x^s = d - ca^d b$ *is equal to 0, then* $x \in \mathcal{A}^d$ *and* $x^d = m$, *where*

$$m = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} [(aw)^d]^2 a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^d b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [(aw)^d]^2 a & [(aw)^d]^2 b \\ ca^d[(aw)^d]^2 a & ca^d[(aw)^d]^2 b \end{bmatrix}. \tag{2}$$

We use the next lemma which is proved in [20]. The expression (3) is called the generalized Banachiewicz–Schur form of x^d. For more details see [1, 2, 4, 8, 14, 21].

Lemma 1.5. [20, Lemma 2.1] *Let* x *be defined as in* (1). *Then the following statements are equivalent:*

(i) $x \in \mathcal{A}^d$ *and* $x^d = r$, *where*

$$r = \begin{bmatrix} a^d + a^d b(x^s)^d ca^d & -a^d b(x^s)^d \\ -(x^s)^d ca^d & (x^s)^d \end{bmatrix}; \tag{3}$$

(ii) $a^\pi b = b(x^s)^\pi, \quad (x^s)^\pi c = ca^\pi \quad$ *and* $\quad y = \begin{bmatrix} aa^\pi & b(x^s)^\pi \\ ca^\pi & x^s(x^s)^\pi \end{bmatrix} \in \mathcal{A}^{qnil}$.

Hartwig et al. [14] presented formulae for the Drazin inverse of a 2×2 block matrix which involve a matrix in the form (3) when the generalized Schur complement is nonsingular, and a matrix in the form (2) when it is equal to zero, under the assumptions $CA^\pi B = 0$ and $AA^\pi B = 0$. Under different conditions and the hypothesis the Schur complement is either nonsingular or zero, these results are generalized in [17, 24].

Deng et al. [11] obtained the explicit representations for the generalized Drazin inverse of operator matrix under the cases that the generalized Schur complement is either nonsingular or zero and
(i) $A^\pi BC = 0$, $CA^\pi B = 0$, $A^\pi AB = A^\pi BD$, or
(ii) $BCA^\pi = 0$, $CA^\pi B = 0$, $CA^\pi A = DCA^\pi$.

Castro-González and Martínez-Serrano [4] investigated conditions under which the Drazin inverse of a block matrix having generalized Schur complement group invertible, can be expressed in terms of a matrix in the Banachiewicz-Schur form and its powers.

In [9], Deng and Wei introduced several explicit representations for the Drazin inverse of block-operator matrix with Drazin invertible Schur complement under certain circumstances.

We introduce new explicit expressions for the generalized Drazin inverse of a block matrix x defined in (1) when the generalized Schur complement is generalized Drazin invertible. Expressions presented in Section 2 contains a matrix in the form (2) under the conditions $a^\pi bc = 0$ and $x^s c = 0$ (or $bca^\pi = 0$ and $bx^s = 0$). In Section 3 we derive formulae for the generalized Drazin inverse x^d which involve the generalized Banachiewicz-Schur form (3). Thus, we extend some results in [11, 19, 23] to more general settings.

2 Expressions for the generalized Drazin inverse

In the following theorem we derive a formula for the generalized Drazin inverse of block matrix x in (1) under some conditions. This formula is rather cumbersome and complicated but the theorem itself will have a number of useful consequences.

Theorem 2.1. *Let x be defined as in (1) where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, m be defined as in (2) and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathcal{A}p)^d$. If*

$$a^\pi bc = 0 \quad \text{and} \quad x^s c = 0, \tag{4}$$

then $x \in \mathcal{A}^d$ and

$$\begin{aligned} x^d &= \sum_{n=0}^{\infty} m^{n+1} \left(\begin{bmatrix} p & 0 \\ 0 & (x^s)^\pi \end{bmatrix} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & ca^n a^\pi b[(x^s)^d]^{n+2} \end{bmatrix} \right) \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & x^s \end{bmatrix}^n \\ &+ \begin{bmatrix} 0 & a^\pi b[(x^s)^d]^2 - (aw)^d b (x^s)^d \\ 0 & (x^s)^d - ca^d (aw)^d b (x^s)^d \end{bmatrix} \\ &+ \sum_{n=1}^{\infty} \begin{bmatrix} 0 & (a - (aw)^d bc) a^{n-1} a^\pi b[(x^s)^d]^{n+2} \\ 0 & ((1-p) - ca^d (aw)^d b) ca^{n-1} a^\pi b[(x^s)^d]^{n+2} \end{bmatrix} \\ &+ \sum_{n=1}^{\infty} \left(\begin{bmatrix} 0 & aa^d (aw)^{n-1} (aw)^\pi b[(x^s)^d]^{n+1} \\ 0 & ca^d (aw)^{n-1} (aw)^\pi b[(x^s)^d]^{n+1} \end{bmatrix} \right. \\ &+ \left. \sum_{k=1}^{\infty} \begin{bmatrix} 0 & aa^d (aw)^{n-1} (aw)^\pi bca^{k-1} a^\pi b[(x^s)^d]^{k+n+2} \\ 0 & ca^d (aw)^{n-1} (aw)^\pi bca^{k-1} a^\pi b[(x^s)^d]^{k+n+2} \end{bmatrix} \right). \end{aligned} \tag{5}$$

Proof. Suppose that $x = y + z$, where

$$y = \begin{bmatrix} a^2 a^d & aa^d b \\ caa^d & ca^d b \end{bmatrix} \qquad \text{and} \qquad z = \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & x^s \end{bmatrix}. \tag{6}$$

Then $zy = 0$, by $a^\pi a^d = 0$ and (4).

In order to show that $y \in \mathcal{A}^d$, let $A_y \equiv a^2 a^d$, $B_y \equiv aa^d b$, $C_y \equiv caa^d$ and $D_y \equiv ca^d b$. Since $(a^2 a^d)^\# = a^d$, then $A_y \in (p\mathcal{A}p)^\#$, $A_y^\pi B_y = a^\pi aa^d b = 0$, $C_y A_y^\pi = caa^d a^\pi = 0$, $y^s = D_y - C_y A_y^\# B_y = 0$ and $W_y = A_y A_y^\# + A_y^\# B_y C_y A_y^\# = w$. Using Lemma 1.4, we deduce that $y \in \mathcal{A}^d$ and $y^d = m$.

If

$$z_1 = \begin{bmatrix} 0 & 0 \\ 0 & x^s \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad z_3 = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix},$$

we have $z = z_1 + z_2 + z_3$, $z_2 z_3 = 0$, $z_3 z_1 = 0$, $z_2^2 = 0$ and $z_1(z_2 + z_3) = 0$. From $aa^\pi \in (p\mathcal{A}p)^{qnil}$ and $\sigma(z_3) \subseteq \sigma_{p\mathcal{A}p}(aa^\pi) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(0)$, we conclude that $z_3 \in \mathcal{A}^{qnil}$. By $z_2 \in \mathcal{A}^{nil}$ and Lemma 1.1, $z_2 + z_3 \in \mathcal{A}^{qnil}$. Observe that $z_1 \in \mathcal{A}^d$ and Lemma 1.3(ii) imply $z \in \mathcal{A}^d$ and

$$z^d = z_1^d + \sum_{n=1}^{\infty} (z_2 + z_3)^n (z_1^d)^{n+1} = z_1^d + \sum_{n=1}^{\infty} z_3^{n-1} (z_2 + z_3)(z_1^d)^{n+1} = z_1^d + \sum_{n=0}^{\infty} z_3^n z_2 (z_1^d)^{n+2}.$$

Now, by Lemma 1.2, $x \in \mathcal{A}^d$ and $x^d = x_1 + x_2$, where

$$x_1 = \sum_{n=0}^{\infty} (y^d)^{n+1} z^n z^\pi \qquad \text{and} \qquad x_2 = \sum_{n=0}^{\infty} y^\pi y^n (z^d)^{n+1}.$$

The equality

$$zz^d = (z_1 + z_2) z_1^d + \sum_{n=0}^{\infty} z_3^{n+1} z_2 (z_1^d)^{n+2} = \begin{bmatrix} 0 & a^\pi b (x^s)^d \\ 0 & x^s (x^s)^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^{n+1} a^\pi b [(x^s)^d]^{n+2} \\ 0 & ca^n a^\pi b [(x^s)^d]^{n+2} \end{bmatrix}$$

gives

$$\begin{aligned} x_1 &= \sum_{n=0}^{\infty} (y^d)^{n+1} \left(\begin{bmatrix} p & -a^\pi b (x^s)^d \\ 0 & (x^s)^\pi \end{bmatrix} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^{n+1} a^\pi b [(x^s)^d]^{n+2} \\ 0 & ca^n a^\pi b [(x^s)^d]^{n+2} \end{bmatrix} \right) z^n \\ &= \sum_{n=0}^{\infty} (y^d)^{n+1} \left(\begin{bmatrix} p & 0 \\ 0 & (x^s)^\pi \end{bmatrix} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & ca^n a^\pi b [(x^s)^d]^{n+2} \end{bmatrix} \right) z^n. \end{aligned} \tag{7}$$

From $yz_2 = 0$, we obtain

$$\begin{aligned} x_2 &= y^\pi z^d + \sum_{n=1}^{\infty} y^\pi y^n \left((z_1^d)^{n+1} + \sum_{k=0}^{\infty} z_3^k z_2 (z_1^d)^{k+n+2} \right) \\ &= y^\pi z_1^d + z_2 (z_1^d)^2 + y^\pi \sum_{n=1}^{\infty} z_3^n z_2 (z_1^d)^{n+2} + \sum_{n=1}^{\infty} y^\pi y^n \left((z_1^d)^{n+1} + \sum_{k=1}^{\infty} z_3^k z_2 (z_1^d)^{k+n+2} \right) \end{aligned} \tag{8}$$

Applying $aa^d(aw) = aw = (aw)aa^d$, we get

$$y^\pi = 1 - yy^d = \begin{bmatrix} p - (aw)^d a & -(aw)^d b \\ -ca^d (aw)^d a & (1-p) - ca^d (aw)^d b \end{bmatrix}$$

and

$$y^n y^\pi = \begin{bmatrix} aa^d (aw)^{n-1} (aw)^\pi a & aa^d (aw)^{n-1} (aw)^\pi b \\ ca^d (aw)^{n-1} (aw)^\pi a & ca^d (aw)^{n-1} (aw)^\pi b \end{bmatrix} \qquad (n = 1, 2, \dots).$$

Hence, by (8), observe that

$$\begin{aligned} x_2 &= \begin{bmatrix} 0 & a^\pi b [(x^s)^d]^2 - (aw)^d b (x^s)^d \\ 0 & (x^s)^d - ca^d (aw)^d b (x^s)^d \end{bmatrix} + y^\pi \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1} a^\pi & 0 \end{bmatrix} \begin{bmatrix} 0 & a^\pi b [(x^s)^d]^{n+2} \\ 0 & 0 \end{bmatrix} \\ &+ \sum_{n=1}^{\infty} y^n y^\pi \left(\begin{bmatrix} 0 & 0 \\ 0 & [(x^s)^d]^{n+1} \end{bmatrix} + \sum_{k=1}^{\infty} \begin{bmatrix} 0 & a^k a^\pi b [(x^s)^d]^{k+n+2} \\ 0 & ca^{k-1} a^\pi b [(x^s)^d]^{k+n+2} \end{bmatrix} \right). \end{aligned} \tag{9}$$

Therefore, (7) and (9) imply (5). $\square$

We can see that the conditions $a^\pi bc = 0$ and $x^s c = 0$ are equivalent with the following geometrical conditions:

$$bc\mathcal{A} \subset a\mathcal{A} \qquad \text{and} \qquad c\mathcal{A} \subset (x^s)^\circ,$$

where $x^\circ = \{y \in \mathcal{A} : xy = 0\}$.

We give an example to illustrate our results.

Example 2.1. Let $\mathcal{A}$ be a Banach algebra, $p \in \mathcal{A}$ be an idempotent, and let $x = \begin{bmatrix} p & b \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$ relative to the idempotent p. Since $a^d = a = p$, $a^\pi = 0$, $x^s = 0 = (x^s)^d$, $(x^s)^\pi = 1 - p$ and $w = p = aw = (aw)^d$, by Theorem 2.1, it follows that $x \in \mathcal{A}^d$ and $x^d = \begin{bmatrix} p & b \\ 0 & 0 \end{bmatrix}$.

The following result is a straightforward application of Theorem 2.1.

Corollary 2.1. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, m be defined as in* (2) *and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathcal{A}p)^d$.*

(i) *If $a^\pi bc = 0$ and $x^s = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = m + m^2 \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} + \sum_{n=2}^{\infty} m^{n+1} \begin{bmatrix} 0 & 0 \\ ca^{n-1}a^\pi & ca^{n-2}a^\pi b \end{bmatrix}.$$

(ii) *If $a^\pi bc = 0$, $ca^\pi b = 0$, $aa^\pi b = a^\pi bd$ and $x^s = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = m\left(1 + \sum_{n=0}^{\infty} m^{n+1} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix}\right).$$

Proof. The part (ii) follows from (i) and $aa^\pi b = a^\pi b(x^s + ca^d b) = 0$. $\square$

Observe that the part (ii) of Corollary 2.1 recovers [11, Theorem 11], because the equality $m - \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} m^2 = m$ holds.

Following the same strategy as in the proof of Theorem 2.1, we obtain a next representation for x^d. For the sake of clarity of presentation, the short proof is given.

Theorem 2.2. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, m be defined as in* (2) *and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathcal{A}p)^d$. If*

$$bca^\pi = 0 \quad \textit{and} \quad bx^s = 0,$$

then $x \in \mathcal{A}^d$ *and*

$$
\begin{aligned}
x^d &= \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^2 ca^\pi - (x^s)^d ca^d (aw)^d a & (x^s)^d - (x^s)^d ca^d (aw)^d b \end{bmatrix} \\
&+ \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^{n+2} ca^{n-1} a^\pi (a - bca^d (aw)^d a) & \left[(x^s)^d\right]^{n+2} ca^{n-1} a^\pi b((1-p) - ca^d (aw)^d b) \end{bmatrix} \\
&+ \sum_{n=1}^{\infty} \left(\begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^{n+1} ca^d (aw)^{n-1} (aw)^\pi a & \left[(x^s)^d\right]^{n+1} ca^d (aw)^{n-1} (aw)^\pi b \end{bmatrix} \right. \\
&+ \left. \sum_{k=1}^{\infty} \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^{k+n+2} ca^{k-1} a^\pi bca^d (aw)^{n-1} (aw)^\pi a & \left[(x^s)^d\right]^{k+n+2} ca^{k-1} a^\pi bca^d (aw)^{n-1} (aw)^\pi b \end{bmatrix} \right) \\
&+ \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & x^s \end{bmatrix}^n \left(\begin{bmatrix} p & 0 \\ 0 & (x^s)^\pi \end{bmatrix} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & \left[(x^s)^d\right]^{n+2} ca^n a^\pi b \end{bmatrix} \right) m^{n+1}. \qquad (10)
\end{aligned}
$$

Proof. Let y and z be defined as in (6). Therefore, $x = y + z$ and $yz = 0$. By Lemma 1.4, we get $y \in \mathcal{A}^d$ and $y^d = m$.

Assume that

$$
z_1 = \begin{bmatrix} 0 & 0 \\ 0 & x^s \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} \quad \text{and} \quad z_3 = \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & 0 \end{bmatrix}.
$$

Now $z = z_1 + z_2 + z_3$, $z_1 \in \mathcal{A}^d$, $z_2 \in \mathcal{A}^{nil}$ and $z_3 \in \mathcal{A}^{qnil}$. The equality $z_3 z_2 = 0$ and Lemma 1.1 give $z_2 + z_3 \in \mathcal{A}^{qnil}$. Because $(z_2 + z_3) z_1 = 0$, by Lemma 1.3(i), $z \in \mathcal{A}^d$ and

$$
z^d = z_1^d + \sum_{n=1}^{\infty} (z_1^d)^{n+1} (z_2 + z_3) z_3^{n-1} = z_1^d + \sum_{n=0}^{\infty} (z_1^d)^{n+2} z_2 z_3^n.
$$

Applying Lemma 1.2, $x \in \mathcal{A}^d$ and $x^d = x_1 + x_2$, where

$$
x_1 = \sum_{n=0}^{\infty} (z^d)^{n+1} y^n y^\pi \qquad \text{and} \qquad x_2 = \sum_{n=0}^{\infty} z^\pi z^n (y^d)^{n+1}.
$$

Using $z_2 y = 0$, we have

$$
\begin{aligned}
x_1 &= z_1^d y^\pi + (z_1^d)^2 z_2 + \sum_{n=1}^{\infty} (z_1^d)^{n+2} z_2 z_3^n y^\pi + \sum_{n=1}^{\infty} \left((z_1^d)^{n+1} + \sum_{k=1}^{\infty} (z_1^d)^{k+n+2} z_2 z_3^k \right) y^n y^\pi \\
&= \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^2 ca^\pi - (x^s)^d ca^d (aw)^d a & (x^s)^d - (x^s)^d ca^d (aw)^d b \end{bmatrix} \\
&+ \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^{n+2} ca^n a^\pi & \left[(x^s)^d\right]^{n+2} ca^{n-1} a^\pi b \end{bmatrix} y^\pi + \sum_{n=1}^{\infty} \left(\begin{bmatrix} 0 & 0 \\ 0 & \left[(x^s)^d\right]^{n+1} \end{bmatrix} \right. \\
&+ \left. \sum_{k=1}^{\infty} \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^{k+n+2} ca^k a^\pi & \left[(x^s)^d\right]^{k+n+2} ca^{k-1} a^\pi b \end{bmatrix} \right) y^n y^\pi. \qquad (11)
\end{aligned}
$$

From (11) and

$$x_2 = \sum_{n=0}^{\infty} z^n \left(\begin{bmatrix} p & 0 \\ 0 & (x^s)^\pi \end{bmatrix} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & \left[(x^s)^d\right]^{n+2} ca^n a^\pi b \end{bmatrix} \right) (y^d)^{n+1},$$

we conclude that (10) hold. $\square$

If we assume that $c = 0$ in Theorem 2.1 and $b = 0$ Theorem 2.2, then $x^s = d$ and we can obtain [3, Theorem 2.3(i)] as a consequence. Recall that [3, Theorem 2.3(i)] is an extension in Banach algebras of results [15, Lemma 2.2] for matrices and [10, Theorem 5.3] for bounded linear operators.

Using Theorem 2.2, we can verify the following corollary.

Corollary 2.2. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, m be defined as in* (2) *and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathcal{A}p)^d$.*

(i) *If $bca^\pi = 0$ and $x^s = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = m + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} m^2 + \sum_{n=2}^{\infty} \begin{bmatrix} 0 & a^{n-1} a^\pi b \\ 0 & ca^{n-2} a^\pi b \end{bmatrix} m^{n+1}.$$

(ii) *If $bca^\pi = 0$, $ca^\pi b = 0$, $ca^\pi a = dca^\pi$ and $x^s = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(1 + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} m^{n+1} \right) m.$$

The item (ii) of Corollary 2.2 covers [11, Theorem 12], by $ca^n a^\pi b = 0$ for $n = 0, 1, 2, \ldots$.

Also remark that Corollary 2.1(i) and Corollary 2.2(i) recover [17, Corollary 3.5] and Lemma 1.4.

Now we state the special cases of Theorem 2.1 and Theorem 2.2. Note that, if $w = aa^d$, then $aw = a^2 a^d \in (p\mathcal{A}p)^\#$ and $(a^2 a^d)^\# = a^d$.

Corollary 2.3. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, and let $bc = 0$. If*

(1) *$x^s c = 0$, then $x \in \mathcal{A}^d$ and*

$$\begin{aligned} x^d &= \sum_{n=0}^{\infty} m_1^{n+1} \begin{bmatrix} p & 0 \\ 0 & (x^s)^\pi \end{bmatrix} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & x^s \end{bmatrix}^n + \begin{bmatrix} 0 & a^\pi b[(x^s)^d]^2 - a^d b (x^s)^d \\ 0 & (x^s)^d - c(a^d)^2 b (x^s)^d \end{bmatrix} \\ &+ \sum_{n=1}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b[(x^s)^d]^{n+2} \\ 0 & ca^{n-1} a^\pi b[(x^s)^d]^{n+2} \end{bmatrix}; \end{aligned}$$

(2) $x^s c = 0$ *and* $a \in (p\mathcal{A}p)^{-1}$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \sum_{n=0}^{\infty} m_2^{n+1} \begin{bmatrix} 0 & 0 \\ 0 & (x^s)^{\pi}(x^s)^n \end{bmatrix} + \begin{bmatrix} 0 & -a^{-1}b(x^s)^d \\ 0 & (x^s)^d - a^{-2}b(x^s)^d \end{bmatrix};$$

(3) $bx^s = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$\begin{aligned} x^d &= \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^2 ca^{\pi} - (x^s)^d ca^d & (x^s)^d - (x^s)^d c(a^d)^2 b \end{bmatrix} \\ &+ \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ \left[(x^s)^d\right]^{n+2} ca^n a^{\pi} & \left[(x^s)^d\right]^{n+2} ca^{n-1}a^{\pi}b \end{bmatrix} \\ &+ \sum_{n=0}^{\infty} \begin{bmatrix} aa^{\pi} & a^{\pi}b \\ ca^{\pi} & x^s \end{bmatrix}^n \begin{bmatrix} p & 0 \\ 0 & (x^s)^{\pi} \end{bmatrix} m_1^{n+1}; \end{aligned}$$

(4) $bx^s = 0$ *and* $a \in (p\mathcal{A}p)^{-1}$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} 0 & 0 \\ -(x^s)^d ca^{-1} & (x^s)^d - (x^s)^d ca^{-2}b \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} p & 0 \\ 0 & (x^s)^n (x^s)^{\pi} \end{bmatrix} m_2^{n+1};$$

(5) $bx^s = 0$ *and* $x^s c = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = m_1 + \begin{bmatrix} 0 & 0 \\ 0 & (x^s)^d \end{bmatrix};$$

(6) $x^s = 0$, *then* $x \in \mathcal{A}^d$ *and* $x^d = m_1$;

where

$$m_1 = \begin{bmatrix} a^d & (a^d)^2 b \\ c(a^d)^2 & c(a^d)^3 b \end{bmatrix} \quad \text{and} \quad m_2 = \begin{bmatrix} a^{-1} & a^{-2}b \\ ca^{-2} & ca^{-3}b \end{bmatrix}.$$

3 Expressions with the generalized Banachiewicz–Schur form

In this section, we give new expressions for x^d in terms of the generalized Drazin inverse of generalized Schur complement, the generalized Banachiewicz-Schur form (3) and its powers.

Theorem 3.1. *Let* x *be defined as in* (1) *where* $a \in (p\mathcal{A}p)^d$ *and the generalized Schur complement* $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, *and let* r *be defined as in* (3)*. If*

$$a^{\pi}bc = 0, \quad a^{\pi}bd = 0, \quad (x^s)^{\pi}ca = 0 \quad \text{and} \quad ab(x^s)^{\pi} = 0, \tag{12}$$

then $x \in \mathcal{A}^d$ *and*

$$x^d = r\left(1 + r\begin{bmatrix} 0 & 0 \\ ca^{\pi} & 0 \end{bmatrix} + \sum_{n=2}^{\infty} r^n \begin{bmatrix} 0 & 0 \\ ca^{n-1}a^{\pi} & ca^{n-2}a^{\pi}b \end{bmatrix}\right). \tag{13}$$

Proof. For

$$y = \begin{bmatrix} a^2a^d & aa^db \\ caa^d & d \end{bmatrix} \qquad \text{and} \qquad z = \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}, \tag{14}$$

we have $x = y + z$ and, by (12), $zy = 0$.

Set $A_y \equiv a^2a^d$, $B_y \equiv aa^db$, $C_y \equiv caa^d$ and $D_y \equiv d$. Then $A_y \in (p\mathcal{A}p)^\#$, $A_y^\# = a^d$, $y^s = D_y - C_yA_y^\# B_y = x^s$, $A_y^\pi B_y = C_yA_y^\pi = 0$, $B_y(y^s)^\pi = a^d(ab(x^s)^\pi) = 0$, $(y^s)^\pi C_y = ((x^s)^\pi ca)a^d = 0$ and $Y_y = \begin{bmatrix} 0 & 0 \\ 0 & x^s(x^s)^\pi \end{bmatrix} \in \mathcal{A}^{qnil}$ implying $y \in \mathcal{A}^d$ and $y^d = r$, by Lemma 1.5.

Assume that

$$z_1 = \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad z_2 = \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix}.$$

Now $z_1z_2 = 0$, $z_1 \in \mathcal{A}^{qnil}$ and $z_2^2 = 0$. Using Lemma 1.1, we deduce that $z = z_1 + z_2 \in \mathcal{A}^{qnil}$.

Applying Lemma 1.3(i), $x \in \mathcal{A}^d$ and $x^d = \sum\limits_{n=0}^{\infty} r^{n+1}z^n = r + r^2z + \sum\limits_{n=2}^{\infty} r^{n+1}z^n$. Since, for $n = 2, 3, \dots$,

$$z^n = zz_1^{n-1} = z\begin{bmatrix} a^{n-1}a^\pi & a^{n-2}a^\pi b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^na^\pi & a^{n-1}a^\pi b \\ ca^{n-1}a^\pi & ca^{n-2}a^\pi b \end{bmatrix},$$

$rz = rz_1$ and $rz^n = r\begin{bmatrix} 0 & 0 \\ ca^{n-1}a^\pi & ca^{n-2}a^\pi b \end{bmatrix}$, we get (13). $\square$

A geometrical reformulation of conditions (12) is as follows:

$$bc\mathcal{A} \subset a\mathcal{A}, \qquad bd\mathcal{A} \subset a\mathcal{A}, \qquad ca\mathcal{A} \subset x^s\mathcal{A} \qquad \text{and} \qquad (x^s)^\circ \subset (ab)^\circ.$$

By Theorem 3.1, we can verify the following corollary.

Corollary 3.1. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^db \in ((1-p)\mathcal{A}(1-p))^d$, and let r be defined as in* (3)*.*

(i) *If equalities* (12)*, $aa^\pi b = 0$ and $ca^\pi b = 0$ hold, then $x \in \mathcal{A}^d$ and*

$$x^d = r\left(1 + \sum_{n=0}^{\infty} r^{n+1}\begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right);$$

(ii) *If $a^\pi b = 0$ and x^s is invertible, then $x \in \mathcal{A}^d$ and*

$$x^d = r_1\left(1 + \sum_{n=0}^{\infty} r_1^{n+1}\begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right);$$

(iii) *If a is invertible, $(x^s)^\pi c = 0$ and $b(x^s)^\pi = 0$, then $x \in \mathcal{A}^d$ and $x = r_2$;*

(iv) *If a and x^s are invertible, then $x \in \mathcal{A}^d$ and $x = r_2$;*

where

$$r_1 = \begin{bmatrix} a^d + a^d b(x^s)^{-1} c a^d & -a^d b(x^s)^{-1} \\ -(x^s)^{-1} c a^d & (x^s)^{-1} \end{bmatrix} \quad \text{and} \quad r_2 = \begin{bmatrix} a^{-1} + a^{-1} b(x^s)^{-1} c a^{-1} & -a^{-1} b(x^s)^{-1} \\ -(x^s)^{-1} c a^{-1} & (x^s)^{-1} \end{bmatrix}.$$

Similarly to Theorem 3.1, we can show the following theorem.

Theorem 3.2. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, and let r be defined as in* (3)*. If*

$$bca^\pi = 0, \quad dca^\pi = 0, \quad (x^s)^\pi ca = 0 \quad \text{and} \quad ab(x^s)^\pi = 0, \tag{15}$$

then $x \in \mathcal{A}^d$ and

$$x^d = \left(1 + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r + \sum_{n=2}^{\infty} \begin{bmatrix} 0 & a^{n-1} a^\pi b \\ 0 & ca^{n-2} a^\pi b \end{bmatrix} r^n \right) r. \tag{16}$$

Proof. Define y and z as in (14). Then $yz = 0$ and, in the analogy way as in the proof of Theorem 3.1, we prove the formula (16). □

Also we can check the next result using Theorem 3.2.

Corollary 3.2. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, r be defined as in* (3)*, and let r_1 be defined as in Corollary 3.1.*

(i) *If equalities* (15)*, $caa^\pi = 0$ and $ca^\pi b = 0$ hold, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(1 + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+1} \right) r.$$

(ii) *If $ca^\pi = 0$ and x^s is invertible, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(1 + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r_1^{n+1} \right) r_1.$$

From Corollary 3.1(ii) or Corollary 3.2(ii), we get an extension of result for the Drazin inverse of a block matrix by Wei [23].

Corollary 3.3. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement $x^s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$, and let r_1 be defined as in Corollary 3.1. If $a^\pi b = 0$, $ca^\pi = 0$ and x^s is invertible, then $x \in \mathcal{A}^d$ and $x = r_1$.*

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