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Star, left-star, and right-star partial orders in Rickart $*$-rings

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Let $\mathcal{A}$ be a unital ring admitting involution. We introduce an order on $\mathcal{A}$ and show that in the case when $\mathcal{A}$ is a Rickart $*$-ring, this order is equivalent to the well-known star partial order. The notion of the left-star and the right-star partial orders is extended to Rickart $*$-rings. Properties of the star, the left-star and the right-star partial orders are studied in Rickart $*$-rings and some known results are generalized. We found matrix forms of elements a and b when $a \leq b$, where $\leq$ is one of these orders. Conditions under which these orders are equivalent to the minus partial order are obtained. The diamond partial order is also investigated.

Keywords: Rickart $*$-ring; star partial order; left-star partial order; right-star partial order; diamond partial order; minus partial order

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1. Introduction

Let $\mathcal{A}$ be a ring. An involution $*$ in a ring $\mathcal{A}$ is a bijection $a \mapsto a^*$ of $\mathcal{A}$ onto itself such that

$$(a^*)^* = a, \quad (ab)^* = b^*a^*, \quad (a+b)^* = a^* + b^*,$$

for every $a, b \in \mathcal{A}$. A ring admitting an involution will be called an involutory ring. For $a \in \mathcal{A}$, we will denote by a° the right annihilator of a, i.e. the set $a^\circ = \{x \in \mathcal{A} : ax = 0\}$. Similarly we denote the left annihilator ${}^\circ a$ of a, i.e. the set ${}^\circ a = \{x \in \mathcal{A} : xa = 0\}$. A ring $\mathcal{A}$ is called a Rickart ring if for every $a \in \mathcal{A}$ there exist some idempotent elements $p, q \in \mathcal{A}$ such that $a^\circ = p \cdot \mathcal{A}$ and ${}^\circ a = \mathcal{A} \cdot q$. An involutory ring $\mathcal{A}$ is a Rickart $*$-ring if the left annihilator ${}^\circ a$ of any element $a \in \mathcal{A}$ is generated by a self-adjoint idempotent $e \in \mathcal{A}$, i.e. ${}^\circ a = \mathcal{A} \cdot e$ where $e = e^* = e^2$. The analogous property for right annihilators is automatically fulfilled in this case. Note that the self-adjoint idempotent e is unique: let f be a self-adjoint idempotent in $\mathcal{A}$ such that $\mathcal{A}e = \mathcal{A}f$. Then, $e = ee \in \mathcal{A}e = \mathcal{A}f$ hence $ef = xff = xf = e$ for some $x \in \mathcal{A}$. We have $e = ef = fe$ and similarly $f = fe = ef$,

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so $e = f$. Note also that if $\mathcal{A}$ is a Rickart ring, then $\mathcal{A}$ has a unity element. The proof is similar to that used for Rickart $*$-rings.[1]

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. The image and kernel of $A \in B(H)$ will be respectively denoted by Im A and Ker$A$. Šemrl defined in [2] an order in $B(H)$, called the minus partial order, in the following way. For $A, B \in B(H)$ we write $A \overset{-}{\leq} B$ when there exist idempotent operators $P, Q \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$, and $AQ = BQ$.

Šemrl proved that this is indeed a partial order on $B(H)$. He also presented some equivalent definitions of this order (see Theorem 2 in [2]). In [3], authors found another equivalent definition which allowed them to consider the algebraic version of the minus partial order. Namely, the following order, called the minus order on a ring $\mathcal{A}$ with the unit 1, was introduced.

Definition 1 Let $\mathcal{A}$ be a ring with the unit 1. For $a, b \in \mathcal{A}$ we write $a \overset{-}{\leq} b$ when there exist idempotent elements $p, q \in \mathcal{A}$ such that

(i) $^\circ a = \mathcal{A} \cdot (1 - p)$,
(ii) $a^\circ = (1 - q) \cdot \mathcal{A}$,
(iii) $pa = pb$, and
(iv) $aq = bq$.

Authors proved in [3] that this is indeed a partial order when $\mathcal{A}$ is a Rickart ring.

Many other partial orders were defined on $B(H)$. For example, Drazin [4] defined the star partial order in the following way:

$$A \underset{*}{\leq} B \quad \text{when} \quad A^*A = A^*B \text{ and } AA^* = BA^*, \tag{1}$$

$A, B \in B(H)$. In [5], authors proved that the following definition is an equivalent definition of this order on $B(H)$.

Definition 2 For $A, B \in B(H)$ we write $A \underset{*}{\leq} B$ when there exist self-adjoint idempotent operators $P, Q \in B(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
(ii) $\operatorname{Ker} A = \operatorname{Ker} Q$,
(iii) $PA = PB$, and
(iv) $AQ = BQ$.

Following the idea of the paper [3], we will in the next section present another equivalent definition of the star partial order on $B(H)$ which we will then generalize to involutory rings with a unit and show that in the case when the ring is a Rickart $*$-ring, the new definition is equivalent to the original, Drazin's definition of the star partial order (1). We will then consider the left-star and the right-star partial orders in a similar way (see the third section) and the diamond partial order (see the fourth section), and study properties of these orders and generalize some known results.

2. Star partial order

Let us first present a new equivalent definition of the star partial order on $B(H)$.

Definition 3 For $A, B \in B(H)$ we write $A \leq B$ when there exist self-adjoint idempotent operators $P, Q \in B(H)$ such that

(i) $^\circ A = B(H) \cdot (I - P)$,
(ii) $A^\circ = (I - Q) \cdot B(H)$,
(iii) $PA = PB$, and
(iv) $AQ = BQ$.

To prove that this is indeed an equivalent definition of the star partial order, we will need the following two lemmas.

Lemma 2.1 *Let $A, B \in B(H)$. Then,*

(i) $A^\circ = B^\circ$ *if and only if* $\operatorname{Ker} A = \operatorname{Ker} B$.
(ii) $^\circ A = {}^\circ B$ *if and only if* $\overline{\operatorname{Im} A} = \overline{\operatorname{Im} B}$.

Proof Let $A, B \in B(H)$ and $A^\circ = B^\circ$. Suppose $x \in \operatorname{Ker} A$ and let P be a self-adjoint idempotent where $\operatorname{Im} P = \operatorname{Ker} A$. So, $Px = x$ and $AP = 0$. It follows that $BP = 0$ and hence $Bx = 0$. So, $\operatorname{Ker} A \subseteq \operatorname{Ker} B$. In the same way, we show that $\operatorname{Ker} B \subseteq \operatorname{Ker} A$ hence $\operatorname{Ker} A = \operatorname{Ker} B$.

Suppose now $\operatorname{Ker} A = \operatorname{Ker} B$ for $A, B \in B(H)$. Let $X \in A^\circ$. So, $AX = 0$ hence $\operatorname{Im} X \subseteq \operatorname{Ker} A$. We have $BXz = 0$ for every $z \in H$ hence $X \in B^\circ$. It follows that $A^\circ \subseteq B^\circ$. Similarly we get $B^\circ \subseteq A^\circ$ hence $A^\circ = B^\circ$.

Note that $^\circ A = {}^\circ B$ if and only if $(A^*)^\circ = (B^*)^\circ$. So, $^\circ A = {}^\circ B$ if and only if $\operatorname{Ker} A^* = \operatorname{Ker} B^*$ which is equivalent to $\overline{\operatorname{Im} A} = \overline{\operatorname{Im} B}$. □

The following lemma will be presented in a broader context of unitary rings, but note that it holds also for idempotent operators in $B(H)$.

Lemma 2.2 *Let $\mathcal{A}$ be a ring with the unit* 1*, and let p be an idempotent element in $\mathcal{A}$. Then*

(i) $p^\circ = (1 - p) \cdot \mathcal{A}$.
(ii) $^\circ p = \mathcal{A} \cdot (1 - p)$.

Proof The proof is simple so we will omit it (see Lemma 2.2 in [3]). □

Corollary 2.1 *Let $A, B \in B(H)$. Then, $A \leq B$ if and only if $A \underset{*}{\leq} B$.*

Proof Let $P, Q \in B(H)$ be idempotents. By Lemma 2.2, $^\circ A = B(H) \cdot (I - P)$ is equivalent to $^\circ A = {}^\circ P$, and $A^\circ = (I - Q) \cdot B(H)$ is equivalent to $A^\circ = Q^\circ$. From Lemma 2.1, we may conclude that $^\circ A = B(H) \cdot (I - P)$ if and only if $\overline{\operatorname{Im} A} = \operatorname{Im} P$, and $A^\circ = (I - Q) \cdot B(H)$ if and only if $\operatorname{Ker} A = \operatorname{Ker} Q$. □

We will now extend the order defined with Definition 3 from $B(H)$ to an involutory ring with the unit 1. The first two conditions from Definition 2 cannot be transferred but since we may define left and right annihilators of any element in a ring, all the conditions from Definition 3 may be used.

Definition 4 Let $\mathcal{A}$ be an involutory ring with the unit 1. For $a, b \in \mathcal{A}$ we write $a \leq b$ when there exist self-adjoint idempotent elements $p, q \in \mathcal{A}$ such that

(i) $^{\circ}a = \mathcal{A} \cdot (1 - p)$,
(ii) $a^{\circ} = (1 - q) \cdot \mathcal{A}$,
(iii) $pa = pb$, and
(iv) $aq = bq$.

The order $\leq$ will be called the star order on $\mathcal{A}$.

We will prove later on (see Theorems 2 and 1) that this order $\leq$ is indeed a partial order when $\mathcal{A}$ is a Rickart $*$-ring, and that at least in Rickart $*$-rings such an order is rightfully named 'the star order'.

Drazin introduced in [4] the star partial order in a broader sense of proper $*$-semigroups. Recall that a special case of a proper $*$-semigroup are all proper involutory rings with 'properness' as customarily defined via $aa^* = 0$ implies $a = 0$. Recall also (see [6] or [1]) that any Rickart $*$-ring has a unit and is a proper involutory ring (i.e. $aa^* = 0$ implies $a = 0$). Theorem 1 will show that on Rickart $*$-rings the order defined with Definition 4 represents the star partial order which was introduced by Drazin in [4]. In the proof of Theorem 1, we will need the following remark and some auxiliary results.

Remark 2.1 Suppose p and q are idempotent elements in $\mathcal{A}$. Then, any $x \in \mathcal{A}$ can be represented in the following form:

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}_{p \times q}.$$

Here, $x_{1,1} = pxq$, $x_{1,2} = px(1-q)$, $x_{2,1} = (1-p)xq$, $x_{2,2} = (1-p)x(1-q)$. If $x = (x_{i,j})_{p \times q}$ and $y = (y_{i,j})_{p \times q}$, then $x + y = (x_{i,j} + y_{i,j})_{p \times q}$. Moreover, if $r \in \mathcal{A}$ is idempotent and $z = (z_{i,j})_{q \times r}$, then $xz = \left(\sum_{k=1}^{2} x_{i,k} z_{k,j}\right)_{p \times r}$. Thus, if we have idempotents in $\mathcal{A}$, then the usual algebraic operations in $\mathcal{A}$ can be interpreted as simple operations between appropriate 2×2 matrices over $\mathcal{A}$. Furthermore,

$$x^* = \begin{bmatrix} x_{1,1}^* & x_{2,1}^* \\ x_{1,2}^* & x_{2,2}^* \end{bmatrix}_{q^* \times p^*}.$$

Similarly, if $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_n$ are idempotents in $\mathcal{A}$ such that $e_1 + e_2 + \cdots + e_n = 1 = f_1 + f_2 + \cdots + f_n$, and for $i \neq j$, $e_i e_j = 0$ and $f_i f_j = 0$ (i.e. these sums of idempotents represent two decompositions of the identity of $\mathcal{A}$), then for any $x \in \mathcal{A}$ we have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)x(f_1 + f_2 + \cdots + f_n) = \sum_{i,j=1}^{n} e_i x f_j.$$

We may write x as a matrix

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f}$$

where $x_{ij} = e_i x f_j$.

Again, the usual algebraic operations in $\mathcal{A}$ can be interpreted as simple operations between appropriate $n \times n$ matrices over $\mathcal{A}$ (see [3]).

Let

$$\mathrm{LP}(a) := \{p \in \mathcal{A} : p = p^2,\ {}^\circ a = \mathcal{A} \cdot (1-p)\} = \{p \in \mathcal{A} : p = p^2,\ {}^\circ a = {}^\circ p\},$$
$$\mathrm{RP}(a) := \{q \in \mathcal{A} : q = q^2,\ a^\circ = (1-q) \cdot \mathcal{A}\} = \{q \in \mathcal{A} : q = q^2,\ a^\circ = q^\circ\}.$$

Since $\mathcal{A}$ is Rickart $*$-ring, there exists the unique self-adjoint idempotent in $\mathrm{LP}(a)$. We will denote it by $\mathrm{lp}(a)$. Similarly, let $\mathrm{rp}(a)$ denote the unique self-adjoint idempotent in $\mathrm{RP}(a)$. Since $a = \mathrm{lp}(a)a = a\mathrm{rp}(a)$, it follows that

$$a \le b \Leftrightarrow a = \mathrm{lp}(a)b = b\mathrm{rp}(a).$$

It is shown in [3] that if $a \in \mathcal{A}$, $p \in \mathrm{LP}(a)$ and $q \in \mathrm{RP}(a)$, then

$$\mathrm{LP}(a) = \left\{ \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p \times p} : p_1 \in p \cdot \mathcal{A} \cdot (1-p) \right\},$$
$$\mathrm{RP}(a) = \left\{ \begin{bmatrix} q & 0 \\ q_1 & 0 \end{bmatrix}_{q \times q} : q_1 \in (1-q) \cdot \mathcal{A} \cdot q \right\}. \tag{2}$$

There are several characterizations of the star partial order (1) on the set of complex matrices. One of them is as follows. If A and B are complex matrices, then

$$A \underset{*}{\le} B \Leftrightarrow A = PB = BQ$$

for some self-adjoint idempotent matrices P and Q. We will show in Theorem 1 that the same characterization is true in Rickart $*$-rings.

First, let us present the following lemmas which will be used frequently.

Lemma 2.3 *Let $\mathcal{A}$ be Rickart $*$-ring and $a \in \mathcal{A}$. Then,*

(i) $(a^*)^\circ = (\mathrm{lp}(a))^\circ$;
(ii) ${}^\circ(a^*) = {}^\circ(\mathrm{rp}(a))$.

Proof The statement (i) follows by the sequence of equivalences:

$$a^*x = 0 \Leftrightarrow x^*a = 0 \Leftrightarrow x^* \in {}^\circ a = {}^\circ\mathrm{lp}(a) \Leftrightarrow x^*\mathrm{lp}(a) = 0 \Leftrightarrow \mathrm{lp}(a)x = 0.$$

The statement (ii) can be proved similarly. $\square$

LEMMA 2.4 *Let $\mathcal{A}$ be Rickart $*$-ring and $a, b \in \mathcal{A}$. The following are equivalent:*

(i) $a = \mathrm{lp}(a)b$;
(ii) $a = pb$, *for some self-adjoint idempotent* p;
(iii) $a^*a = a^*b$.

Proof (i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii): Suppose $a = pb$ where p is self-adjoint idempotent. Then, $a = pa$ and $a^*a = a^*pb = (pa)^*b = a^*b$.
(iii) $\Rightarrow$ (i): Now, $a^*(b-a) = 0$. By Lemma 2.3, $\mathrm{lp}(a)(b-a) = 0$ so $a = \mathrm{lp}(a)b$. □

Similarly, we obtain the following lemma.

LEMMA 2.5 *Let $\mathcal{A}$ be Rickart $*$-ring and $a, b \in \mathcal{A}$. The following are equivalent:*

(i) $a = b\,\mathrm{rp}(a)$;
(ii) $a = bq$, *for some self-adjoint idempotent* q;
(iii) $aa^* = ba^*$.

THEOREM 1 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$. The following conditions are equivalent:*

(i) $a \leq b$;
(ii) *there exist self-adjoint idempotents p and q such that* $a = pb = bq$;
(iii) $a^*a = a^*b$ *and* $aa^* = ba^*$;
(iv)
$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} \tag{3}$$
where $p = \mathrm{lp}(a)$ *and* $q = \mathrm{rp}(a)$;
(v) *there exist self-adjoint idempotents p and q such that* (3) *holds.*

Proof (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows by Lemmas 2.4 and 2.5.

(i) $\Rightarrow$ (iv): Suppose that $a \leq b$. Then $a = pb = bq$ where $p = \mathrm{lp}(a)$ and $q = \mathrm{rp}(a)$. It is clear that $a = paq$ so a has the requisite matrix form. According to Remark 2.1 representation of b follows by

$$\begin{aligned} pbq &= bqq = bq = a, \\ pb(1-q) &= bq(1-q) = 0, \\ (1-p)bq &= (1-p)pb = 0. \end{aligned}$$

(iv) $\Rightarrow$ (v) is trivial.
(v) $\Rightarrow$ (ii): We have

$$pb = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = a$$

and similarly $a = bq$. □

The next theorem can be proved directly in a similar way as Theorem 3.3 in [3]. Since any Rickart $*$-ring is a proper involutory ring and since Drazin introduced the star partial order (1) in a broader sense of proper $*$-semigroups, the next theorem is a direct consequence of Theorem 1 ((i) $\Leftrightarrow$ (iii)).

THEOREM 2 *Let $\mathcal{A}$ be a Rickart $*$-ring. The order $\leq$, introduced with Definition 4, is a partial order in $\mathcal{A}$.*

From now on, we may and will denote the star partial order on a Rickart $*$-ring by $\leq$.

Remark 2.2 A C^*-algebra that is a Rickart $*$-ring is called a Rickart C^*-algebra. Recall (see [1]) that the algebra $B(H)$ is an example of a Rickart C^*-algebra.

Remark 2.3 Statement (ii) of Theorem 1 shows that the conditions (i) and (ii) in Definition 4 are redundant in the case when $\mathcal{A}$ is a Rickart $*$-ring. Therefore, according to Lemma 2.1, the conditions (i) and (ii) in Definition 2 are also redundant.

Remark 2.4 Statements (iv) and (v) of Theorem 1 give the characterization of all elements b above a given element a under the star partial order.

COROLLARY 2.2 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$. Then, $a \leq b$ if and only if $b - a \leq b$.*

Proof Suppose that $a \leq b$. By Theorem 1, $a = pb = bq$ for some self-adjoint idempotents p and q. Now we have $b-a = b-pb = p'b$ and $b-a = b-bq = bq'$ where $p' = 1-p$ and $q' = 1-q$ are self-adjoint idempotents. Hence, by Theorem 1 again, we conclude that $b-a \leq b$. Since $a + (b-a) = b$, the opposite direction follows. □

THEOREM 3 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$. Then, $a \leq b$ if and only if there exist decompositions of the identity of the ring $\mathcal{A}$*

$$1 = e_1 + e_2 + e_3, \ \ 1 = f_1 + f_2 + f_3$$

where $e_1 = \mathrm{lp}(a)$, $e_2 = \mathrm{lp}(b-a)$, $f_1 = \mathrm{rp}(a)$, $f_2 = \mathrm{rp}(b-a)$, and e_3, f_3 are self-adjoint idempotents, such that

$$a = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}. \tag{4}$$

Proof The proof proceeds along the same lines as the proof of Theorem 3.4 in [3]. In this setting, the proof is slightly simpler. For the sake of completion, let us present it. 'If' part follows by Theorem 1. Suppose now that $a \leq b$. By Theorem 1, we have

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q},$$

where $p = \mathrm{lp}(a)$ and $q = \mathrm{rp}(a)$. Set $e_1 = p$, $e_2 = \mathrm{lp}(b_1) = \mathrm{lp}(b-a)$ and $e_3 = 1 - e_1 - e_2$. Since $pb_1 = p(1-p)b(1-q) = 0$ and ${}^{\circ}b_1 = {}^{\circ}(\mathrm{lp}(b_1))$, we have $p\mathrm{lp}(b_1) = 0$, i.e.

$e_1e_2 = 0$. Also, $e_2e_1 = (e_1e_2)^* = 0$. Now it is easy to check that $1 = e_1 + e_2 + e_3$ is decomposition of the identity. Similarly, we can show that $1 = f_1 + f_2 + f_3$, where $f_1 = q$, $f_2 = \mathrm{rp}(b - a)$, $f_3 = 1 - f_1 - f_2$, is decomposition of the identity. Because of $a = e_1af_1$ and $b - a = e_2(b - a)f_2$, we conclude that a and b have representations (4). □

In the next lemma, we obtain the matrix representations of a and b with respect to $\mathrm{lp}(a)$ and $\mathrm{rp}(a)$ when $a \overset{-}{\leq} b$. The result is, in some way, analogous to Theorem 1 in [7] where the case of complex matrices is considered. Our proof is completely different.

Lemma 2.6 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$. Then, $a \overset{-}{\leq} b$ if and only if there exist $p_1 \in p \cdot \mathcal{A} \cdot (1 - p)$, $q_1 \in (1 - q) \cdot \mathcal{A} \cdot q$, and $b_4 \in (1 - p) \cdot \mathcal{A} \cdot (1 - q)$ such that*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 + p_1b_4q_1 & p_1b_4 \\ b_4q_1 & b_4 \end{bmatrix}_{p\times q} \tag{5}$$

where $p = \mathrm{lp}(a)$ and $q = \mathrm{rp}(a)$.

Proof It is clear that a has matrix representation given in (5). Suppose that $b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times q}$.

By definition, $a \overset{-}{\leq} b$ if and only if there exist $p' \in \mathrm{LP}(a)$ and $q' \in \mathrm{RP}(a)$ such that $a = p'b$ and $a = bq'$. According to the characterizations (2), this is equivalent to

$$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = a = p'b = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times q} = \begin{bmatrix} b_1 + p_1b_3 & b_2 + p_1b_4 \\ 0 & 0 \end{bmatrix}_{p\times q},$$

$$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = a = bq' = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times q} \begin{bmatrix} q & 0 \\ q_1 & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} b_1 + b_2q_1 & 0 \\ b_3 + b_4q_1 & 0 \end{bmatrix}_{p\times q}.$$

This is true if and only if $b_2 = -p_1b_4$, $b_3 = -b_4q_1$ and $b_1 = a_1 + p_1b_4q_1$. We can take $p_1 = -p_1$ and $q_1 = -q_1$. The proof is complete. □

It is clear that the star partial order implies the minus partial order. We will now investigate under what conditions the reverse holds. Many of these results are motivated by the corresponding results for complex matrices.[7,8] Even in the case of matrices, the proofs are not elementary. In some cases, the presented proofs are even simpler than the matrix proofs which involve finite-dimensional linear algebra methods and/or use a Moore–Penrose inverse. Of course, we cannot use these methods.

Theorem 4 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$ such that $a \overset{-}{\leq} b$. The following conditions are equivalent:*

(i) *$a \overset{*}{\leq} b$;*
(ii) *ba^* and a^*b are self-adjoint;*
(iii) *ba^* and a^*b are normal.*

Proof (i) ⇒ (ii): Since $a \overset{*}{\leq} b$, we have $aa^* = ba^*$ and $a^*a = a^*b$ so ba^* and a^*b are self-adjoint.

(ii) ⇒ (iii) is trivial.

(iii) $\Rightarrow$ (i): Suppose that $a \overset{-}{\leq} b$ and that ba^* and a^*b are normal. Then, $ba^*ab^* = ab^*ba^*$. Using the representations given in (5), we obtain

$$ba^*ab^* = \begin{bmatrix} (a_1 + p_1b_4q_1)a_1^*a_1(a_1 + p_1b_4q_1)^* & (a_1 + p_1b_4q_1)a_1^*a_1(b_4q_1)^* \\ b_4q_1a_1^*a_1(a_1 + p_1b_4q_1)^* & b_4q_1a_1^*a_1(b_4q_1)^* \end{bmatrix}_{p\times p},$$

$$ab^*ba^* = \begin{bmatrix} a_1(a_1 + p_1b_4q_1)^*(a_1 + p_1b_4q_1)a_1^* + a_1(b_4q_1)^*b_4q_1a_1^* & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

It follows that $0 = b_4q_1a_1^*a_1(b_4q_1)^* = (b_4q_1a_1^*)(b_4q_1a_1^*)^*$. Since $\mathcal{A}$ is proper, we have $b_4q_1a_1^* = 0$. Hence, $b_4q_1 \in {}^\circ(a_1^*) = {}^\circ(a^*) = {}^\circ(\mathrm{rp}(a)) = {}^\circ q$. Since $q_1 \in \mathcal{A} \cdot q$, we obtain $b_4q_1 = 0$. Similarly, we obtain that $p_1b_4 = 0$, so

$$b = \begin{bmatrix} a_1 & 0 \\ 0 & b_4 \end{bmatrix}_{p\times q}.$$

By Theorem 1, $a \leq b$. □

Lemma 2.7 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a \in \mathcal{A}$. Then,*

(i) $\mathrm{lp}(a) = \mathrm{rp}(a^*)$;
(ii) $\mathrm{lp}(a) = \mathrm{lp}(aa^*)$;
(iii) $\mathrm{rp}(a^*) = \mathrm{rp}(aa^*)$;
(iv) $\mathrm{lp}(aa^*) = \mathrm{rp}(aa^*) = \mathrm{lp}(a) = \mathrm{rp}(a^*)$.

Proof

(i) The proof follows directly from (i) of Lemma 2.3.
(ii) Every Rickart $*$-ring is proper, i.e. $a^*a = 0$ implies $a = 0$ for every $a \in \mathcal{A}$. Let $a^*ax = 0$. Then $(ax)^*(ax) = 0$ and hence $ax = 0$. Similarly, $xaa^* = 0$ implies $xa = 0$. So, $xaa^* = 0$ if and only if $xa = 0$. This means ${}^\circ(aa^*) = {}^\circ a$ so $\mathrm{lp}(aa^*) = \mathrm{lp}(a)$.
(iii) Similarly to (ii).
(iv) Follows by (i), (ii) and (iii). □

Theorem 5 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$ such that $a \overset{-}{\leq} b$. The following conditions are equivalent:*

(i) $a \leq b$;
(ii) $aa^* \leq bb^*$;
(iii) $a^*a \leq b^*b$.

Proof (i) $\Rightarrow$ (ii): Suppose that $a \leq b$. Then $aa^* = ba^* = ab^*$ and $a^*a = a^*b = b^*a$. Therefore,

$$(aa^*)^*(bb^*) = aa^*bb^* = aa^*ab^* = aa^*aa^* = (aa^*)^*(aa^*).$$

Likewise, $(bb^*)(aa^*)^* = (aa^*)(aa^*)^*$, so $aa^* \leq bb^*$.

(ii) $\Rightarrow$ (i): Let $p = \mathrm{lp}(a)$ and $q = \mathrm{rp}(a)$. Then

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \quad \text{and} \quad aa^* = \begin{bmatrix} a_1a_1^* & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

Suppose that $aa^* \leq bb^*$. From Theorem 1 ((i) $\Leftrightarrow$ (iv)) and Lemma 2.7, it follows that

$$bb^* = \begin{bmatrix} a_1a_1^* & 0 \\ 0 & c \end{bmatrix}_{p\times p}, \tag{6}$$

where $c \in (1-p)\cdot\mathcal{A}\cdot(1-p)$. By Lemma 2.6, the condition $a \overset{-}{\leq} b$ gives

$$b = \begin{bmatrix} a_1 + p_1b_4q_1 & p_1b_4 \\ b_4q_1 & b_4 \end{bmatrix}_{p\times q}. \tag{7}$$

It follows that

$$bb^* = \begin{bmatrix} (a_1+p_1b_4q_1)(a_1+p_1b_4q_1)^* + (p_1b_4)(p_1b_4)^* & (a_1+p_1b_4q_1)(b_4q_1)^* + p_1b_4b_4^* \\ b_4q_1(a_1+p_1b_4q_1)^* + b_4(p_1b_4)^* & b_4q_1(b_4q_1)^* + b_4b_4^* \end{bmatrix}_{p\times p}. \tag{8}$$

From (6) and (8), we conclude that

$$\begin{aligned} a_1a_1^* &= (a_1+p_1b_4q_1)(a_1+p_1b_4q_1)^* + (p_1b_4)(p_1b_4)^* \\ &= (a_1+p_1b_4q_1)a_1^* + (a_1+p_1b_4q_1)(b_4q_1)^*p_1^* + (p_1b_4)(p_1b_4)^* \\ &= (a_1+p_1b_4q_1)a_1^* + (-p_1b_4b_4^*)p_1^* + (p_1b_4)(p_1b_4)^* \\ &= (a_1+p_1b_4q_1)a_1^* = a_1a_1^* + p_1b_4q_1a_1^*. \end{aligned} \tag{9}$$

Hence, $p_1b_4q_1a_1^* = 0$. By Lemma 2.3 (ii), we conclude $0 = p_1b_4q_1q = p_1b_4q_1$, since $q_1 \in (1-q)\cdot\mathcal{A}\cdot q$. Now, from (9) we obtain that $(p_1b_4)(p_1b_4)^* = 0$. By properness, $p_1b_4 = 0$. From $(a_1+p_1b_4q_1)(b_4q_1)^* + p_1b_4b_4^* = 0$, we have $a_1(b_4q_1)^* = 0$. Thus, $q(b_4q_1)^* = 0$, so $b_4q_1 = 0$. Now, (7) becomes

$$b = \begin{bmatrix} a_1 & 0 \\ 0 & b_4 \end{bmatrix}_{p\times q}.$$

By Theorem 1, $a \leq b$.

(i) $\Leftrightarrow$ (iii): Similarly to (i) $\Leftrightarrow$ (ii). □

LEMMA 2.8 *Let $\mathcal{A}$ be Rickart $*$-ring. If $a \in \mathcal{A}$ is normal, then* $\mathrm{lp}(a) = \mathrm{rp}(a)$.

Proof By Lemma 2.7, $\mathrm{rp}(a) = \mathrm{rp}(a^*a) = \mathrm{rp}(aa^*) = \mathrm{rp}(a^*) = \mathrm{lp}(a)$. □

THEOREM 6 *Let $\mathcal{A}$ be a Rickart $*$-ring. Let $a, b \in \mathcal{A}$ where a is normal and $a \overset{-}{\leq} b$. Then, the following conditions are equivalent:*

(i) *$a \leq b$;*
(ii) *a and b commute;*
(iii) *a^* and b commute;*
(iv) *a and b^* commute;*
(v) *a^* and b^* commute.*

Proof We will prove only the equivalence of (i) and (ii). Other equivalences can be proved in a similar way.

(i) $\Rightarrow$ (ii): Suppose that a is normal and $a \leq b$. Theorem 1 and Lemma 2.8 give

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $p = \mathrm{lp}(a) = \mathrm{rp}(a)$. It is easily seen that

$$ab = ba = \begin{bmatrix} a_1^2 & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

(ii) $\Rightarrow$ (i): Since a is normal and $a \overset{-}{\leq} b$, Lemmas 2.8 and 2.6 give

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \quad \text{and} \quad b = \begin{bmatrix} a_1 + p_1 b_4 q_1 & p_1 b_4 \\ b_4 q_1 & b_4 \end{bmatrix}_{p\times p}$$

where $p = \mathrm{lp}(a) = \mathrm{rp}(a)$, $p_1 \in p \cdot \mathcal{A} \cdot (1-p)$ and $q_1 \in (1-p) \cdot \mathcal{A} \cdot p$. Suppose that $ab = ba$. It follows that

$$\begin{bmatrix} a_1^2 + a_1 p_1 b_4 q_1 & a_1 p_1 b_4 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} a_1^2 + p_1 b_4 q_1 a_1 & 0 \\ b_4 q_1 a_1 & 0 \end{bmatrix}_{p\times p}$$

and hence $a_1 p_1 b_4 = 0$. Since $a_1^\circ = p^\circ$ and $p_1 \in p \cdot \mathcal{A} \cdot (1-p)$, it follows that $p_1 b_4 = 0$. Similarly, from $b_4 q_1 a_1 = 0$ we obtain $b_4 q_1 = 0$. Thus,

$$b = \begin{bmatrix} a_1 & 0 \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

and $a \leq b$. □

Recall that an element $w \in \mathcal{A}$ is a partial isometry if $ww^*w = w$.

Theorem 7 *Let $\mathcal{A}$ be a Rickart $*$-ring and let $a, b \in \mathcal{A}$ be partial isometries. Then $a \leq b$ if and only if $a \overset{-}{\leq} b$.*

Proof The only if part is trivial. Suppose that $a \overset{-}{\leq} b$. We need to show that $a \leq b$. By Theorem 5, it is sufficient to show that $aa^* \leq bb^*$. By assumption, there exist idempotents p and q such that $a = pb = bq$. This gives $a = bq = bb^*bq = bb^*a$, so

$$(aa^*)(aa^*)^* = aa^* = bb^*aa^* = (bb^*)(aa^*)^*$$

and

$$(aa^*)^*(aa^*) = aa^* = aa^*bb^* = (aa^*)^*(bb^*).$$

Thus, $aa^* \leq bb^*$. □

We will present at the end of this section some inheritance properties of the star partial order. First, let us extend a property of the minus partial order from $B(H)$ (see [2]) to any ring with a unit.

LEMMA 2.9 *Let $\mathcal{A}$ be a ring with a unit and $a, b \in \mathcal{A}$. If $b^2 = b$ and $a \overset{-}{\leq} b$, then $a^2 = a$.*

Proof Let b be an idempotent element and $a \overset{-}{\leq} b$. By definition, there exist idempotents $p \in \mathrm{LP}(a)$ and $q \in \mathrm{RP}(a)$ such that $a = pb = bq$. We have $a = bq = b^2 q = ba$ so $1 - b \in {}^\circ a = {}^\circ p$, i.e. $p = bp$. Therefore, $p = p^2 = pbp = ap$, so

$$a = pa = apa = aa = a^2. \qquad \square$$

With the next theorem, we will show that if b is an idempotent, a self-adjoint idempotent, or a partial isometry, then every a satisfying $a \leq b$ inherits the same property.

THEOREM 8 *Let $\mathcal{A}$ be a Rickart $*$-ring and $a, b \in \mathcal{A}$. Suppose $a \leq b$.*

(i) *If $b^2 = b$, then $a^2 = a$.*
(ii) *If $b^2 = b = b^*$, then $a^2 = a = a^*$.*
(iii) *If $bb^*b = b$, then $aa^*a = a$.*

Proof

(i) Suppose $a \leq b$. Let first $b^2 = b$. Since $a \leq b$, we have $a \overset{-}{\leq} b$ and hence, by Lemma 2.9, $a^2 = a$.

(ii) Let now $b^2 = b = b^*$. As before we may conclude that $a^2 = a$ so we have to prove only that a is a self-adjoint element. Since $a \leq b$, we have $a = pb = bq$ where $p = \mathrm{lp}(a)$ and $q = \mathrm{rp}(a)$. From the proof of Lemma 2.9, we know that $p = bp$, so

$$a = pa = bpa = b^* p^* a = (pb)^* a = a^* a.$$

Thus, $a = a^*$.

(iii) Finally, let $b = bb^*b$. Since $a \leq b$, there exist self-adjoint idempotent elements $p, q \in \mathcal{A}$ such that $pa = aq = a$ and

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q}.$$

We have

$$\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q} = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q} \begin{bmatrix} a_1^* & 0 \\ 0 & b_1^* \end{bmatrix}_{q \times p} \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q}$$

hence $a_1 = a_1 a_1^* a_1$. So, a is a partial isometry since $a_1 = paq = a$.

$\square$

3. Left-star and right-star partial orders

Let $M_{m \times n}$ be the set of all $m \times n$ complex matrices and let as before $\operatorname{Im} A$ denote the image (the column space) of $A \in M_{m \times n}$. The left-star and the right-star partial orders were introduced by Baksalary and Mitra in [9] in the following way.

Definition 5 The left-star partial order on $M_{m\times n}$ is a relation defined by

$$A *\leq B \quad \text{when} \quad A^*A = A^*B \quad \text{and} \quad \operatorname{Im} A \subseteq \operatorname{Im} B.$$

Definition 6 The right-star partial order on $M_{m\times n}$ is a relation defined by

$$A \leq* B \quad \text{when} \quad AA^* = BA^* \quad \text{and} \quad \operatorname{Im} A^* \subseteq \operatorname{Im} B^*.$$

Baksalary and Mitra proved that for $A, B \in M_{m,n}$, $A \underset{*}{\leq} B$ implies $A *\leq B$ and $A \leq* B$, and $A *\leq B$ or $A \leq* B$ implies $A \overset{-}{\leq} B$. The reverse implications are in general not true.

Following Šemrl's approach [2] of defining the minus partial order on $B(H)$ and using known facts about the relation between the star, the left-star, the right-star and the minus partial orders on the set of complex matrices, authors introduced in [10] the following order on $B(H)$.

Definition 7 For $A, B \in B(H)$ we write $A *\leq B$ when there exist a self-adjoint idempotent operator P and an idempotent operator $Q \in B(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
(ii) $\operatorname{Ker} A = \operatorname{Ker} Q$,
(iii) $PA = PB$, and
(iv) $AQ = BQ$.

It was proved in [10] that that for $A, B \in B(H)$, $A *\leq B$ if and only if $A^*A = A^*B$ and $\operatorname{Im} A \subseteq \operatorname{Im} B$, so the order from Definition 7 is called the left-star partial order on $B(H)$. Similarly, the following definition introduces the right-star partial order on $B(H)$.

Definition 8 For $A, B \in B(H)$ we write $A \leq* B$ when there exist an idempotent operator P and a self-adjoint idempotent operator $Q \in B(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
(ii) $\operatorname{Ker} A = \operatorname{Ker} Q$,
(iii) $PA = PB$, and
(iv) $AQ = BQ$.

For $A, B \in B(H)$, we have $A \leq* B$ in the sense of Definition 8 if and only if $AA^* = BA^*$ and $\operatorname{Im} A^* \subseteq \operatorname{Im} B^*$ (see [10]).

Let us now present an equivalent definition of the left-star partial order on $B(H)$.

Definition 9 For $A, B \in B(H)$ we write $A *\leq B$ when there exist a self-adjoint idempotent operator $P \in B(H)$ and an idempotent operator Q such that

(i) ${}^\circ A = B(H) \cdot (I - P)$,
(ii) $A^\circ = (I - Q) \cdot B(H)$,
(iii) $PA = PB$, and
(iv) $AQ = BQ$.

Because this is indeed an equivalent definition to Definition 7, it follows from Lemmas 2.1 and 2.2. Note that we may in a similar way introduce an equivalent definition of the right-star partial order.

Let us now extend these notions of the left-star and the right-star orders to involutory rings with a unit.

Definition 10 Let $\mathcal{A}$ be an involutory ring with the unit 1. For $a, b \in \mathcal{A}$ we write $a \ast\!\le b$ when there exist a self-adjoint idempotent $p \in \mathcal{A}$ and an idempotent $q \in \mathcal{A}$ such that

(i) $^\circ a = \mathcal{A} \cdot (1 - p)$,
(ii) $a^\circ = (1 - q) \cdot \mathcal{A}$,
(iii) $pa = pb$, and
(iv) $aq = bq$.
The order $\ast\!\le$ will be called the left-star order on $\mathcal{A}$.

Definition 11 Let $\mathcal{A}$ be an involutory ring with the unit 1. For $a, b \in \mathcal{A}$ we write $a \le\!\ast\, b$ when there exist an idempotent $p \in \mathcal{A}$ and a self-adjoint idempotent $q \in \mathcal{A}$ such that

(i) $^\circ a = \mathcal{A} \cdot (1 - p)$,
(ii) $a^\circ = (1 - q) \cdot \mathcal{A}$,
(iii) $pa = pb$, and
(iv) $aq = bq$.
The order $\le\!\ast$ will be called the right-star order on $\mathcal{A}$.

Remark 3.1 By Lemmas 2.4 and 2.5, the condition (i) in Definition 10 and the condition (ii) in Definition 11 are redundant in the case when $\mathcal{A}$ is a Rickart $\ast$-ring. Also, the condition (i) in Definition 7 and the condition (ii) in Definition 8 are redundant.

Remark 3.2 It is easy to check that $a \ast\!\le b$ if and only if $a^* \le\!\ast\, b^*$. This is provided by $x^\circ = y^\circ \Leftrightarrow {}^\circ(x^*) = {}^\circ(y^*)$.

Remark 3.3 Let $\mathcal{A}$ be an involutory ring with the unit 1. Comparing Definition 1 with Definitions 10 and 11, it is clear that $a \ast\!\le b \Rightarrow a \overset{-}{\le} b$ and $a \le\!\ast\, b \Rightarrow a \overset{-}{\le} b$.

The next theorem can be proved in a similar way as Theorem 1.

Theorem 9 *Let $\mathcal{A}$ be a Rickart $\ast$-ring and $a, b \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) $a \ast\!\le b$;
(ii) *there exist a self-adjoint idempotent p and an idempotent $q \in \mathrm{RP}(a)$ such that $a = pb = bq$;*
(iii) *$a^*a = a^*b$ and $a = bq$ for some idempotent $q \in \mathrm{RP}(a)$;*
(iv)

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \textit{and} \quad b = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q}, \tag{10}$$

where $p = \mathrm{lp}(a)$ and $q \in \mathrm{RP}(a)$;

(v) *there exist a self-adjoint idempotent p and an idempotent $q \in \mathrm{RP}(a)$ such that (10) holds.*

Of course, the analogous theorem for right-star order holds.

The following theorem states that orders introduced with Definitions 10 and 11 are partial orders when $\mathcal{A}$ is a Rickart $*$-ring.

THEOREM 10 *Let $\mathcal{A}$ be a Rickart $*$-ring. The orders $*\leq$ and $\leq *$, defined with Definitions 10 and 11, are partial orders on $\mathcal{A}$.*

Proof The proof proceeds along the same lines as the proof of Theorem 3.3 in [3]. For the sake of completion, let us present it in its entirety.

We give the proof only for $*\leq$ relation. The fact that $\leq *$ is partial order can be proved analogously. Since $\mathcal{A}$ is Rickart $*$-ring, the relation $*\leq$ is reflexive. To prove antisymmetry, suppose that $a \leq b$ and $b \leq a$. By Theorem 9, we have matrix representations (10) and there exists the self-adjoint idempotent r such that $b = ra$. We have

$$b - a = (1-p)b(1-q) = (1-p)ra(1-q) = (1-p)raq(1-q) = 0.$$

To show the transitivity of $*\leq$, suppose that $a*\leq b$ and $b*\leq c$. Then, there exist self-adjoint idempotents $p = \mathrm{lp}(a)$, $r = \mathrm{lp}(b)$, and idempotents $q \in \mathrm{RP}(a)$, $s \in \mathrm{RP}(b)$ such that a and b have matrix forms as in (10) and $b = rc = cs$. Suppose that

$$r = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}_{p\times p}, \quad s = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}_{q\times q} \quad \text{and} \quad c = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p\times q}.$$

Since

$$0 = b(1-s) = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} \begin{bmatrix} q - s_1 & -s_2 \\ -s_3 & 1-q-s_4 \end{bmatrix}_{q\times q} = \begin{bmatrix} a_1(q-s_1) & -a_1 s_2 \\ -b_1 s_3 & b_1(1-q-s_4) \end{bmatrix}_{p\times q},$$

we have $a_1(q - s_1) = 0$, $a_1 s_2 = 0$, and $b_1 s_3 = 0$. Since $a_1^\circ = q^\circ$, we obtain that $0 = q(q - s_1) = q - s_1$ and $0 = qs_2 = s_2$. From $b = cs$, we conclude that

$$\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}_{p\times q} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p\times q} \begin{bmatrix} q & 0 \\ s_3 & s_4 \end{bmatrix}_{q\times q} = \begin{bmatrix} c_1 + c_2 s_3 & c_2 s_4 \\ c_3 + c_4 s_3 & c_4 s_4 \end{bmatrix}_{p\times q}.$$

Hence,

$$a_1 = c_1 + c_2 s_3 \quad \text{and} \quad 0 = c_3 + c_4 s_3. \tag{11}$$

Let $q' = \begin{bmatrix} q & 0 \\ s_3 & 0 \end{bmatrix}_{q\times q}$. By formulas (2), we obtain $q' \in \mathrm{RP}(a)$. As $b_1 s_3 = 0$, we also have $bq' = a$. From (11) and

$$cq' = \begin{bmatrix} c_1 + c_2 s_3 & 0 \\ c_3 + c_4 s_3 & 0 \end{bmatrix}_{p\times q}$$

it follows that $cq' = a$.

Similar consideration shows that $r_1 = p$ and $r_3 = 0$. Since r and p are self-adjoint, we conclude that $r_2 = 0$. Now, from $b = rc$ it is easy to show that $a = pc$. By definition, we conclude that $a*\leq c$. □

The following theorem is a generalization of the corresponding results for complex matrices (see Theorem 4.2. in [9] and Theorem 2.1 in [11]).

THEOREM 11 *Let $\mathcal{A}$ be Rickart $*$-ring and $a, b \in \mathcal{A}$. Then, the following conditions are equivalent:*

(i) $a \ast\!\leq b$;

(ii) $a \overset{-}{\leq} b$ *and* $a^* b$ *is self-adjoint;*

(iii)

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ b_4 q_1 & b_4 \end{bmatrix}_{p \times q},$$

where $p = \mathrm{lp}(a)$ *and* $q = \mathrm{rp}(a)$.

Proof (i) $\Rightarrow$ (ii) follows by Remark 3.3 and Theorem 9 ((i) $\Rightarrow$ (iii)).
(ii) $\Rightarrow$ (iii): Suppose that $a \overset{-}{\leq} b$ and $a^* b$ is self-adjoint. By Lemma 2.6, it follows that

$$a^* b = \begin{bmatrix} a_1^* & 0 \\ 0 & 0 \end{bmatrix}_{q \times p} \begin{bmatrix} a_1 + p_1 b_4 q_1 & p_1 b_4 \\ b_4 q_1 & b_4 \end{bmatrix}_{p \times q} = \begin{bmatrix} a_1^* a_1 + a_1^* p_1 b_4 q_1 & a_1^* p_1 b_4 \\ 0 & 0 \end{bmatrix}_{q \times q},$$

where $p = \mathrm{lp}(a)$ and $q = \mathrm{rp}(a)$. Since $a^* b$ is self-adjoint, we have $a_1^* p_1 b_4 = 0$, and thus $p_1 b_4 \in (a_1^*)^\circ = \mathrm{rp}(a^*)^\circ = \mathrm{lp}(a)^\circ = p^\circ$. As $p_1 \in p\mathcal{A}(1-p)$ we have $p_1 b_4 = 0$, so the representation of b follows.

(iii) $\Rightarrow$ (i): Suppose that a and b have the given representations and let $q' = \begin{bmatrix} q & 0 \\ -q_1 & 0 \end{bmatrix}_{q \times q}$. By (2), $q' \in \mathrm{RP}(a)$. Direct computations show that $a^* a = a^* b$ and $a = bq'$. By Theorem 9, it follows that $a \ast\!\leq b$. □

THEOREM 12 *Let $\mathcal{A}$ be Rickart $*$-ring and $a, b \in \mathcal{A}$. Then, the following conditions are equivalent:*

(i) $a \leq\!\ast\, b$;

(ii) $a \overset{-}{\leq} b$ *and* $b a^*$ *is self-adjoint;*

(iii)

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad b = \begin{bmatrix} a_1 & p_1 b_4 \\ 0 & b_4 \end{bmatrix}_{p \times q},$$

where $p = \mathrm{lp}(a)$ *and* $q = \mathrm{rp}(a)$.

Proof The proof is similar to the proof of Theorem 11. □

THEOREM 13 *Let $\mathcal{A}$ be Rickart $*$-ring and let $a, b \in \mathcal{A}$ be normal elements. Then*

$$a \ast\!\leq b \quad \Leftrightarrow \quad a \leq\!\ast\, b \quad \Leftrightarrow \quad a \leq b.$$

Proof Suppose that a and b are normal and $a \mathrel{*\!\leq} b$. By Lemma 2.8, $\mathrm{lp}(a) = \mathrm{rp}(a)$. By Theorem 11, it follows that

$$bb^* = \begin{bmatrix} a_1 & 0 \\ b_4q_1 & b_4 \end{bmatrix}_{p\times p} \begin{bmatrix} a_1^* & (b_4q_1)^* \\ 0 & b_4^* \end{bmatrix}_{p\times p} = \begin{bmatrix} a_1a_1^* & a_1(b_4q_1)^* \\ b_4q_1a_1^* & b_4q_1(b_4q_1)^* + b_4b_4^* \end{bmatrix}_{p\times p},$$

$$b^*b = \begin{bmatrix} a_1^*a_1 + (b_4q_1)^*b_4q_1 & (b_4q_1)^*b_4 \\ b_4^*b_4q_1 & b_4^*b_4 \end{bmatrix}_{p\times p},$$

where $p = \mathrm{lp}(a)$. Since a and b are normal, it follows that $a_1a_1^* = a_1^*a_1$ and $a_1a_1^* = a_1^*a_1 + (b_4q_1)^*b_4q_1$. Therefore, $(b_4q_1)^*b_4q_1 = 0$ and since $\mathcal{A}$ is proper ring, we obtain $b_4q_1 = 0$. By Theorem 1 (iv) $\Rightarrow$ (i), it follows that $a \leq b$. Since the star order induces the left-star order, we have $a \mathrel{*\!\leq} b$ if and only if $a \leq b$. In the same manner, we can show that $a \mathrel{\leq\!*} b$ if and only if $a \leq b$. $\square$

An element a in a ring $\mathcal{A}$ is called regular if there exists $x \in \mathcal{A}$ such that $axa = a$. Recall that a von Neumann regular ring $\mathcal{A}$ is a ring where every $a \in \mathcal{A}$ is regular. We call an element $a \in \mathcal{A}$ a $*$-regular element or Moore–Penrose invertible element with respect to * if there is $x \in \mathcal{A}$ with

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

If such x exists, we write $x = a^\dagger$ and call it the Moore–Penrose inverse of a. A ring $\mathcal{A}$ where every element is $*$-regular will be called a $*$-regular ring. It is known (see [4] or [12]) that in proper involutory rings, an element a is a $*$-regular element if and only if aa^* and a^*a are both regular. It follows that a proper involutory ring $\mathcal{A}$ is $*$-regular whenever every element of $\mathcal{A}$ is regular (for example, $M_n(\mathbb{C})$). Since every Rickart $*$-ring is a proper involutory ring, it follows that every von Neumann regular ring which is also a Rickart $*$-ring is a $*$-regular ring.

Let now $\mathcal{C}$ be a C^*-algebra with the unit 1. The subset of $\mathcal{C}$ consisting of all Moore–Penrose invertible elements of $\mathcal{C}$ will be denoted by $\mathcal{C}^\dagger$. Inspired by a paper of Baksalary and Mitra [9], Liu, Benítez and Zhong generalized in [13] the left-star and the right-star partial orders from the set of all complex $m \times n$ matrices to a $\mathcal{C}^\dagger$ in the following way.

For $a \in \mathcal{C}^\dagger$ let $a_l^\pi = 1 - a^\dagger a$ and $a_r^\pi = 1 - aa^\dagger$. Observe that a_l^π and a_r^π are self-adjoined idempotents. For $a, b \in \mathcal{C}^\dagger$ let

$$a \mathrel{*\!\leq} b \quad \text{if} \quad a^*a = a^*b \text{ and } b_r^\pi a = 0$$

and

$$a \mathrel{\leq\!*} b \quad \text{if} \quad aa^* = ba^* \text{ and } ab_l^\pi = 0.$$

It is easy to prove (see [10]) that if $A, B \in B(H)$ are Moore–Penrose invertible elements, then $B_r^\pi A = 0$ if and only if $\operatorname{Im} A \subseteq \operatorname{Im} B$, and $AB_l^\pi = 0$ if and only if $\operatorname{Im} A^* \subseteq \operatorname{Im} B^*$. Recall that an operator in $B(H)$ has a Moore–Penrose inverse if and only if its image is closed. It follows that on the set of operators from $B(H)$ with a closed image, the above orders are equivalent respectively to the left-star and the right-star partial order.

Let us now extend this observation to $*$-regular rings.

Definition 12 Let $\mathcal{A}$ be a $*$-regular ring with the unit 1. Then $a \mathrel{*\!\leq} b$ if $a^*a = a^*b$ and $b_r^\pi a = 0$.

Definition 13 Let $\mathcal{A}$ be a $*$-regular ring with the unit 1. Then $a \leq * b$ if $aa^* = ba^*$ and $ab_l^\pi = 0$.

Here, b_r^π and b_l^π are defined as before.

THEOREM 14 *Let $\mathcal{A}$ be a Rickart $*$-ring which is also a von Neumann regular ring. Then, Definition 10 is equivalent to Definition 12, and Definition 11 is equivalent to Definition 13.*

Proof Let us prove only that Definition 10 is equivalent to Definition 12. The other equivalence may be proved similarly.

Suppose $a * \leq b$ in the sense of Definition 10. So, there exists a self-adjoint idempotent $p \in \mathcal{A}$ such that ${}^\circ a = \mathcal{A} \cdot (1-p)$. Hence, $a = pa$ and $a^* = a^* p$. Since $pa = pb$, it follows that

$$a^* a = a^* pa = a^* pb = a^* b.$$

From $a * \leq b$, we have $a \overset{-}{\leq} b$ (see Definition 1). Theorem 2.6 in [3] states that in a von Neumann regular ring with the unit, we have $a \overset{-}{\leq} b$ (in the sense of Definition 1) if and only if there exists $x \in \mathcal{A}$ such that $ax = bx$, $xa = xb$ where $axa = a$ (see [14] for the original definition of the minus partial order). Since $a \overset{-}{\leq} b$, such x exists. From $ax = bx$, it follows $a = bxa$. Recall that $\mathcal{A}$ is a $*$-regular ring so there exists the Moore–Penrose inverse $b^\dagger$ of b. This yields

$$bb^\dagger a = bb^\dagger bxa = bxa = a.$$

So, $b_r^\pi a = 0$ hence Definition 10 implies Definition 12.

Conversely, suppose that for $a, b \in \mathcal{A}$, we have $a^* a = a^* b$ and $b_r^\pi a = 0$. There exist self-adjoint idempotent elements $p, r \in \mathcal{A}$ such that ${}^\circ a = \mathcal{A} \cdot (1-p)$,

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times r} \quad \text{and} \quad a^* = \begin{bmatrix} a_1^* & 0 \\ 0 & 0 \end{bmatrix}_{r \times p}.$$

Let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times r}.$$

Since $a^* a = a^* b$, we may, as in the proof of Theorem 1, show that $b_2 = 0$ and $a_1 = b_1$. From

$$p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

it follows that $pa = pb$.

Let us now prove that there exists an idempotent $q \in \mathcal{A}$ such that $a^\circ = (1-q) \cdot \mathcal{A}$ and $aq = bq$. Recall that if $a \overset{-}{\leq} b$, then such idempotent q exists. To conclude the proof we will show that from our assumptions $a^* a = a^* b$ and $bb^\dagger a = a$, it follows that there exists $x \in \mathcal{A}$ such that $ax = bx$, $xa = xb$ where $axa = a$.

Since a is a regular element, there exists $c \in \mathcal{A}$ such that $aca = a$. It is known [15] (see also [16]) that the Moore–Penrose inverse $a^\dagger$ of a von Neumann regular element a in an involutory ring can be characterized by the invertibility of the element $a^* a + 1 - ca$. Moreover,

$$a^\dagger = ((a^* a + 1 - ca)^*)^{-1} a^*. \tag{12}$$

Let $z \in (a^*)^\circ$. Then, $a^*z = 0$ which yields by equation (12) that $a^\dagger z = 0$ and hence $z \in (a^\dagger)^\circ$. Let $z \in (a^\dagger)^\circ$. Again, by using Equation (12), we may conclude that $z \in (a^*)^\circ$ hence $(a^*)^\circ = (a^\dagger)^\circ$.

From $a^*a = a^*b$, it follows that $a^*(b-a) = 0$ hence $(b-a) \in (a^*)^\circ = (a^\dagger)^\circ$. We have $a^\dagger(b-a) = 0$ and hence $a^\dagger a = a^\dagger b$. It follows that $aa^\dagger a = aa^\dagger b$ which yields $a = aa^\dagger b$. So,

$$ab^\dagger b = aa^\dagger bb^\dagger b = aa^\dagger b = a$$

and

$$ab^\dagger a = aa^\dagger bb^\dagger a = aa^\dagger a = a.$$

Since $bb^\dagger a = ab^\dagger b = ab^\dagger a = a$, we may conclude from Lemma 1 in [17] (compare items (iv) and (v)) that there exists $x \in \mathcal{A}$ such that $axa = a$, $xax = x$, $xa = xb$ and $ax = bx$. □

Remark 3.4 We may prove Theorem 14 in an alternative way, using Lemmas 2.4 and 2.5: Suppose $a \ast\!\leq b$ in the sense of Definition 10. So, there exist self-adjoint idempotent p such that $a = pb$ and there exist idempotent q such that and $a = bq$. By Lemma 2.4, $a^*a = a^*b$. Also, $bb^\dagger a = bb^\dagger bq = bq = a$, hence $b_r^\pi a = 0$. Conversely, suppose that for $a, b \in \mathcal{A}$ we have $a^*a = a^*b$ and $b_r^\pi a = 0$, i.e. $a = bb^\dagger a$. By Lemma 2.4, it follows that there exist self-adjoint idempotent p such that $a = pb$ and thus $a = pa$. Let $q = b^\dagger a$. Then $bq = bb^\dagger a = a$ and $q^2 = b^\dagger ab^\dagger a = b^\dagger pbb^\dagger a = b^\dagger pa = b^\dagger a = q$. It remains to show that $q^\circ = a^\circ$. But, $b^\dagger ax = 0 \Rightarrow bb^\dagger ax = 0 \Rightarrow ax = 0 \Rightarrow b^\dagger ax = 0$.

4. Diamond partial order

Baksalary and Hauke introduced in [18] another partial order on $M_{m\times n}$ which is also related to the minus partial order and the star partial order.

Definition 14 For $A, B \in M_{m\times n}$ we write $A \overset{\diamond}{\leq} B$ when $AB^*A = AA^*A$, $\operatorname{Im} A \subseteq \operatorname{Im} B$ and $\operatorname{Im} A^* \subseteq \operatorname{Im} B^*$. The order $\overset{\diamond}{\leq}$ will be called the diamond order on $M_{m\times n}$.

Let $\mathcal{A}$ be the set of all Moore–Penrose invertible elements in $B(H)$. It is known that $A \in \mathcal{A}$ if and only if $\operatorname{Im} A$ is closed. Suppose we define the diamond order on $\mathcal{A}$ in the same way as in Definition 14. It may be proved that the diamond order is also a partial order on $\mathcal{A}$ but we will omit the proof since a more general result will be presented in the continuation.

Note that for $A, B \in \mathcal{A}$ we have $B_r^\pi A = 0$ if and only if $\operatorname{Im} A \subseteq \operatorname{Im} B$, and $AB_l^\pi = 0$ if and only if $\operatorname{Im} A^* \subseteq \operatorname{Im} B^*$. This observation provides a motivation for the following definition.

Definition 15 Let $\mathcal{A}$ be a $*$-regular ring with the unit 1, and $a, b \in \mathcal{A}$. For $a, b \in \mathcal{A}$ we write $a \overset{\diamond}{\leq} b$ when $ab^*a = aa^*a$, $b_r^\pi a = 0$, and $ab_l^\pi = 0$. The order $\overset{\diamond}{\leq}$ will be called the diamond order on $\mathcal{A}$.

The order defined with Definition 15 is a partial order. Indeed, observe that this is a corollary of Lemma 5, Theorem 2 and Corollary 1 in [19]. For the sake of completeness, let us provide an alternative, more direct proof. Let $\mathcal{A}$ be a $*$-regular ring with the unit 1.

Clearly, $\overset{\diamond}{\leq}$ is reflexive. Let $a \overset{\diamond}{\leq} b$ and $b \overset{\diamond}{\leq} a$, $a, b \in \mathcal{A}$. Then, $a = bb^\dagger a = ab^\dagger b$ and $b = aa^\dagger b = ba^\dagger a$. Also, it is not hard to prove (see for example [18]) that $a^\dagger ba^\dagger = a^\dagger$ is equivalent to $ab^*a = aa^*a$. Hence,

$$a = aa^\dagger a = aa^\dagger ba^\dagger a = ba^\dagger a = b.$$

For the proof of transitivity, let us first observe the following. Suppose $b_r^\pi a = 0$ and $ab_l^\pi = 0$. So, $a = bb^\dagger a = ab^\dagger b$. Let $d = b^\dagger ab^\dagger$. Then

$$a = bb^\dagger a = bb^\dagger ab^\dagger b = bdb.$$

Let now $a \overset{\diamond}{\leq} b$ and $b \overset{\diamond}{\leq} c$, for some $a, b, c \in \mathcal{A}$. Since $a = bb^\dagger a = ab^\dagger b$ and $b = cc^\dagger b = bc^\dagger c$, we have

$$a = bb^\dagger a = cc^\dagger bb^\dagger a = cc^\dagger a$$

and

$$a = ab^\dagger b = ab^\dagger bc^\dagger c = ac^\dagger c.$$

It follows that $c_r^\pi a = 0$ and $ac_l^\pi = 0$. Since $a \overset{\diamond}{\leq} b$, there exists $d \in \mathcal{A}$ such that $a = bdb$. From $bc^*b = bb^*b$ and $ab^*a = aa^*a$, we have

$$ac^*a = bdbc^*bdb = bdbb^*bdb = ab^*a = aa^*a,$$

hence $a \overset{\diamond}{\leq} c$.

Suppose that $A, B \in M_{m\times n}$. It is clear that $\operatorname{Im} A \subseteq \operatorname{Im} B$ if and only if $A = BL$ for some $L \in M_{n\times n}$. Also, $\operatorname{Im} A^* \subseteq \operatorname{Im} B^*$ if and only if $A = MB$, for some $M \in M_{m\times m}$. Note that, in the setting of Hilbert space operators, the condition $\operatorname{Im} A \subseteq \operatorname{Im} B$ does not imply $A = BL$ unless $\operatorname{Im} B = \overline{\operatorname{Im} B}$.

Definition 16 Let $\mathcal{A}$ be an involutory ring. For $a, b \in \mathcal{A}$ we write $a \overset{d}{\leq} b$ when $ab^*a = aa^*a$, $a\mathcal{A} \subseteq b\mathcal{A}$ and $\mathcal{A}a \subseteq \mathcal{A}b$.

In a very recent paper [19], authors proved that the order, defined with Definition 16, is indeed a partial order when $\mathcal{A}$ is a $*$-regular ring. Recall that every $*$-regular ring is proper (of course, the converse is not true). Indeed, suppose that $aa^* = 0$. Then, $a = aa^\dagger a = a(a^\dagger a)^* = aa^*(a^\dagger)^* = 0$. We will show that the order from Definition 16 is a partial order for every proper involutory ring.

First, let us compare Definitions 15 and 16.

THEOREM 15 *Let $\mathcal{A}$ be $*$-regular ring with the unit 1. Then, the orders defined by Definitions 15 and 16 are the same.*

Proof We claim that $a\mathcal{A} \subseteq b\mathcal{A}$ if and only if $b_r^\pi a = 0$. If $a\mathcal{A} \subseteq b\mathcal{A}$ then $a = bx$ for some $x \in \mathcal{A}$, so $b_r^\pi a = (1 - bb^\dagger)a = (1 - bb^\dagger)bx = bx - bx = 0$. On the other hand, suppose that $b_r^\pi a = 0$. Then, $a = bb^\dagger a$, so $a\mathcal{A} \subseteq b\mathcal{A}$. Similarly, $\mathcal{A}a \subseteq \mathcal{A}b$ if and only if $ab_l^\pi = 0$. This completes the proof. $\square$

THEOREM 16 *Let $\mathcal{A}$ be a proper involutory ring with the unit 1. The order $\overset{d}{\leq}$, defined with Definition 16, is a partial order on $\mathcal{A}$.*

Proof Obviously, $\overset{d}{\leq}$ is reflexive. Suppose that $a \overset{d}{\leq} b$ and $b \overset{d}{\leq} a$. Then $aa^*a = ab^*a$, $bb^*b = ba^*b$ and there exist $x, y \in \mathcal{A}$ such that $a = bx = yb$. Therefore, $bb^*a = bb^*bx = ba^*bx = ba^*a$ and $ab^*b = ybb^*b = yba^*b = aa^*b$. We have

$$(a-b)(a-b)^*(a-b) = aa^*a - aa^*b - ab^*a + ab^*b - ba^*a + ba^*b + bb^*a - bb^*b = 0.$$

Hence, $(a-b)(a-b)^*((a-b)(a-b)^*)^* = 0$. By properness, it follows that $a - b = 0$.

To show transitivity, suppose that $a \overset{d}{\leq} b$ and $b \overset{d}{\leq} c$. We see at once that $a\mathcal{A} \subseteq c\mathcal{A}$ and $\mathcal{A}a \subseteq \mathcal{A}c$. By assumption $aa^*a = ab^*a$, $bb^*b = bc^*b$ and there exist $x, y \in \mathcal{A}$ such that $a = bx = yb$. Therefore,

$$ac^*a = ybc^*bx = ybb^*bx = ab^*a = aa^*a.$$

It follows $a \overset{d}{\leq} c$. □

It was proved in [18] that for $A, B \in M_{m\times n}$

$$A \overset{\diamond}{\leq} B \quad \text{if and only if} \quad A^\dagger \overset{-}{\leq} B^\dagger \tag{13}$$

where $A^\dagger$ and $B^\dagger$ are respectively Moore–Penrose inverses of A and B.

It is known (see [20]) that in every von Neumann regular ring $a \overset{-}{\leq} b$ if and only if $a = ab^-a = ab^-b = bb^-a$ for some (and thus every) b^- such that $bb^-b = b$. This is true especially when for b^- we take $b^\dagger$. This observation may be used to prove that the relation (13) between the minus and the diamond partial order is valid also in the setting of Rickart $*$-rings which are also von Neumann rings. Recently [19], it has already been proved that (13) is valid for every $*$-regular ring hence we will omit our proof.

Let us conclude this section with an observation about a relation between the star partial order and the diamond partial order in Rickart $*$-rings.

THEOREM 17 *Let $\mathcal{A}$ be Rickart $*$-ring and $a, b \in \mathcal{A}$. If*

$$a \leq b \quad \textit{then} \quad a \overset{d}{\leq} b.$$

Proof Suppose that $a \leq b$. By Theorem 1 there exist self-adjoint idempotents p and q such that $a = pb = bq$, hence $\mathcal{A}a \subseteq \mathcal{A}b$ and $a\mathcal{A} \subseteq b\mathcal{A}$. Also,

$$ab^*a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \begin{bmatrix} a_1^* & 0 \\ 0 & b_1^* \end{bmatrix}_{q\times p} \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} a_1a_1^*a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = aa^*a.$$

Therefore, $a \overset{d}{\leq} b$. □

5. Concluding remarks

The orders defined with Definitions 4, 10 and 11 are a proper extension of the well-known partial order on the set of self-adjoint idempotents.

THEOREM 18 *Let $\mathcal{A}$ be an involutory ring with the unit* 1. *Suppose $a, b \in \mathcal{A}$ are self-adjoint idempotent elements and let $\leq$ be an order defined with Definition* 4*, or* 10*, or* 11. *Then, $a \leq b$ if and only if $ab = ba = a$.*

The proof of this theorem is the same as the proof of Theorem 2.5 in [3].

It is known that the set of self-adjoint idempotents of a Rickart $*$-ring forms a lattice (see [1], Section 3, Proposition 7). A natural question to ask is whether the partially ordered set $(\mathcal{A}, \leq)$, where $\mathcal{A}$ is a Rickart $*$-ring and $\leq$ is the star partial order, is a lattice. It is easy to see that this is not true even for the case of matrices (two different invertible matrices do not have supremum). The question that we pose here is whether two different elements in a Rickart $*$-ring have infimum. We leave this as an open problem.

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