Formulae for the generalized Drazin inverse of a block matrix in terms of Banachiewicz–Schur forms

Dijana Mosić and Dragan S. Djordjević*

Abstract

We introduce new expressions for the generalized Drazin inverse of a block matrix with the generalized Schur complement being generalized Drazin invertible in a Banach algebra under some conditions. We generalized some recent results for Drazin inverse and group inverse of complex matrices.

Key words and phrases: generalized Drazin inverse, Schur complement, Banachiewicz–Schur form, block matrix.
2010 *Mathematics subject classification*: 46H05, 47A05, 15A09.

1 Introduction

The research on the representations of the Drazin inverse for block matrices is an important problem in the theory and applications of generalized inverses of matrices (see [2, 8, 11, 12, 15, 16]). Generalized inverses of block matrices have important applications in automatics, probability, statistics, mathematical programming, numerical analysis, game theory, econometrics, control theory and so on [3, 4]. In 1979, Campbell and Meyer proposed the problem of finding a formula for the Drazin inverse of a 2×2 matrix in terms of its various blocks, where the blocks on the diagonal are required to be square matrices [4]. At the present time, there is not known a complete solution to this problem.

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. For $a \in \mathcal{A}$, let $\sigma(a)$ be the spectrum of a. We denote by $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$ the sets of all nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$, respectively.

*The author are supported by the Ministry of Education and Science, Republic of Serbia, grant no. 174007.

Recall that an element $b \in \mathcal{A}$ is the generalized Drazin inverse (or Koliha–Drazin inverse) of $a \in \mathcal{A}$ provided that

$$bab = b, \qquad ab = ba, \qquad a - a^2 b \in \mathcal{A}^{qnil}.$$

If the generalized Drazin inverse of a exists, it is unique and denoted by a^d [13], and a is generalized Drazin invertible. By $\mathcal{A}^d$ we denote the set of all generalized Drazin invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}^d$, then the spectral idempotent a^π of a corresponding to the set $\{0\}$ is given by $a^\pi = 1 - aa^d$. A particular case of the generalized Drazin inverse is the (ordinary) Drazin inverse for which $a - a^2 b \in \mathcal{A}^{nil}$. The group inverse is the Drazin inverse for which the condition $a - a^2 b \in \mathcal{A}^{nil}$ is replaced with $a = aba$. By $a^\#$ we denote the group inverse of a. The set of all group invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^\#$.

We use the next lemma, recalling that part (i) is proved by Castro–González and Koliha [5], and part (ii) for bounded linear operators is proved in [9].

Lemma 1.1. *Let $b \in \mathcal{A}^d$ and $a \in \mathcal{A}^{qnil}$.*

(i) [5, Corollary 3.4] *If $ab = 0$, then $a+b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n$.*

(ii) [9, Theorem 2.2] *If $ba = 0$, then $a+b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} a^n (b^d)^{n+1}$.*

If $a \in \mathcal{A}$, and if $p = p^2 \in \mathcal{A}$ is an idempotent, then a has the following block matrix representation

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{1}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$, and let the generalized Schur complement be denoted by $s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. Notice that if pap is invertible in $p\mathcal{A}p$, then s becomes the (ordinary) Schur complement ([20]).

The following useful lemma is proved in [14] for elements of a Banach algebra.

Lemma 1.2. [14, Lemma 2.1] *Let x be defined as in* (1). *Then the following statements are equivalent:*

(i) $x \in \mathcal{A}^d$ *and* $x^d = r$, *where*

$$r = \begin{bmatrix} a^d + a^d b s^d c a^d & -a^d b s^d \\ -s^d c a^d & s^d \end{bmatrix}; \tag{2}$$

(ii) $a^\pi b = b s^\pi, \quad s^\pi c = c a^\pi \quad \text{and} \quad y = \begin{bmatrix} a a^\pi & b s^\pi \\ c a^\pi & s s^\pi \end{bmatrix} \in \mathcal{A}^{qnil}$.

The expression (2) is called the generalized Banachiewicz–Schur form of x. For more detailed see [1, 2, 6, 7, 10, 18].

A general formula for block triangular matrices was given by Meyer and Rose [15], and Hartwig and Shoaf [12]. Hence, a challenge in this field is to extend the result of Meyer and Rose to the case of arbitrary matrices.

Under different conditions, Castro-González and Martínez-Serrano [6] presented several explicit representations for the Drazin inverse of a block matrix with the group invertible generalized Schur complement.

Deng and Wei [8] studied conditions under which the Drazin inverse of block-operator matrix having generalized Schur complement Drazin invertible, can be expressed in terms of a matrix in the Banachiewicz-Schur form and its powers.

In this paper, we presente new explicit formulae for the generalized Drazin inverse of a block matrix x defined in (1) in the case that the generalized Schur complement is generalized Drazin invertible. These expressions involve the generalized Banachiewicz-Schur form (2). Several special cases are analyzed: we notice that some of them extend results from [6, 17, 19] to more general settings.

2 Generalized Drazin inverse

Throughout this paper, we assume that a block matrix x is defined as in (1), where $a \in (p\mathcal{A}p)^d$, and the generalized Schur complement satisfies $s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$.

In the following theorem we provide the explicit expression for the generalized Drazin inverse of block matrix x in (1) in terms of a^d, s^d, and the generalized Banachiewicz–Schur forms r defined as in (2).

Theorem 2.1. *Let x be defined as in* (1), *and let r be defined as in* (2). *If*

$$ca^\pi = 0 \quad \text{and} \quad abs^\pi = 0, \tag{3}$$

then $x \in \mathcal{A}^d$ *and*

$$x^d = r + \sum_{n=0}^{\infty} \begin{bmatrix} aa^\pi & bs^\pi \\ 0 & ss^\pi \end{bmatrix}^n \begin{bmatrix} 0 & a^\pi bss^d \\ s^\pi c & s^\pi ca^d bss^d \end{bmatrix} r^{n+2}. \tag{4}$$

Proof. Assume that $x = y + z$, where

$$y = \begin{bmatrix} aa^\pi & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} = \begin{bmatrix} aa^\pi & bs^\pi \\ 0 & ss^\pi \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix}.$$

Using equalities $a^\pi a^d = 0$ and (3), we get $zy = 0$.

Since $aa^\pi \in (p\mathcal{A}p)^{qnil}$, $ss^\pi \in ((1-p)\mathcal{A}(1-p))^{qnil}$ and $\sigma(y) \subseteq \sigma_{p\mathcal{A}p}(aa^\pi) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(ss^\pi)$, we conclude that $y \in \mathcal{A}^{qnil}$.

In order to show that $z \in \mathcal{A}^d$, let $z = z_1 + z_2$, where

$$z_1 = \begin{bmatrix} a^2a^d & aa^d bss^d \\ ss^d caa^d & ss^d dss^d \end{bmatrix} \quad \text{and} \quad z_2 = \begin{bmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & s^\pi ca^d bss^d \end{bmatrix}.$$

Then $z_1z_2 = 0$ and $z_2^2 = 0$, i.e. $z_2 \in \mathcal{A}^{nil}$. If we set $A_{z_1} \equiv a^2a^d$, $B_{z_1} \equiv aa^d bss^d$, $C_{z_1} \equiv ss^d caa^d$ and $D_{z_1} \equiv ss^d dss^d$, we have $A_{z_1} \in (p\mathcal{A}p)^\#$, $A_{z_1}^\# = a^d$, $A_{z_1}^\pi B_{z_1} = a^\pi aa^d bss^d = 0$, $C_{z_1} A_{z_1}^\pi = 0$, $S_{z_1} = D_{z_1} - C_{z_1} A_{z_1}^\# B_{z_1} = s^2 s^d \in ((1-p)\mathcal{A}(1-p))^\#$, $S_{z_1}^\# = s^d$, $S_{z_1}^\pi C_{z_1} = s^\pi ss^d caa^d = 0$, $B_{z_1} S_{z_1}^\pi = 0$ and $Y_{z_1} = 0$. By Lemma 1.2, we deduce that $z_1 \in \mathcal{A}^d$ and $z_1^d = r$. Applying Lemma 1.1(ii), we obtain that $z \in \mathcal{A}^d$ and $z^d = r + z_2 r^2$.

Using Lemma 1.1(ii) again, $x \in \mathcal{A}^d$ and $x^d = \sum\limits_{n=0}^{\infty} y^n (z^d)^{n+1}$. Since $z_1z_2 = 0$ and $z_1^d = r$, we have $rz_2 = r^2 z_1 z_2 = 0$ which gives

$$x^d = \sum_{n=0}^{\infty} y^n (1 + z_2 r) r^{n+1}.$$

From $yr = 0$ we obtain

$$x^d = (1 + z_2 r) r + \sum_{n=1}^{\infty} y^n z_2 r^{n+2} = r + \sum_{n=0}^{\infty} y^n z_2 r^{n+2}$$

which yields (4). □

A geometrical reformulation of conditions $ca^\pi = 0$ and $abs^\pi = 0$ is as follows:

$$(a^d)^\circ \subset c^\circ \qquad \text{and} \qquad s^\circ \subset (ab)^\circ,$$

where $a^\circ = \{x \in \mathcal{A} : ax = 0\}$.

The following corollary is a straightforward application of Theorem 2.1.

Corollary 2.1. *Let x be defined as in* (1), $a \in (p\mathcal{A}p)^{\#}$ *and let* $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$. *If* $ca^{\pi} = 0$ *and* $abs^{\pi} = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} p - a^{\pi}bs^{\#}ca^{\#} & a^{\pi}bs^{\#} \\ s^{\pi}ca^{\#} & 1-p \end{bmatrix} r_1 + \begin{bmatrix} bs^{\pi}ca^{\#} & 0 \\ 0 & 0 \end{bmatrix} r_1^2;$$

where

$$r_1 = \begin{bmatrix} a^{\#} + a^{\#}bs^{\#}ca^{\#} & -a^{\#}bs^{\#} \\ -s^{\#}ca^{\#} & s^{\#} \end{bmatrix}.$$

If we assume that $s^{\circ} \subset b^{\circ}$ and $a^{\circ} \subset (sc)^{\circ}$, we have the following representation of generalized Drazin inverse x^d.

Theorem 2.2. *Let x be defined as in* (1), *and let r be defined as in* (2). *If*

$$bs^{\pi} = 0 \quad \textit{and} \quad sca^{\pi} = 0,$$

then $x \in \mathcal{A}^d$ *and*

$$x^d = r + \sum_{n=0}^{\infty} \begin{bmatrix} aa^{\pi} & 0 \\ ca^{\pi} & ss^{\pi} \end{bmatrix}^n \begin{bmatrix} 0 & a^{\pi}b \\ s^{\pi}caa^d & s^{\pi}ca^d b \end{bmatrix} r^{n+2}.$$

Proof. Let $x = y + z$, where

$$y = \begin{bmatrix} aa^{\pi} & 0 \\ ca^{\pi} & ss^{\pi} \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix}.$$

Then $zy = 0$ and we can finish the proof in the same way as in the proof of Theorem 2.1. □

By Theorem 2.2, we can verify the next expression for x^d in the case when a and s are group invertible.

Corollary 2.2. *Let x be defined as in* (1) *where* $a \in (p\mathcal{A}p)^{\#}$ *and* $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$, *and let r_1 be defined as in Corollary 2.1. If* $bs^{\pi} = 0$ *and* $sca^{\pi} = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} p - a^{\pi}bs^{\#}ca^{\#} & a^{\pi}bs^{\#} \\ s^{\pi}ca^{\#} & 1-p \end{bmatrix} r_1 + \begin{bmatrix} 0 & 0 \\ 0 & ca^{\pi}b \end{bmatrix} r_1^3.$$

Using Corollary 2.1 and Corollary 2.2, we can verify the following result.

Corollary 2.3. *Let x be defined as in* (1), $a \in (p\mathcal{A}p)^{\#}$ *and* $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$, *and let r_1 be defined as in Corollary 2.1. If* $ca^{\pi} = 0$ *and* $bs^{\pi} = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} p - a^{\pi}bs^{\#}ca^{\#} & a^{\pi}bs^{\#} \\ s^{\pi}ca^{\#} & 1-p \end{bmatrix} r_1.$$

Note that Corollary 2.3 recovers [6, Theorem 2.5] for the Drazin inverse of complex matrices.

If we assume that the generalized Schur complement s is invertible in Theorem 2.1 and Theorem 2.2, then we can prove the next result.

Corollary 2.4. *Let x be defined as in* (1)*, where $a \in (p\mathcal{A}p)^d$ and the generalized Schur complement satisfies $s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $ca^\pi = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(1 + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r_2^{n+1}\right) r_2,$$

where

$$r_2 = \begin{bmatrix} a^d + a^d bs^{-1}ca^d & -a^d bs^{-1} \\ -s^{-1}ca^d & s^{-1} \end{bmatrix}.$$

Following the same strategy as in the proof of the preceding results, we derive a new representation for x^d.

Theorem 2.3. *Let x be defined as in* (1)*, and let r be defined as in* (2)*. If*

$$a^\pi b = 0 \quad \text{and} \quad s^\pi ca = 0,$$

then $x \in \mathcal{A}^d$ and

$$x^d = r + \sum_{n=0}^{\infty} r^{n+2} \begin{bmatrix} 0 & bs^\pi \\ ss^d ca^\pi & ss^d ca^d bs^\pi \end{bmatrix} \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & ss^\pi \end{bmatrix}^n. \tag{5}$$

Proof. If we write $x = y + z$, where

$$y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} = \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} a^2 a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix},$$

then we have that $yz = 0$ and $y \in \mathcal{A}^{qnil}$.

Suppose that $z = z_1 + z_2$, where

$$z_1 = \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & ss^d dss^d \end{bmatrix} \quad \text{and} \quad z_2 = \begin{bmatrix} 0 & aa^d bs^\pi \\ ss^d ca^\pi & ss^d ca^d bs^\pi \end{bmatrix}.$$

Now we can check that $z_2 z_1 = 0$ and $z_2^2 = 0$. Using Lemma 1.2 we get $z_1 \in \mathcal{A}^d$ and $z_1^d = r$. From Lemma 1.1(i) it follows $z \in \mathcal{A}^d$ and $z^d = r + r^2 z_2$.

By Lemma 1.1(i), we conclude that $x \in \mathcal{A}^d$ and

$$x^d = \sum_{n=0}^{\infty} (z^d)^{n+1} y^n = r + \sum_{n=0}^{\infty} r^{n+2} z_2 y^n$$

implying (5). $\square$

Observe that conditions $a^\pi b = 0$ and $s^\pi ca = 0$ are equivalent with the following geometrical conditions:

$$b\mathcal{A} \subset a\mathcal{A} \qquad \text{and} \qquad ca\mathcal{A} \subset s\mathcal{A},$$

where $a\mathcal{A} = \{ax : x \in \mathcal{A}\}$.

As a consequence of Theorem 2.3, we obtain the next result.

Corollary 2.5. *Let x be defined as in* (1), *$a \in (p\mathcal{A}p)^\#$, $s = d - ca^\# b \in ((1-p)\mathcal{A}(1-p))^\#$, and let r_1 be defined as in Corollary 2.1. If $a^\pi b = 0$ and $s^\pi ca = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = r_1 \begin{bmatrix} p - a^\# bs^\# ca^\pi & a^\# bs^\pi \\ s^\# ca^\pi & 1-p \end{bmatrix} + r_1^2 \begin{bmatrix} a^\# bs^\pi c & 0 \\ 0 & 0 \end{bmatrix}.$$

The following theorem gives a formula for the generalized Drazin inverse of x in (1) under assumptions $c\mathcal{A} \subset s\mathcal{A}$ and $bs\mathcal{A} \subset a\mathcal{A}$.

Theorem 2.4. *Let x be defined as in* (1), *and let r be defined as in* (2). *If*

$$s^\pi c = 0 \quad \textit{and} \quad a^\pi bs = 0,$$

then $x \in \mathcal{A}^d$ and

$$x^d = r + \sum_{n=0}^{\infty} r^{n+2} \begin{bmatrix} 0 & aa^d bs^\pi \\ ca^\pi & ca^d bs^\pi \end{bmatrix} \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & ss^\pi \end{bmatrix}^n.$$

Proof. Similarly as Theorem 2.3, we can prove this theorem using the representation $x = y + z$, where

$$y = \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & s^\pi s \end{bmatrix} \qquad \text{and} \qquad z = \begin{bmatrix} a^2 a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix}.$$

$\square$

From Theorem 2.4, we can obtain the next corollary.

Corollary 2.6. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^\#$ and $s = d - ca^\# b \in ((1-p)\mathcal{A}(1-p))^\#$, and let r_1 be defined as in Corollary 2.1. If $s^\pi c = 0$ and $a^\pi bs = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = r_1 \begin{bmatrix} p - a^\# bs^\# ca^\pi & a^\# bs^\pi \\ s^\# ca^\pi & 1-p \end{bmatrix} + r_1^3 \begin{bmatrix} 0 & 0 \\ 0 & ca^\pi b \end{bmatrix}.$$

Now, by Corollary 2.5 and Corollary 2.6, we prove the result which covers [6, Theorem 2.2].

Corollary 2.7. *Let x be defined as in* (1)*, $a \in (p\mathcal{A}p)^{\#}$, $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$, and let r_1 be defined as in Corollary 2.1. If $s^{\pi}c = 0$ and $a^{\pi}b = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = r_1 \begin{bmatrix} p - a^{\#}bs^{\#}ca^{\pi} & a^{\#}bs^{\pi} \\ s^{\#}ca^{\pi} & 1-p \end{bmatrix}.$$

Using Theorems 2.1–2.4, we can prove the next corollary which recovers [6, Corollary 2.3] and [17, Theorem 3.2].

Corollary 2.8. *Let x be defined as in* (1) *where $a \in (p\mathcal{A}p)^{\#}$, the generalized Schur complement satisfies $s = d - ca^{\#}b \in ((1-p)\mathcal{A}(1-p))^{\#}$, and let r_1 be defined as in Corollary 2.1. If $a^{\pi}b = 0 = bs^{\pi}$ and $ca^{\pi} = 0 = s^{\pi}c$, then $x \in \mathcal{A}^d$ and $x^d = r_1$.*

The next result is a consequence of Theorem 2.3 and Theorem 2.4.

Corollary 2.9. *Let x be defined as in* (1)*, where $a \in (p\mathcal{A}p)^d$ and $s = d - ca^db \in ((1-p)\mathcal{A}(1-p))^{-1}$, and let r_2 be defined as in Corollary 2.4. If $a^{\pi}b = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = r_2 \left(1 + \sum_{n=0}^{\infty} r_2^{n+1} \begin{bmatrix} 0 & 0 \\ ca^{\pi}a^n & 0 \end{bmatrix} \right).$$

Next, by Corollary 2.4 and Corollary 2.9, we obtain an extension of the result for the Drazin inverse of a block matrix by Wei [19] to a block matrix of Banach algebra.

Corollary 2.10. *Let x be defined as in* (1)*, where $a \in (p\mathcal{A}p)^d$ and $s = d - ca^db \in ((1-p)\mathcal{A}(1-p))^{-1}$, and let r_2 be defined as in Corollary 2.4. If $ca^{\pi} = 0$ and $a^{\pi}b = 0$, then $x \in \mathcal{A}^d$ and $x^d = r_2$.*

Some representations for the generalized Drazin inverse of triangular matrices are presented in the following theorems.

Theorem 2.5. *Let $x = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, where $a \in (p\mathcal{A}p)^d$ and $s = d \in ((1-p)\mathcal{A}(1-p))^d$.*

(i) *If $ca^{\pi} = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = r_3 + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ s^n s^{\pi} c & 0 \end{bmatrix} r_3^{n+2};$$

(i) *If* $s^\pi c = 0$*, then* $x \in \mathcal{A}^d$ *and*

$$x^d = r_3 + \sum_{n=0}^{\infty} r_3^{n+2} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix};$$

where

$$r_3 = \begin{bmatrix} a^d & 0 \\ -s^d c a^d & s^d \end{bmatrix}.$$

Proof. If we suppose that $b = 0$ in Theorem 2.1 and Theorem 2.4, we check these formulae. □

If the hypothesis $c = 0$ is assumed in Theorem 2.2 and Theorem 2.3, we can verify the next result.

Theorem 2.6. *Let* $x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$*, where* $a \in (p\mathcal{A}p)^d$ *and* $s = d \in ((1-p)\mathcal{A}(1-p))^d$*.*

(i) *If* $bs^\pi = 0$*, then* $x \in \mathcal{A}^d$ *and*

$$x^d = r_4 + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r_4^{n+2};$$

(i) *If* $a^\pi b = 0$*, then* $x \in \mathcal{A}^d$ *and*

$$x^d = r_4 + \sum_{n=0}^{\infty} r_4^{n+2} \begin{bmatrix} 0 & bs^\pi s^n \\ 0 & 0 \end{bmatrix};$$

where

$$r_4 = \begin{bmatrix} a^d & -a^d b s^d \\ 0 & s^d \end{bmatrix}.$$

Finally, we give an example to illustrate our results.

Example 2.1. Let $\mathcal{A}$ be a Banach algebra and let $x = \begin{bmatrix} p & 0 \\ c & 1-p \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$. Hence, $a^d = a = p$, $a^\pi = 0$, $s = s^d = 1-p$, $s^\pi = 0$ and $b = 0$. Applying Theorem 2.1 or Theorem 2.5, we conclude that $x \in \mathcal{A}^d$ and $x^d = \begin{bmatrix} p & 0 \\ -c & 1-p \end{bmatrix}$.

References

[1] J.K. Baksalary, G.P.H. Styan, *Generalized inverses of partitioned matrices in Banachiewicz-Schur form*, Linear Algebra Appl. 354 (2002) 41-47.

[2] J. Benítez, N. Thome, *The generalized Schur complement in group inverses and* $(k+1)$*-potent matrices*, Linear Multilinear Algebra 54 (2006) 405–413.

[3] S.L. Campbell, Singular systems of differential equations, Pitman, San Francisco - London - Melbourne, 1980.

[4] S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Pitman, London, 1979.

[5] N. Castro–González, J. J. Koliha, *New additive results for the g-Drazin inverse*, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085-1097.

[6] N. Castro-González, M.F. Martínez-Serrano, Drazin inverse of partitioned matrices in terms of Banachiewicz-Schur forms, Linear Algebra Appl. 432 (2010) 1691–1702.

[7] C. Deng, *A note on the Drazin inverses with Banachiewicz-Schur forms*, Appl. Math. Comput. 213 (2009) 230–234.

[8] C. Deng, Y. Wei, *Representations for the Drazin inverse of* 2×2 *block-operator matrix with singular Schur complement* Linear Algebra Appl. 435 (2011) 2766–2783.

[9] D.S. Djordjević, Y. Wei, *Additive results for the generalized Drazin inverse*, J. Austral. Math. Soc. 73 (2002) 115-125.

[10] R.E. Hartwig, X. Li, Y. Wei, *Representations for the Drazin inverse of* 2×2 *block matrix*, SIAM J. Matrix Anal. Appl. 27 (2006) 757–771.

[11] R.E. Hartwig, G. Wang and Y. Wei, *Some additive results on Drazin inverse*, Linear Algebra Appl. 322 (2001) 207-217.

[12] R.E. Hartwig, J. Shoaf, *Group inverses and Drazin inverse of bidiagonal and triangular Teoplitz matrices*, J. Aust. Math. Soc. Ser. A 24 (1977) 1034.

[13] J.J. Koliha, A generalized Drazin inverse, Glasgow Math. J. 38 (1996) 367–381.

[14] M.Z. Kolundžija, D. Mosić, D.S. Djordjević, *Further results on the generalized Drazin inverse of block matrices in Banach algebras*, Bull. Malays. Math. Sci. Soc. (to appear).

[15] C.D. Meyer, N.J. Rose, *The index and the Drazin inverse of block triangular matrices*, SIAM J. Appl. Math. 33 (1977) 17.

[16] D. Mosić, D.S. Djordjević, *Reprezentation for the generalized Drazin inverse of block matrices in Banach algebras*, Appl. Math. Comput. 218 (2012) 12001–12007.

[17] X. Sheng, G. Chen, *Some generalized inverses of partitioned matrix and quotient identity of generalized Schur complement*, Appl. Math. Comput. 196 (2008) 174-184.

[18] Y. Tian, Y. Takane, *More on generalized inverses of partitioned matrices with Banachiewicz-Schur forms*, Linear Algebra Appl. 430 (2009) 1641-1655.

[19] Y. Wei, *Expressions for the Drazin inverse of* 2×2 *block matrix*, Linear Multilinear Algebra 45 (1998) 131–146.

[20] F. Zhang (Ed.), *The Schur Complement and Its Applications*, Springer, Berlin, 2005.

Address:

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

E-mail

D. Mosić: `dijana@pmf.ni.ac.rs`
D. S. Djordjević: `dragan@pmf.ni.ac.rs`