

On Polynomially Riesz Operators

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Abstract. A bounded linear operator A on a Banach space X is said to be “polynomially Riesz”, if there exists a nonzero complex polynomial p such that the image $p(A)$ is Riesz. In this paper we give some characterizations of these operators.

1. Introduction

Let \mathbb{C} denote the set of all complex numbers and let X and Y be infinite dimensional Banach spaces. We denote by $B(X)$ the set of all linear bounded operators on X and by $K(X)$ the set of all compact operators on X . We write $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not invertible}\}$ for the *spectrum* of $A \in B(X)$.

For $A \in B(X)$ let $N(A)$ denote the null-space and $R(A)$ the range of A . We set $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim X/R(A) = \text{codim} R(A)$. Sets of *upper and lower semi-Fredholm* operators, respectively, are defined as $\Phi_+(X) = \{A \in B(X) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed}\}$, and $\Phi_-(X) = \{A \in B(X) : \beta(A) < \infty\}$. For upper and lower semi-Fredholm operators the index is defined by $i(A) = \alpha(A) - \beta(A)$. If $A \in \Phi_+(X) \setminus \Phi_-(X)$, then $i(A) = -\infty$, and if $A \in \Phi_-(X) \setminus \Phi_+(X)$, then $i(A) = +\infty$. The set of Fredholm operators is defined as $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

An operator $A \in B(X)$ is relatively regular (or g -invertible) if there exists $B \in B(X)$ such that $ABA = A$. It is well-known that A is relatively regular if and only if $R(A)$ and $N(A)$ are closed and complemented subspaces of X . We say that an operator $A \in B(X)$ is *left Fredholm*, and write $A \in \Phi_l(X)$, if A is a relatively regular upper semi-Fredholm operator, while we say that A is *right Fredholm*, and write $A \in \Phi_r(X)$, if A is a relatively regular lower semi-Fredholm operator. In other words, A is left Fredholm if $R(A)$ is a closed and complemented subspace of X and $\alpha(A) < \infty$, while A is right Fredholm if $N(A)$ is a complemented subspace of X and $\beta(A) < \infty$. An operator $A \in B(X)$ is called a *Weyl* operator if it is Fredholm of index zero. We shall say that $A \in B(X)$ is *upper semi-Weyl* if it is upper semi-Fredholm and $i(A) \leq 0$, while A is *lower semi-Weyl* if it is lower semi-Fredholm and $i(A) \geq 0$. An operator $A \in B(X)$ is *left (right) Weyl* if A is left (right) Fredholm and $i(A) \leq 0$ ($i(A) \geq 0$).

Denote by $\text{asc}(A)$ ($\text{dsc}(A)$) the ascent (the descent) of $A \in B(X)$, i.e. the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$ ($R(A^n) = R(A^{n+1})$). If such n does not exist, then $\text{asc}(A) = \infty$ ($\text{dsc}(A) = \infty$).

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An operator $A \in B(X)$ is *Browder* if it is Fredholm and has finite ascent and finite descent. Let us mention that Browder operators are known in the literature also as Riesz-Schauder operators [4]. It is well known that every Browder operator is Weyl ([10], Proposition 38.6(a); [9], Theorem 7.9.3). An operator $A \in B(X)$ is called *upper semi-Browder* if it is upper semi-Fredholm of finite ascent, and *lower semi-Browder* if it is lower semi-Fredholm of finite descent. We shall say that $A \in B(X)$ is *left Browder* if it is left Fredholm of finite ascent, and *right Browder* if it is right Fredholm of finite descent [17], [18].

The Calkin algebra over X is the quotient algebra $C(X) = B(X)/K(X)$, and let $\Pi : B(X) \rightarrow C(X)$ denote the natural homomorphism. An operator $A \in B(X)$ is Riesz, $A \in R(X)$, if $\{\lambda \in \mathbb{C} : A - \lambda \in \Phi(X)\} = \mathbb{C} \setminus \{0\}$. Recall that $A \in R(X)$ if and only if $\Pi(A)$ is quasinilpotent in $C(X)$ ([1], p. 179 and 180). Clearly, $K(X) \subset R(X)$. Recall that ([1], Theorem 3.112. (i), (ii))

$$A, B \in R(X), AB = BA \implies A + B \in R(X), \tag{1.1}$$

$$A \in R(X), B \in B(X), AB = BA \implies AB \in R(X). \tag{1.2}$$

Also, for $A \in B(X)$, it is known ([1], Corollary 3.114)

$$A \in R(X) \iff A' \in R(X'), \tag{1.3}$$

where $A' \in B(X')$ is the adjoint operator of A .

If we introduce Fredholm, Weyl and Browder spectra $\theta \in \{\sigma_{ess}, \omega_{ess}, \beta_{ess}\}$ in the obvious way: $\sigma_{ess}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}$, $\omega_{ess}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\}$ and $\beta_{ess}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Browder}\}$, $A \in B(X)$, we shall also write

$$\theta^+, \theta^-, \theta^{left}, \theta^{right}$$

for the corresponding upper, lower, left and right versions.

We recall that for $\theta \in \{\sigma_{ess}, \omega_{ess}, \beta_{ess}\}$ ([1], Theorem 3.111)

$$A \in R(X) \iff \theta(A) = \{0\}. \tag{1.4}$$

For $A \in B(X)$ it is well known that ([13], Corollary 19.20)

$$\beta_{ess}(A) = \sigma_{ess}(A) \cup \text{acc } \sigma(A), \tag{1.5}$$

where $\text{acc } \sigma(A)$ is the set of accumulation points of $\sigma(A)$.

Also we recall the following result ([17], Theorem 10).

Theorem 1.1. *If $T \in B(X)$, then for each $*$ = +, -, left, right there is inclusion*

$$\partial\beta_{ess}(T) \subset \partial\beta_{ess}^*(T) \subset \partial\omega_{ess}^*(T) \subset \partial\sigma_{ess}^*(T) \subset \beta_{ess}^*(T) \subset \beta_{ess}(T).$$

For $A \in B(X)$ set $N^\infty(A) = \cup_n N(A^n)$ for the hyper-kernel of A and $R^\infty(A) = \cap_n R(A^n)$ for the hyper-range of A . If $\lambda \in \mathbb{C}$ is an eigenvalue of A , the $\dim N^\infty(A - \lambda)$ is called the algebraic multiplicity and denoted by $\text{mult}(A, \lambda)$.

We recall the following result which was proved by H. Baklouti ([2], Theorem 2.1):

Theorem 1.2. *Let $A \in B(X)$ and let p be a non-zero complex polynomial. If μ be in $\sigma(p(A)) \setminus \beta_{ess}(p(A))$, then μ is an eigenvalue of $p(A)$ and*

$$N^\infty(\mu - p(A)) = \bigoplus_{\substack{\lambda \in \sigma(A) \\ p(\lambda) = \mu}} N^\infty(\lambda - A).$$

In particular, $\text{mult}(p(A), \mu) = \sum_{\substack{\lambda \in \sigma(A) \\ p(\lambda) = \mu}} \text{mult}(A, \lambda)$.

Set $k(A) = \dim N(A)/(N(A) \cap R(A^\infty))$. An operator $A \in B(X)$ is called *essentially Kato* if $R(A)$ is closed and $k(A) < \infty$ ([13], Definition 21.4, Theorem 21.3; [15], Theorem 2.1). Clearly, every semi-Fredholm operator is an essentially Kato operator. The corresponding spectrum is

$$\sigma_{Ke}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not essentially Kato}\}.$$

In [15] this spectrum was investigated and called Browder's essential generalized spectrum of A . Recall the following result for $A \in B(X)$ ([13], Theorem 21.11):

$$\partial\sigma_{ess}(A) \subset \sigma_{Ke}(A) \subset \sigma_{ess}(A). \quad (1.6)$$

Let \mathcal{A} be a complex Banach algebra. For $K \subset \mathbb{C}$, ∂K denotes the topological boundary of K .

If K is a compact set, $K \subseteq \mathbb{C}$, we shall write ηK for the *connected hull* of K , where the complement $\mathbb{C} \setminus \eta K$ is the unique unbounded component of the complement $\mathbb{C} \setminus K$ ([8]; [9], Definition 7.10.1). A hole of K is a component of $\eta K \setminus K$. Generally ([8], Theorem 1.2, Theorem 1.3; [9], Theorem 7.10.3), for compact subsets $H, K \subseteq \mathbb{C}$,

$$\partial H \subseteq K \subseteq H \implies \partial H \subseteq \partial K \subseteq K \subseteq H \subseteq \eta K = \eta H, \quad (1.7)$$

and H can be obtained from K by filling in some holes of K .

Remark that for finite $K \subseteq \mathbb{C}$ it follows that $\eta K = K$. Therefore, for compact subsets $H, K \subseteq \mathbb{C}$,

$$\eta K = \eta H \implies (H \text{ is finite} \iff K \text{ is finite}), \quad (1.8)$$

and in that case $H = K$.

2. Polynomially Riesz operators

We shall write

$$H +_{comm} K = \{c + d : (c, d) \in H \times K, cd = dc\}$$

for the commuting sum and

$$H \cdot_{comm} K = \{cd : (c, d) \in H \times K, cd = dc\}$$

for the commuting product of subsets $H, K \subseteq \mathcal{A}$.

We say that $S \subseteq \mathcal{A}$ is a *commutative ideal* if

$$S +_{comm} S \subseteq S, \mathcal{A} \cdot_{comm} S \subseteq S.$$

We shall write $\text{Poly} = \mathbb{C}[z]$ for the algebra of complex polynomials. If $S \subseteq \mathcal{A}$ is an arbitrary set we shall write that $a \in \text{Poly}^{-1}(S)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in S$. If $S \subseteq \mathcal{A}$ is a commutative ideal, the set

$$\mathcal{P}_a^S = \{p \in \text{Poly} : p(a) \in S\}$$

of polynomials p for which $p(a) \in S$ will be an ideal of the algebra Poly . Since the natural numbers are well ordered there will be a unique polynomial p of minimal degree with leading coefficient 1 contained in \mathcal{P}_a^S which we call the *minimal polynomial* of a ; we shall write $p = \pi_a \equiv \pi_a^S$. Then \mathcal{P}_a^S is generated by $p = \pi_a$, i.e. $\mathcal{P}_a^S = \pi_a \cdot \text{Poly}$.

According to (1.1) and (1.2) we conclude that the set of Riesz operators $R(X)$ is a commutative ideal in the algebra $B(X)$.

We shall say that an operator $A \in B(X)$ is *polynomially Riesz* and write $A \in \text{Poly}^{-1}R(X)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(A) \in R(X)$. These operators have been recently discussed in [20].

Some parts of the following theorem are contained in Theorem 11.1 in [19]. For the sake of completeness we give the whole proof and start with the following simple lemma.

Lemma 2.1. *If $A, B \in B(X)$ commute, A is Fredholm and AB is Riesz, then B is Riesz.*

Proof. Suppose that $AB = BA \in R(X)$ and $A \in \Phi(X)$. Then $\Pi(A)$ is invertible in the Calkin algebra $C(X)$ ([4], Theorem 3.2.8), while $\Pi(A)\Pi(B) = \Pi(AB)$ is quasinilpotent in $C(X)$ and $(\Pi(A))^{-1}$ commute with $\Pi(B)$ and hence also with $\Pi(A)\Pi(B)$. It follows that $\Pi(B) = (\Pi(A))^{-1}\Pi(A)\Pi(B)$ is quasinilpotent in $C(X)$, and so $B \in R(X)$. \square

We remark that the assertion of the previous lemma also holds if the condition of commutativity of operators A and B is replaced by a weaker condition that $AB - BA$ belongs to the perturbation class of the set of Fredholm operators $\text{Ptrb}(\Phi(X))$ ($\text{Ptrb}(\Phi(X)) = \{S \in B(X) : S + T \in \Phi(X) \text{ for all } T \in \Phi(X)\}$).

Theorem 2.2. *Let $A \in B(X)$. Then $A \in \text{Poly}^{-1}R(X)$ if and only if $\beta_{\text{ess}}(A)$ is finite and in that case*

$$\beta_{\text{ess}}(A) = \pi_A^{-1}(0),$$

where π_A is the minimal polynomial of A .

Proof. Suppose that $A \in \text{Poly}^{-1}R(X)$. Then $\pi_A(A) \in R(X)$ and from (1.4) and [6] it follows that $\pi_A(\beta_{\text{ess}}(A)) = \beta_{\text{ess}}(\pi_A(A)) = \{0\}$, and therefore,

$$\beta_{\text{ess}}(A) \subset \pi_A^{-1}(0). \tag{2.1}$$

To prove the opposite suppose that $\beta_{\text{ess}}(A)$ is finite and let $\beta_{\text{ess}}(A) = \{\lambda_1, \dots, \lambda_n\}$. For $p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$ we have $\{0\} = p(\beta_{\text{ess}}(A)) = \beta_{\text{ess}}(p(A))$, and so, $p(A) \in R(X)$ by (1.4).

Let $A \in \text{Poly}^{-1}R(X)$ and let λ be a zero of the minimal polynomial π_A . Then $\pi_A(z) = (z - \lambda)q(z)$ and therefore,

$$\pi_A(A) = (A - \lambda)q(A) = q(A)(A - \lambda) \in R(X). \tag{2.2}$$

We show that $\lambda \in \sigma_{\text{ess}}(A)$. If $\lambda \notin \sigma_{\text{ess}}(A)$, then $A - \lambda$ is Fredholm, and from Lemma 2.1 it follows that $q(A) \in R(X)$ which contradicts the fact that the polynomial π_A is minimal. Therefore, $\pi_A^{-1}(0) \subset \sigma_{\text{ess}}(A)$, which together with (2.1) gives $\sigma_{\text{ess}}(A) = \beta_{\text{ess}}(A) = \pi_A^{-1}(0)$. This completes the proof. \square

Theorem 2.3. *Let $A \in B(X)$. If θ is one of $\sigma_{K_e}, \sigma_{\text{ess}}, \sigma_{\text{ess}}^+, \sigma_{\text{ess}}^-, \sigma_{\text{ess}}^{\text{left}}, \sigma_{\text{ess}}^{\text{right}}, \omega_{\text{ess}}, \omega_{\text{ess}}^+, \omega_{\text{ess}}^-, \omega_{\text{ess}}^{\text{left}}, \omega_{\text{ess}}^{\text{right}}, \beta_{\text{ess}}, \beta_{\text{ess}}^+, \beta_{\text{ess}}^-, \beta_{\text{ess}}^{\text{left}}, \beta_{\text{ess}}^{\text{right}}$, then*

$$A \in \text{Poly}^{-1}R(X) \iff \theta(A) \text{ is finite,}$$

and in that case

$$\theta(A) = \pi_A^{-1}(0),$$

where π_A is the minimal polynomial of A .

Proof. From Theorem 1.1, (1.6), (1.7) and (1.8) it follows that if one of the mentioned spectra is finite, then any other of them is also finite and they are equal. Now the rest of the assertion follows from Theorem 2.2. \square

Corollary 2.4. *If $A \in \text{Poly}^{-1}R(X)$, then for all $\lambda \in \pi_A^{-1}(0)$, $A - \lambda$ is not essentially Kato, and for all $\lambda \notin \pi_A^{-1}(0)$, $A - \lambda$ is Browder.*

Proof. Follows from Theorem 2.3. \square

Corollary 2.5. *If $A \in \text{Poly}^{-1}R(X)$, then $\sigma(A)$ is at most countable.*

Proof. From (1.5) it follows that the set $\sigma(A) \setminus \beta_{\text{ess}}(A)$ consists of the isolated points of $\sigma(A)$ and consequently, it is at most countable. Since, by Theorem 2.2, $\beta_{\text{ess}}(A)$ is finite, it follows that $\sigma(A)$ is at most countable. \square

We recall the following concept introduced by A. Jeribi and N. Moalla in [11], Definition 1.2:

Definition 2.6. An operator $A \in B(X)$ is called *generalized Riesz* if there exists E a finite subset of \mathbb{C} such that

- (i) For all $\lambda \in \mathbb{C} \setminus E$, $A - \lambda$ is a Fredholm operator on X .
- (ii) For all $\lambda \in \mathbb{C} \setminus E$, $A - \lambda$ has finite ascent and finite descent.
- (iii) All $\lambda \in \sigma(A) \setminus E$ are eigenvalues of finite multiplicity, and have no accumulation point except possibly points of E .

Theorem 2.7. Let $A \in B(X)$. Then A is generalized Riesz if and only if A has the finite Browder spectrum.

Proof. Suppose that A is generalized Riesz. Then there exists a finite subset E of \mathbb{C} such that for every $\lambda \in \mathbb{C} \setminus E$, $A - \lambda$ is a Fredholm operator on X with finite ascent and finite descent, i.e. $A - \lambda$ is Browder. Therefore, $\beta_{ess}(A) \subset E$ and hence, $\beta_{ess}(A)$ is a finite subset of \mathbb{C} .

To prove the converse, suppose that $\beta_{ess}(A)$ is a finite subset of \mathbb{C} . For all $\lambda \in \mathbb{C} \setminus \beta_{ess}(A)$, $A - \lambda$ is Browder. Let $\lambda \in \sigma(A) \setminus \beta_{ess}(A)$. Then $A - \lambda$ is a Weyl operator and $\text{asc}(A - \lambda) = \text{dsc}(A - \lambda) = p < \infty$. From $i(A - \lambda) = 0$ it follows that $\alpha(A - \lambda) = \beta(A - \lambda)$ and since $\lambda \in \sigma(A)$ we get $\alpha(A - \lambda) > 0$, i.e. λ is an eigenvalue of A . Since $N^\infty(A - \lambda) = N((A - \lambda)^p)$ and $(A - \lambda)^p \in \Phi(X)$ ([9] (6.4.4.1)) we get $\dim N^\infty(A - \lambda) < \infty$. From (1.5) it follows that all $\lambda \in \sigma(A) \setminus \beta_{ess}(A)$ have no accumulation point except possibly points of $\beta_{ess}(A)$. Therefore, A is generalized Riesz with $E = \beta_{ess}(A)$. \square

Corollary 2.8. Let $A \in B(X)$. Then A is generalized Riesz if and only if A is polynomially Riesz.

Proof. Follows from Theorem 2.2 and Theorem 2.7. \square

The following two results extend Corollary 2.1 in [2] and Proposition 3.1 in [11].

Theorem 2.9. Let $A \in \text{Poly}^{-1}R(X)$ and let $\mu \in \sigma(\pi_A(A)) \setminus \{0\}$. Then

$$N^\infty(\mu - \pi_A(A)) = \bigoplus_{\substack{\lambda \in \sigma(A) \\ \pi_A(\lambda) = \mu}} N^\infty(\lambda - A)$$

and

$$\text{mult}(\pi_A(A), \mu) = \sum_{\substack{\lambda \in \sigma(A) \\ \pi_A(\lambda) = \mu}} \text{mult}(A, \lambda).$$

Proof. From $\pi_A(A) \in R(X)$ it follows that $\beta_{ess}(\pi_A(A)) = \{0\}$ and the assertion follows from Theorem 1.2. \square

Theorem 2.10. Let $A \in B(X)$. Then $A \in \text{Poly}^{-1}R(X)$ if and only if $A' \in \text{Poly}^{-1}R(X')$ and the minimal polynomial of A is equal to the minimal polynomial of A' .

Moreover, if $A \in \text{Poly}^{-1}R(X)$, then for every $\lambda \in \sigma(A) \setminus \pi_A^{-1}(0)$,

$$\text{mult}(A, \lambda) = \text{mult}(A', \lambda).$$

Proof. For a nonzero complex polynomial $p(z)$, from (1.3) it follows that $p(A) \in R(X)$ if and only if $p(A') = p(A)' \in R(X')$. Therefore, $A \in \text{Poly}^{-1}R(X)$ if and only if $A' \in \text{Poly}^{-1}R(X')$ and $\pi_A = \pi_{A'}$.

Let $A \in \text{Poly}^{-1}R(X)$ and let $\lambda \in \sigma(A) \setminus \pi_A^{-1}(0)$. From Theorem 2.2 it follows that $\lambda \in \sigma(A) \setminus \beta_{ess}(A)$, and hence $A - \lambda$ is Browder and λ is an eigenvalue of A . Consequently, $N^\infty(A - \lambda) = N((A - \lambda)^p)$ and $N^\infty(A' - \lambda) = N((A' - \lambda)^p)$ where $p = \text{asc}(A - \lambda) = \text{asc}(A' - \lambda)$. Since $A - \lambda$ is Weyl, it follows that $(A - \lambda)^p$ is Weyl ([9], Theorem 6.5.2). Hence and according to [4], Proposition 1.2.7, it follows

$$\begin{aligned} \text{mult}(A, \lambda) &= \dim N((A - \lambda)^p) = \alpha((A - \lambda)^p) = \beta((A - \lambda)^p) \\ &= \alpha((A' - \lambda)^p) = \dim N((A' - \lambda)^p) = \text{mult}(A', \lambda). \end{aligned}$$

\square

Gilfeather ([5], Theorem 1) proved that if A is a polynomially compact operator on a Banach space, then every λ zero of the minimal polynomial of A is the limit of the eigenvalues of A or else there exists a closed infinite dimensional A -invariant subspace X_λ such that the reduction of $A - \lambda$ on X_λ is quasinilpotent. We shall show that polynomially Riesz operators have the same property. In order to prove this we need the following simple assertion.

Lemma 2.11. *Let $A \in B(X)$ and let $X = X_1 \oplus \cdots \oplus X_n$ where X_i is a closed A -invariant subspace of X and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on X_i , $i = 1, \dots, n$. Then*

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_1) \cup \cdots \cup \sigma_{\text{ess}}(A_n), \quad (2.3)$$

and A is Riesz if and only if A_1, \dots, A_n are Riesz.

Proof. For $\lambda \in \mathbb{C}$, from

$$N(A - \lambda) = N(A_1 - \lambda) \oplus \cdots \oplus N(A_n - \lambda)$$

and

$$R(A - \lambda) = R(A_1 - \lambda) \oplus \cdots \oplus R(A_n - \lambda)$$

it follows that $A - \lambda$ is Fredholm if and only if $A_i - \lambda$ is Fredholm, $i = 1, \dots, n$. This implies (2.3) and therefore, $\sigma_{\text{ess}}(A) = \{0\}$ if and only if $\sigma_{\text{ess}}(A_i) = \{0\}$, $i = 1, \dots, n$, which means that A is Riesz if and only if A_1, \dots, A_n are Riesz. \square

Theorem 2.12. *Let $A \in \text{Poly}^{-1}R(X)$. Then each $\lambda \in \pi_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else there exists a closed A -invariant subspace X_λ of X which is infinite dimensional and $\sigma(A_\lambda) = \{\lambda\}$ where A_λ is the reduction of A on X_λ .*

Proof. From Theorem 2.2 and Theorem 2.7 it follows that each $\lambda \in \pi_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else λ is an isolated point of $\sigma(A)$. Let $\lambda \in \pi_A^{-1}(0)$ and let λ be an isolated point of $\sigma(A)$. Then there exist open sets Ω_1 and Ω_2 which boundaries are simple closed rectifiable curves and such that $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $\lambda \in \Omega_1$, the closure of Ω_1 contains no other point of $\sigma(A)$ and $\sigma(A) \subset \Omega_1 \cup \Omega_2$. Using the spectral projection $P_\lambda = \frac{1}{2\pi i} \int_{\partial\Omega_1} (zI - A)^{-1} dz$ we have $X = X_\lambda \oplus X_\mu$, where $X_\lambda = P_\lambda X$ and $X_\mu = (I - P_\lambda)X$ are closed A -invariant subspaces of X , $A = A_\lambda \oplus A_\mu$, where A_λ (A_μ) is the reduction of A on X_λ (X_μ), and also $\sigma(A_\lambda) = \{\lambda\}$ and $\sigma(A_\mu) = \sigma(A) \setminus \{\lambda\}$. From $\pi_A(A) \in R(X)$ it follows that $\pi_A(A_\mu)$ is Riesz ([4], Lemma 3.5.1; [1], Theorem 3.113 (iii)). Since $(A_\mu - \lambda)^{-1}$ commutes with $\pi_A(A_\mu)$, by [1], Theorem 3.112 (ii), we get that $(A_\mu - \lambda)^{-1} \pi_A(A_\mu)$ is Riesz, that is there exists a polynomial q which degree is less than the degree of the polynomial π_A such that $q(A_\mu)$ is Riesz. We shall prove that $\dim X_\lambda = \infty$. Suppose the opposite that $\dim X_\lambda < \infty$. Then $q(A_\lambda)$ is compact and hence it is Riesz. Since $q(A) = q(A_\lambda) \oplus q(A_\mu)$, from Lemma 2.11 it follows that $q(A)$ is Riesz which contradicts the fact that the polynomial π_A is minimal. Thus X_λ is infinite dimensional and the proof is complete. \square

Gilfeather ([5], Theorem 1) described the structure of polynomially compact operators proving that every polynomially compact operator on a Banach space is the finite direct sum of translates of operators which have property that the finite power of the operators is compact. The structure of polynomially Riesz operators on Hilbert spaces was described by Y. M. Han, S. H. Lee and W. Y. Lee ([7], Lemma 3): every polynomially Riesz operators on a Hilbert space is the finite direct sum of translates of Riesz operators. This assertion holds also for polynomially Riesz operators on Banach spaces:

Theorem 2.13. *If $A \in \text{Poly}^{-1}R(X)$ and $\pi_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$, then the Banach space X is decomposed into the direct sum $X = X_1 \oplus \cdots \oplus X_n$ where X_i is closed A -invariant subspace of X , and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on X_i and $A_i - \lambda_i$ is Riesz, $i = 1, \dots, n$.*

Proof. Let $A \in \text{Poly}^{-1}R(X)$ and $\pi_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$. There exist open sets $\Omega_1, \dots, \Omega_n$ such that $\lambda_i \in \Omega_i$, $\partial\Omega_i$ is a rectifiable simple closed curve, $i = 1, \dots, n$, $\sigma(A) \subset \cup_{i=1}^n \Omega_i$, $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ for $i \neq j$ and if λ_i is an isolated

point of $\sigma(A)$, then $\sigma(A) \cap \Omega_i = \{\lambda_i\}$. For the spectral projections $P_i = \int_{\partial\Omega_i} (zI - A)^{-1} dz$ and $X_i = P_i X$ we have $X = X_1 \oplus \cdots \oplus X_n$ and X_i is a closed A -invariant subspace of X , $i = 1, \dots, n$. Also $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on X_i and $\sigma(A_i) = \sigma(A) \cap \Omega_i$. By Theorem 2.3, $\sigma_{\text{ess}}(A) = \{\lambda_1, \dots, \lambda_n\}$ and according to (2.3) we conclude $\sigma_{\text{ess}}(A_i) = \{\lambda_i\}$, $i = 1, \dots, n$. Hence $\sigma_{\text{ess}}(A_i - \lambda_i) = \{0\}$, that is $A_i - \lambda_i$ is Riesz, $i = 1, \dots, n$. \square

Gilfeather ([5], Theorem 2) described also the structure of polynomially compact normal operators on a Hilbert space. In the following theorem we extend Gilfeather's result on polynomially Riesz normal operators.

Theorem 2.14. *Let H be a Hilbert space and let $A \in B(H)$ be normal and polynomially Riesz. Let $\pi_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$. Then for each $i \in \{1, \dots, n\}$, $\lambda \in \pi_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else it is an isolated eigenvalues with the infinite dimensional eigenspace.*

Then H is decomposed into the orthogonal direct sum $H = H_1 \oplus \cdots \oplus H_n$ and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on H_i , and $A_i - \lambda_i$ is compact, $i = 1, \dots, n$. Moreover, if λ_i is an isolated point of $\sigma(A)$, then $A_i = \lambda_i I$.

Proof. Applying the functional calculus as in the proof of Theorem 2.13, we get that the spectral projections P_i , $i = 1, \dots, n$, are orthogonal and $P_i P_j = 0$, $i \neq j$. Therefore, $H_i = P_i X$ are closed A -invariant subspaces of H such that $H_i \perp H_j$, $i \neq j$, and hence H is decomposed into the orthogonal direct sum $H = H_1 \oplus \cdots \oplus H_n$. If A_i is the reduction of A on H_i , then from Theorem 2.13 we have that $A_i - \lambda_i$ is Riesz. Let $\Pi_i : B(H_i) \rightarrow C(H_i)$ denote the natural homomorphism where $C(H_i)$ is the Calkin algebra over H_i , $C(H_i) = B(H_i)/K(H_i)$, $i = 1, \dots, n$. Since $C(H_i)$ is a C^* algebra and since $\Pi_i(A_i - \lambda_i)$ is normal and quasinilpotent, it follows that $\Pi_i(A_i - \lambda_i) = 0$, that is $A_i - \lambda_i$ is compact.

From Theorem 2.12 it follows that each $\lambda_i \in \pi_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else it is an isolated point of $\sigma(A)$ in which case $\sigma(A_i) = \{\lambda_i\}$ and H_i is infinite dimensional. Since $\sigma(A_i - \lambda_i) = \{0\}$ and $A_i - \lambda_i$ is normal, it follows that $A_i - \lambda_i = 0$, i.e. $A_i = \lambda_i I$. As H_i is infinite dimensional, λ_i is an eigenvalue with the infinite dimensional eigenspace. \square

If H is a Hilbert space, an operator $A \in B(H)$ is called hyponormal if $\|A^*x\| \leq \|Ax\|$ for all $x \in H$, that is $A^*A - AA^* \geq 0$. Gilfeather proved for $A \in B(H)$ ([5], Proposition 4):

$$A \text{ is hyponormal, } \sigma(A) \text{ is countable} \implies A \text{ is normal.} \quad (2.4)$$

The conclusion of (2.4) extends to *paranormal* operators, i.e. a paranormal operator $A \in B(H)$ with countable spectrum is normal [12, 14], where A is paranormal if $\|Ax\|^2 \leq \|A^2x\|$ for all unit vectors $x \in H$. Evidently, $A \in B(H)$ hyponormal implies A paranormal. The following corollary is an extension of Corollary 2 in [5] to polynomially Riesz paranormal operators.

Corollary 2.15. *Every polynomially Riesz paranormal operator in $B(H)$ is normal.*

Proof. If $A \in \text{Poly}^{-1}\text{R}(H)$ and $\pi_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$, then it follows from Theorem 2.7 that $H = \bigoplus_{i=1}^n H_i$, where each H_i is a closed A -invariant subspace of H , and $A = \bigoplus_{i=1}^n A_i$. Here each $A_i - \lambda_i$ is Riesz, the operator A_i has at best a countable spectrum (with λ_i as its only possible limit point) and all points of $\sigma(A_i)$ other than the point λ_i are eigenvalues of the operator. Recall that the restriction of a paranormal operator to an invariant subspace is paranormal and (as noted above) a paranormal operator with countable spectrum is normal. Hence each A_i , and consequently A , is normal. \square

The following theorem is an extension of Theorem 2.6 in [11] and Theorem 5.2, Chapter V in [16].

Theorem 2.16. *Let $B \in B(X)$. Then the following conditions are equivalent:*

(2.16.1) B is Browder.

(2.16.2) There exist $n \in \mathbb{N}$, $T \in B(X)$ and $A \in K(X)$ such that

$$TB^n = B^nT = I - A.$$

(2.16.3) There exist $n \in \mathbb{N}$, $T \in B(X)$, $A \in \text{Poly}^{-1}K(X)$ and $\lambda \in \mathbb{C}$ such that $\pi_A(\lambda) \neq 0$ and

$$TB^n = B^nT = \lambda - A.$$

(2.16.4) There exist $n \in \mathbb{N}$, $T \in B(X)$, $A \in \text{Poly}^{-1}R(X)$ and $\lambda \in \mathbb{C}$ such that $\pi_A(\lambda) \neq 0$ and

$$TB^n = B^nT = \lambda - A. \quad (2.5)$$

Proof. (2.16.1) \implies (2.16.2): Follows from [16], Theorem 5.2, p. 123-124.

(2.16.2) \implies (2.16.3): Suppose that there exist $n \in \mathbb{N}$, $T \in B(X)$ and $A \in K(X)$ such that $TB^n = B^nT = I - A$. Then $A \in \text{Poly}^{-1}K(X)$ with $\pi_A(z) = z$ and hence $\pi_A(1) \neq 0$. Thus, the statement (2.16.3) holds for $\lambda = 1$.

(2.16.3) \implies (2.16.4): Follows from the inclusion $K(X) \subset R(X)$.

(2.16.4) \implies (2.16.1): If (2.16.4) holds, then from Corollary 2.4 it follows that $\lambda - A$ is Browder and from (2.5), according to [9], Theorem 7.10.2, we get B^n is Browder. Again by [9], Theorem 7.10.2 we conclude that B is Browder. \square

If $A \in B(X, Y)$, $B \in B(Y, X)$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, it is well know that ([3], Chapter 5.1, Lemma 7)

$$BA - \lambda \text{ is left (right) invertible} \iff AB - \lambda \text{ is left (right) invertible}, \quad (2.6)$$

$$BA - \lambda \text{ is left (right) Fredholm} \iff AB - \lambda \text{ is left (right) Fredholm}. \quad (2.7)$$

Theorem 2.17. Let $A \in B(X, Y)$, $B \in B(Y, X)$ and let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then

$$BA - \lambda \text{ is left Browder} \iff AB - \lambda \text{ is left Browder}, \quad (2.8)$$

and

$$BA - \lambda \text{ is right Browder} \iff AB - \lambda \text{ is right Browder}. \quad (2.9)$$

Proof. From [17], Theorem 5, for $T \in B(X)$ we have

$$\beta_{ess}^{left}(T) = \sigma_{ess}^{left}(T) \cup \text{acc } \sigma^{left}(T) \quad (2.10)$$

From (2.6) it follows that

$$\sigma^{left}(BA) \cup \{0\} = \sigma^{left}(AB) \cup \{0\}, \quad (2.11)$$

while (2.7) implies

$$\sigma_{ess}^{left}(BA) \cup \{0\} = \sigma_{ess}^{left}(AB) \cup \{0\}. \quad (2.12)$$

From (2.11) it follows that

$$\text{acc } \sigma^{left}(BA) = \text{acc } \sigma^{left}(AB). \quad (2.13)$$

Now from (2.10), (2.12) and (2.13) we conclude

$$\beta_{ess}^{left}(BA) \cup \{0\} = \beta_{ess}^{left}(AB) \cup \{0\}, \quad (2.14)$$

which implies (2.8).

Similarly, we obtain the equality

$$\beta_{ess}^{right}(BA) \cup \{0\} = \beta_{ess}^{right}(AB) \cup \{0\}, \quad (2.15)$$

which implies (2.9). \square

If $A \in B(X, Y)$ and $B \in B(Y, X)$, from (2.14) and (2.15) it follows

$$\beta_{ess}(BA) \cup \{0\} = \beta_{ess}(AB) \cup \{0\}. \quad (2.16)$$

A. Jeribi and N. Moalla ([11], Proposition 3.3) proved that if BA is polynomially compact, then AB and BA are generalized Riesz operators. We can improve on this:

Theorem 2.18. *Let $A \in B(X, Y)$ and $B \in B(Y, X)$. Then*

$$BA \in \text{Poly}^{-1}R(X) \iff AB \in \text{Poly}^{-1}R(Y) \quad (2.17)$$

and in that case

$$\pi_{BA}^{-1}(0) \cup \{0\} = \pi_{AB}^{-1}(0) \cup \{0\} \quad (2.18)$$

and

$$\sigma(BA) \setminus (\pi_{BA}^{-1}(0) \cup \{0\}) = \sigma(AB) \setminus (\pi_{AB}^{-1}(0) \cup \{0\}). \quad (2.19)$$

Proof. From Theorem 2.2 and (2.16) we get (2.17) and (2.18). (2.19) follows from (2.6) and (2.18). \square

We remark that if $A \in B(X, Y)$, $B \in B(Y, X)$ and $BA \in \text{Poly}^{-1}R(X)$, then for all $\lambda \in \sigma(BA) \setminus (\pi_{BA}^{-1}(0) \cup \{0\})$ it follows

$$\text{mult}(BA, \lambda) = \text{mult}(AB, \lambda). \quad (2.20)$$

For a proof of the equality (2.20) we refer the reader to the proof of Proposition 3.3 in [11].

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