FACTORIZATIONS OF WEIGHTED EP BANACH SPACE OPERATORS AND BANACH ALGEBRA ELEMENTS

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Abstract. Weighted EP Banach space operators and Banach algebra elements are characterized using different kinds of factorizations. The results presented extend well-known characterizations of (weighted) EP matrices, (weighted) EP Hilbert space operators and (weighted) EP C^*-algebra elements.

1. Introduction and Preliminaries

A square matrix A is said to be EP if $N(A) = N(A^*)$. Since a necessary and sufficient condition for a matrix A to be EP is the fact that A commutes with its Moore-Penrose inverse, the notion under consideration has been extended to Hilbert space operators and C^*-algebra elements, see [15, 11, 12, 13, 1].

In the context of Banach algebras the notion of Moore-Penrose inverse was introduced by V. Rakočević in [16] (see also [2]). In the recent past EP Banach space operators and EP Banach algebra elements, i.e., Moore-Penrose invertible operators or elements of an algebra such that they commute with their Moore-Penrose inverse, were introduced and characterized, see [2, 4, 3]. It is worth noting that EP objects generalizes normal and hermitian objects, see [4, Theorems 3.1 and 3.3].

In addition, in the recent work by Y. Tian and H. Wang [17] the notion of weighted EP matrices (matrices that commute with their weighted Moore-Penrose inverse) was introduced. What is more, weighted EP Banach algebra elements were characterized in [5].

Note that one of the main lines of research concerning EP objects consists in characterizing them through factorizations. In fact, EP matrices, EP Hilbert and Banach space operators and EP C^*-algebra and Banach algebra elements have been characterized through several different kind of factorizations.

The objective of this work is to characterize weighted EP Banach space operators and weighted EP Banach algebra elements using factorizations; actually, three different kinds of factorizations will be considered (see sections 2-4). Several reasons motivate this research. First of all, considering the general notion of weighted EP Banach algebra element, results proved for (weighted) EP matrices, Hilbert space operators and C^*-algebra elements can be recovered and generalized. It is also worth noting that due to the lack of involution on a Banach

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algebra, and in particular on the Banach algebra of bounded and linear maps defined on a Banach space, the proofs not only are different from the ones known for matrices, Hilbert space operators or C^*-algebra elements, but also they give a new insight into the cases where the involution does exist. In fact, the results and proofs presented do not depend on a particular norm, the Euclidean norm, but they hold for any norm. In particular, the results considered in this work also apply to (weighted) EP matrices defined using an arbitrary norm on a finite dimensional vector space, extending in this way the results of [17] to any norm.

From now on X and Y will denote two Banach spaces and $L(X,Y)$ will stand for the Banach space of all bounded and linear maps defined on X and with values in Y. As usual, when $X=Y$, $L(X,Y)$ will be denoted by $L(X)$. If $T \in L(X,Y)$, then $N(T) \subseteq X$ and $R(T) \subseteq Y$ will stand for the null space and the range of T, respectively. In addition, $I_X \in L(X)$ will denote the identity map. Moreover, X^* will denote the dual space of X and if $T \in L(X)$, then $T^* \in L(X^*)$ will stand for the adjoint of $T \in L(X)$.

On the other hand, A will denote a complex unital Banach algebra with unit 1. In addition, the set of all invertible elements of A will be denoted by A^{-1}. If $a \in A$, then $L_a\colon A \to A$ and $R_a\colon A \to A$ will denote the maps defined by left and right multiplication, respectively: $L_a(x) = ax$ and $R_a(x) = xa$, where $x \in A$. Moreover, the following notation will be used: $N(L_a) = a^{-1}(0)$, $R(L_a) = aA$, $N(R_a) = a_{-1}(0)$, $R(R_a) = Aa$.

Recall that an element $a \in A$ is called *regular,* if it has a *generalized inverse,* namely if there exists $b \in A$ such that $a = aba$. Furthermore, a generalized inverse b of a regular element $a \in A$ will be called *normalized,* if b is regular and a is a generalized inverse of b, equivalently, $a = aba$ and $b = bab$. Note that if b is a generalized inverse of a, then $c = bab$ is a normalized generalized inverse of a.

Next follows the key notion in the definition of (weighted) Moore-Penrose invertible Banach algebra elements.

Definition 1.1. Given a unital Banach algebra A, an element $a \in A$ is said to be *hermitian,* if $\parallel \exp(ita) \parallel = 1$, for all $t \in \mathbb{R}$.

Concerning equivalent definitions and the main properties of hermitian Banach algebra elements and Banach space operators, see for example [14, 6, 9]. Recall that if A is a C^*-algebra, then $a \in A$ is hermitian if and only if a is self-adjoint, see [6, Proposition 20, Chapter I, Section 12]. Given a unital Banach algebra A, the set of all Hermitian elements of A will be denoted by $H(A)$.

Now the notion of Moore-Penrose invertible Banach algebra element will be recalled.

Definition 1.2. Let A be a unital Banach algebra and consider $a \in A$. If there exists $x \in A$ such that x is a normalized generalized inverse of a and ax and xa are hermitian, then x will be said to be the Moore-Penrose inverse of a, and it will be denoted by $a^\dagger$.

Recall that according to [16, Lemma 2.1], there is at most one Moore-Penrose inverse. Concerning the Moore-Penrose inverse in Banach algebras, see [16, 2, 4, 3]. For the original definition of the Moore-Penrose inverse for matrices, see [15].

Next the definition of EP Banach algebra elements will be recalled.

Definition 1.3. Given a unital Banach algebra A, the element $a \in A$ is said to be *EP,* if $a^\dagger$ exists and commutes with a.

To recall the notion of weighted Moore-Penrose invertible Banach algebra elements, some preparation is needed.

Let A be a complex unital Banach algebra and consider $a \in A$. The element a will be said to be *positive*, if $V(a) \subset \mathbb{R}_+$, where $V(a) = \{f(a) \colon f \in A^*, \parallel f \parallel \leq 1, f(1) = 1\}$ ([6, Definition 5, Chapter V, Section 38]). Denote A_+ the set of all positive elements of A. Note that necessary and sufficient for $a \in A$ to be positive is that a is hermitian and $\sigma(a) \subset \mathbb{R}_+$ ([6, Definition 5, Chapter V, Section 38]). Recall that according to [6, Lemma 7, Chapter V, Section 38], if $c \in A_+$, then there exists $d \in A_+$ such that $d^2 = c$. Moreover, according to [10, Theorem], the square root of c is unique. In particular, the square root of c will be denoted by $c^{1/2}$. For the definition and equivalent conditions of positive C^*-algebra elements, see [7, Definition 3.1 and Theorem 3.6, Chapter VIII, Section 3].

Given a complex unital Banach algebra A and $u \in A^{-1} \cap A_+$, denote by $A^u = (A^u, \parallel \cdot \parallel_u)$ the complex unital Banach algebra with underlying space A and norm $\parallel x \parallel_u = \parallel u^{1/2} x u^{-1/2} \parallel$. In the following definition weighted Moore-Penrose invertible elements will be introduced.

Definition 1.4. Let A be a complex unital Banach algebra and consider e and f two positive and invertible elements in A. The element $a \in A$ will be said to be *weighted Moore-Penrose invertible with weights e and f,* if there exists $b \in A$ such that b is a normalized generalized inverse of a and ab (respectively ba) is a hermitian element of $(A^e, \parallel \cdot \parallel_e)$ (respectively $(A^f, \parallel \cdot \parallel_f)$.

If the weighted Moore-Penrose inverse of a exists, then it is unique and so it will be denoted by $a^\dagger_{e,f}$ (see [5, Proposition 2.5]). According to [5], if A is a C^*-algebra, then the conditions in Definition 1.4 are equivalent to the ones that characterize the usual weighted Moore-Penrose inverse.

Next the definition of weighted EP Banach algebra element will be recalled.

Definition 1.5. Given a unital Banach algebra A and e, $f \in A$ two invertible and positive elements, $a \in A$ is said to be *weighted EP with weights e and f,* if $a^\dagger_{e,f}$ exists and commutes with a.

Remark 1.6. Let A be a unital Banach algebra and consider $a \in A$.
(i). Recall that $a \in A$ is said to be *group invertible,* if there exists $b \in A$, a normalized generalized inverse of a, such that $ab = ba$. It is well known that if the group inverse of a exists, then it is unique; in addition, in this case it is denoted by $a^\sharp$. Note that given e, $f \in A$ two invertible and positive elements, necessary and sufficient for $a \in A$ to be weighted EP with weights e and f is that a is group invertible and $aa^\sharp$ is a hermitian element of $(A^e, \parallel \cdot \parallel_e)$ and $a^\sharp a$ is a hermitian element of $(A^f, \parallel \cdot \parallel_f)$. Naturally, in this case, $a^\sharp = a^\dagger_{e,f}$. So that, roughly speaking, weighted EP elements are group invertible elements with two extra condition, i.e., the idempotents $aa^\sharp$ and $a^\sharp a$ must be hermitian elements of two associated Banach algebras.

(ii). Note that in Definition 1.5 the weights e, $f \in A$ need not to be ordered. In fact, according to a result of hermitian elements ([9, Theorem 4.4(i)]), $a \in A$ is weighted EP with weights e and f if and only if $a \in A$ is weighted EP with weights f and e (see Corollary 4.4 and Theorem 4.6).

To study the factorization that will be considered in the next section, the notion of weighted Moore-Penrose inverse for operators defined between different Banach spaces needs to be introduced. However, first some preliminary results will be recalled.

Remark 1.7. Let X and Y be two Banach spaces and let $T \in L(X,Y)$. If $S \in L(Y,X)$ is a normalized generalized inverse of T, then it is not difficult to prove that $TS \in L(Y)$ and $ST \in L(X)$ are idempotents, $R(TS) = R(T)$, $N(TS) = N(S)$, $R(ST) = R(S)$, $N(ST) = N(T)$, $X = R(S) \oplus N(T)$ and $Y = R(T) \oplus N(S)$.

Next the notion of weighted Moore–Penrose inverse for operators defined between different Banach spaces will be introduced.

Definition 1.8. Let X and Y be two Banach spaces and let $T \in L(X,Y)$. Consider $E \in L(Y)$ and $F \in L(X)$ two invertible and positive operators. If there exists $S \in L(Y,X)$ such that $TST = T$, $STS = S$, $TS \in H(L(Y)^E)$ and $ST \in H(L(X)^F)$, then T will be said to be *weighted Moore-Penrose invertible with weights E and F.*

In first place it will be shown that given $T \in L(X,Y)$, there is at most one operator $S \in L(Y,X)$ satisfying Definition 1.8. In the following Lemma, ideas similar to the ones in [16, Lemma 2.1] will be used. In addition, it will be used the following result. Given Z a Banach space and E, $F \in L(Z)$ two hermitian idempotents such that $R(E) = R(F)$, then $E = F$ ([14, Theorem 2.2]).

Lemma 1.9. *Let X and Y be two Banach spaces and let $T \in L(X,Y)$. Consider $E \in L(Y)$ and $F \in L(X)$ two invertible and positive operators. If $S_i \in L(Y,X)$ complies the four conditions of Definition 1.8, $i = 1, 2$, then $S_1 = S_2$.*

Proof. Since $R(TS_1) = R(T) = R(TS_2)$ and $TS_i \in H(L(Y)^E)$, $i = 1, 2$, according to [14, Theorem 2.2], $TS_1 = TS_2$. In addition, since $R(I_X - S_1T) = N(S_1T) = N(T) = N(S_2T) = R(I_X - S_2T)$ and $ST_i \in H(L(X)^F)$, $i = 1, 2$, according to [9, Theorem 44(i)] and [14, Theorem 2.2], $I_X - S_1T = I_X - S_2T$, equivalently $S_1T = S_2T$. Thus,
$$S_1 = S_1TS_1 = S_1TS_2 = S_2TS_2 = S_2.$$
□

The following result will be used in the next section.

Lemma 1.10. *Let X and Y be two Banach spaces and let $T \in L(X,Y)$. Consider $E \in L(Y)$ and $F \in L(X)$ two invertible and positive operators. Necessary and sufficient for $T^{\dagger}_{E,F}$ to exit is that there exist two idempotents $P \in H(L(Y)^E)$ and $Q \in H(L(X)^F)$ such that $R(P) = R(T)$ and $N(Q) = N(T)$.*

Proof. Adapt the proof of [5, Theorem 2.7] to the conditions of this lemma. □

2. Factorization $a = bc$

In this section, given a unital Banach algebra A and e, $f \in A$ invertible and positive, weighted EP elements of the form $a = bc$ will be characterized, where a, b, $c \in A$, a is weighted Moore-Penrose invertible with weights e and f, $b^{-1}(0) = \{0\}$ and $cA = A$. However, in first place the Banach space operator case will be studied.

Theorem 2.1. *Let X and Y be two Banach spaces and consider $E, H \in L(X)$ and $F \in L(Y)$ three invertible positive operators. Let $T \in L(X)$ such that $T^{\dagger}_{E,H}$ exists and suppose that there exist $C \in L(X, Y)$ and $B \in L(Y, X)$ such that C is surjective, B is injective and $T = BC$. Then the following statements hold.*

(i) *There exists $B^{\dagger}_{E,F} \in L(X, Y)$ such that $B^{\dagger}_{E,F} B = I_Y$.*

(ii) *There exists $C^{\dagger}_{F,H} \in L(Y, X)$ such that $CC^{\dagger}_{F,H} = I_Y$.*

(iii) *$T^{\dagger}_{E,H} = C^{\dagger}_{F,H} B^{\dagger}_{E,F}$, $TT^{\dagger}_{E,H} = BB^{\dagger}_{E,F}$, $T^{\dagger}_{E,H} T = C^{\dagger}_{F,H} C$, $B^{\dagger}_{E,F} = CT^{\dagger}_{E,H}$, $C^{\dagger}_{F,H} = T^{\dagger}_{E,H} B$, $TC^{\dagger}_{F,H} = B$ and $B^{\dagger}_{E,F} T = C$.*

Proof. Note that according to [5, Theorem 2.7], there exist two idempotents P,$Q \in L(X)$ such that $P \in H(L(X)^E)$, $Q \in H(L(X)^H)$, $R(P) = R(T)$ and $N(Q) = N(T)$.

(i). Since B is injective and $R(P) = R(T) = R(B)$, according to Lemma 1.10, $B^{\dagger}_{E,F}$ exists. In fact, $I_Y \in H(L(Y)^H)$ and $P \in H(L(X)^E)$ are two idempotents such that $N(B) = N(I_Y)$ and $R(B) = R(P)$. In addition, $R(I_Y - B^{\dagger}_{E,F} B) = N(B^{\dagger}_{E,F} B) = N(B) = 0$.

(ii). Similarly, since $I_Y \in H(L(Y)^F)$ and $Q \in H(L(X)^H)$ are two idempotents such that $R(C) = Y = R(I_Y)$ and $N(C) = N(T) = N(Q)$, according to Lemma 1.10, $C^{\dagger}_{F,H}$ exists. Further, since $CC^{\dagger}_{F,H}$ and $I_Y \in H(L(Y)^F)$ are idempotents such that $R(CC^{\dagger}_{F,H}) = R(C) = R(I_Y)$, according to [14, Theorem 2.2], $CC^{\dagger}_{F,H} = I_Y$.

(iii). Consider $S = C^{\dagger}_{F,H} B^{\dagger}_{E,F}$. It is not difficult to prove that $T = TST$, $S = STS$, $TS = BB^{\dagger}_{E,F}$, $ST = C^{\dagger}_{F,H} C$. However, since $BB^{\dagger}_{E,F} \in H(L(X)^E)$ and $C^{\dagger}_{F,H} C \in H(L(X)^H)$ are idempotent operators, according to Definition 1.8 and Lemma 1.9, $T^{\dagger}_{E,H} = C^{\dagger}_{F,H} B^{\dagger}_{E,F}$. The remaining identities can be derived from what has been proved and statements (i)-(ii). □

Remark 2.2. (a). Let X be a Banach space and consider $E, H \in L(X)$ two invertible positive operators. Let $T \in L(X)$ such that $T^{\dagger}_{E,H}$. Note that the decomposition of T as in Theorem 2.1 is always possible. In fact, since $R(T)$ is closed, $T = \overline{T}\pi$, where $\pi \colon X \to X/N(T) = Y$ is the canonical quotient map and $\overline{T} \colon Y \to X$ is the factorization of T. In addition, consider any invertible and positive operator $F \in L(Y)$, for example $F = I_Y$.
(b). Note that in Theorem 2.1 the identity $T^{\dagger}_{E,H} = C^{\dagger}_{F,H} B^{\dagger}_{E,F}$ is a particular reverse order law for $T = BC$.

In the following theorem weighted EP operators of the form $T = BC$ will be characterized.

Theorem 2.3. *Under the same hypotheses of Theorem 2.1, the following statements are equivalent.*

(i) *T is weighted EP with weights E and H.*
(ii) $BB^{\dagger}_{E,F} = C^{\dagger}_{F,H}C$.
(iii) $R(B) = R(C^{\dagger}_{F,H})$ *and* $N(B^{\dagger}_{E,F}) = N(C)$.
(iv) $(I_X - C^{\dagger}_{F,H}C)B = 0$ *and* $C(I_X - BB^{\dagger}_{E,F}) = 0$.
(v) $B^{\dagger}_{E,F}(I_X - C^{\dagger}_{F,H}C) = 0$ *and* $(I_X - BB^{\dagger}_{E,F})C^{\dagger}_{F,H} = 0$.
(vi) *There exists an isomorphism $U \in L(Y)$ such that $C = UB^{\dagger}_{E,F}$ and $B = C^{\dagger}_{F,H}U$.*
(vii) *There exist a surjective map $U_1 \in L(Y)$ and an injective map $U_2 \in L(Y)$ such that $C = U_2B^{\dagger}_{E,F}$ and $B = C^{\dagger}_{F,H}U_1$.*
(viii) *There exist $U_3, U_4, U_5, U_6 \in L(Y)$ such that $C = U_5B^{\dagger}_{E,F}$, $B^{\dagger}_{E,F} = U_6C$, $B = C^{\dagger}_{F,H}U_3$ and $C^{\dagger}_{F,H} = BU_4$.*
(ix) *There exist a surjective map $U_7 \in L(Y)$ and an injective map $U_8 \in L(Y)$ such that $B^{\dagger}_{E,F} = U_8C$ and $C^{\dagger}_{F,H} = BU_7$.*

Proof. According to Theorem 2.1, statements (i) and (ii) are equivalents. In addition, according to Remark 1.7, statement (ii) implies statement (iii). On the other hand, if statement (iii) holds, then $R(BB^{\dagger}_{E,F}) = R(B) = R(C^{\dagger}_{F,H}) = R(C^{\dagger}_{F,H}C)$ and $N(BB^{\dagger}_{E,F}) = N(B^{\dagger}_{E,F}) = N(C) = N(C^{\dagger}_{F,H}C)$. However, since $BB^{\dagger}_{E,F}$ and $C^{\dagger}_{F,H}C$ are idempotents, statement (ii) holds.

Suppose that the statement (iii) holds. Since $N(I_X - C^{\dagger}_{F,H}C) = R(C^{\dagger}_{F,H}C) = R(C^{\dagger}_{F,H}) = R(B)$ and $R(I_X - BB^{\dagger}_{E,F}) = N(BB^{\dagger}_{E,F}) = N(B^{\dagger}_{E,F}) = N(C)$, statement (iv) holds.

On the other hand, observe that if statement (iv) holds, then $R(T) = R(B) \subseteq N(I_X - C^{\dagger}_{F,H}C) = R(C^{\dagger}_{F,H}) = R(T^{\dagger}_{E,H})$ and $N(T^{\dagger}_{E,H}) = N(B^{\dagger}_{E,F}) = R(I_X - BB^{\dagger}_{E,F}) \subseteq N(C) = N(T)$. However, since according to Remark 1.7

$$X = R(T) \oplus N(T^{\dagger}_{E,H}) = R(T^{\dagger}_{E,H}) \oplus N(T),$$

it is not difficult to prove that $R(T) = R(T^{\dagger}_{E,H})$ and $N(T^{\dagger}_{E,H}) = N(T)$. As a result, $R(TT^{\dagger}_{E,H}) = R(T^{\dagger}_{E,H}T)$ and $N(TT^{\dagger}_{E,H}) = N(T^{\dagger}_{E,H}T)$. Therefore, $TT^{\dagger}_{E,H} = T^{\dagger}_{E,H}T$.

A similar argument proves that statement (iii) implies statement (v), which in turn implies statement (i).

Next suppose that statement (i) holds and consider $U = CB \in L(Y)$. Then, according to statement (iv), $C = UB^{\dagger}_{E,F}$ and $B = C^{\dagger}_{F,H}U$. In order to prove that $U \in L(Y)$ is an isomorphism, define $Z = B^{\dagger}_{E,F}C^{\dagger}_{F,H} \in L(Y)$. Now well, according to statement (iv) and to Theorem 2.1(ii), $UZ = I_Y$. In addition, according again to statement (iv) and Theorem 2.1(i), $ZU = I_Y$.

Clearly, statement (vi) implies statements (vii)-(ix). On the other hand, it is not difficult to prove that statement (vii) (respectively statements (viii) and (ix)) implies statement (iii). □

Next the Banach algebra case will be studied. Firstly some preliminary facts need to be considered.

Theorem 2.4. *Let A be a unital Banach algebra and consider three invertible and positive elements $e, f, h \in A$ and $a \in A$ such that $a^{\dagger}_{e,h}$ exists. Suppose that there exist $b, c \in A$ such that $(b)^{-1}(0) = \{0\}$, $cA = A$ and $a = bc$. Then, the following statements hold:*

(i) *There exists $b^{\dagger}_{e,f} \in A$ such that $b^{\dagger}_{e,f}b = 1$.*
(ii) *There exists $c^{\dagger}_{f,h} \in A$ such that $cc^{\dagger}_{f,h} = 1$.*
(iii) *$a^{\dagger}_{e,h} = c^{\dagger}_{f,h}b^{\dagger}_{e,f}$, $aa^{\dagger}_{e,h} = bb^{\dagger}_{e,f}$, $a^{\dagger}_{e,h}a = c^{\dagger}_{f,h}c$, $b^{\dagger}_{e,f} = ca^{\dagger}_{e,h}$, $c^{\dagger}_{f,h} = a^{\dagger}_{e,h}b$, $ac^{\dagger}_{f,h} = b$ and $b^{\dagger}_{e,f}a = c$.*

Proof. Consider the maps $L_a, L_b, L_c \in L(A)$. Note that $L_a = L_bL_c$, $N(L_b) = \{0\}$ and $R(L_c) = A$. In addition, according to [5, Theorem 2.4], L_e, L_f and $L_h \in L(A)$ are three positive and invertible operators. Now well, according to [5, Theorem 2.8], $(L_a)^{\dagger}_{L_e,L_h}$ exists and $(L_a)^{\dagger}_{L_e,L_h} = L_{a^{\dagger}_{e,h}} \in L(A)$. Consequently, according to Theorem 2.1, $(L_b)^{\dagger}_{L_e,L_f}$ exists and $(L_b)^{\dagger}_{L_e,L_f}L_b = I_A$, $(L_c)^{\dagger}_{L_f,L_h}$ exists and $L_c(L_c)^{\dagger}_{L_f,L_h} = I_A$, $L_{a^{\dagger}_{e,h}} = (L_c)^{\dagger}_{L_f,L_h}(L_b)^{\dagger}_{L_e,L_f}$, $(L_b)^{\dagger}_{L_e,L_f} = L_{ca^{\dagger}_{e,h}}$ and $(L_c)^{\dagger}_{L_f,L_h} = L_{a^{\dagger}_{e,h}b}$.

Let $c' = (L_c)^{\dagger}_{L_f,L_h}(1) = a^{\dagger}_{e,h}b$. Since $L_c(L_c)^{\dagger}_{L_f,L_h} = I_A$, $cc' = 1$. In particular, $cc'c = c$, $c'cc' = c'$ and $c'c = a^{\dagger}_{e,h}bc = a^{\dagger}_{e,h}a \in H(A^h)$. Hence, $c^{\dagger}_{f,h}$ exists, $c^{\dagger}_{f,h} = c' = a^{\dagger}_{e,h}b$ and $cc^{\dagger}_{f,h} = 1$.

If $b' = ca^{\dagger}_{e,h}$, then $b'b = c(a^{\dagger}_{e,h}b) = cc^{\dagger}_{f,h} = 1$. As a result, $bb'b = b$ and $b'bb' = b'$. However, since $bb' = bca^{\dagger}_{e,h} = aa^{\dagger}_{e,h} \in H(A^e)$, $b^{\dagger}_{e,f}$ exists, $b^{\dagger}_{e,f} = b' = ca^{\dagger}_{e,h}$ and $b^{\dagger}_{e,f}b = 1$.

Set $a' = c^{\dagger}_{f,h}b^{\dagger}_{e,f}$. According to what has been proved, $aa'a = a$, $a'aa' = a'$, $aa' = bb^{\dagger}_{e,f}$ and $a'a = c^{\dagger}_{f,h}c$. Therefore, according to [5, Theorem 2.5] $a^{\dagger}_{e,h} = a' = c^{\dagger}_{f,h}b^{\dagger}_{e,f}$. The remaining identities can be derived from statements (i)-(ii). $\square$

In the following theorem, weighted Moore-Penrose invertible Banach algebra elements of the form $a = bc$ will characterized.

Theorem 2.5. *Under the same hypotheses of Theorem 2.4, the following statements are equivalent.*

(i) *The element a is weighted EP with weights e and h.*
(ii) *$bb^{\dagger}_{e,f} = c^{\dagger}_{f,h}c$.*
(iii) *$bA = c^{\dagger}_{f,h}A$ and $(b^{\dagger}_{e,f})^{-1}(0) = c^{-1}(0)$.*
(iv) *$(1 - c^{\dagger}_{f,h}c)b = 0$ and $c(1 - bb^{\dagger}_{e,f}) = 0$.*
(v) *$b^{\dagger}_{e,f}(1 - c^{\dagger}_{f,h}c) = 0$ and $(1 - bb^{\dagger}_{e,f})c^{\dagger}_{f,h} = 0$.*
(vi) *There exists $u \in A^{-1}$ such that $c = ub^{\dagger}_{e,f}$ and $b = c^{\dagger}_{f,h}u$.*
(vii) *There exist $u_1, u_2 \in A$ such that $u_1A = A$, $u_2^{-1}(0) = \{0\}$, $c = u_2b^{\dagger}_{e,f}$ and $b = c^{\dagger}_{f,h}u_1$.*

(viii) *There exist* $u_3, u_4, u_5, u_6 \in A$ *such that* $c = u_5 b^{\dagger}_{e,f}$, $b^{\dagger}_{e,f} = u_6 c$, $b = c^{\dagger}_{f,h} u_3$ *and* $c^{\dagger}_{f,h} = b u_4$.

(ix) *There exist* $u_7, u_8 \in A$ *such that* $u_7 A = A$, $u_8^{-1}(0) = \{0\}$, $b^{\dagger}_{e,f} = u_8 c$ *and* $c^{\dagger}_{f,h} = b u_7$.

(x) $a \in c^{\dagger}_{f,h} A \cap A b^{\dagger}_{e,f}$.

(xi) $a^{\dagger}_{e,h} \in bA \cap Ac$.

(xii) $bA^{-1} = c^{\dagger}_{f,h} A^{-1}$ *and* $A^{-1} c = A^{-1} b^{\dagger}_{e,f}$.

(xiii) $b_{-1}(0) = (c^{\dagger}_{f,h})_{-1}(0)$ *and* $Ac = Ab^{\dagger}_{e,f}$.

Proof. As in the proof of Theorem 2.4, let $T = L_a$, $B = L_b$, $C = L_c \in L(A)$ and note that $T = BC$, $N(B) = \{0\}$ and $R(C) = A$. In addition, $E = L_e$, $F = L_f$, $H = L_h \in L(A)$ are three invertible positive operators ([5, Lemma 2.4]). Therefore, according to [5, Theorem 2.8] and Theorem 2.1, T, B, C are weighted Moore–Penrose invertible and $T^{\dagger}_{E,H} = L_{a^{\dagger}_{e,h}}$, $B^{\dagger}_{E,F} = L_{b^{\dagger}_{e,f}}$ and $C^{\dagger}_{F,H} = L_{c^{\dagger}_{f,h}}$. Now well, according to Theorem 2.3, statements (i)-(v) are equivalent. To prove the equivalence of statements (i) and (vi)-(ix), adapt the proof of Theorem 2.3 to the case under consideration.

Suppose that statement (i) holds. Note that a straightforward calculation, using in particular statement (vi), proves that that $a \in c^{\dagger}_{f,h} A \cap A b^{\dagger}_{e,f}$. On the other hand, if statement (x) holds, then, according to Theorem 2.4, $a \in a^{\dagger}_{e,h} A \cap A a^{\dagger}_{e,h}$. In particular, $aA \subseteq a^{\dagger}_{e,h} A$ and $(a^{\dagger}_{e,h})^{-1}(0) \subseteq a^{-1}(0)$. However, an argument similar to the one in Theorem 2.3 (statement (iv) implies statement (i)), using in particular [5, Remark 2.6], proves that statement (i) holds.

Note that according to [5, Remark 2.6(i)], $(a^{\dagger}_{e,h})^{\dagger}_{h,e} = a$. In addition, according to Theorem 2.4, $(b^{\dagger}_{e,f})^{\dagger}_{f,e} = b$ and $(c^{\dagger}_{f,h})^{\dagger}_{h,f} = c$. What is more, it is clear that a is weighted EP with weights e and h if and only if $a^{\dagger}_{e,h}$ is weighted EP with weights h and e. Therefore, to prove the equivalence between statements (i) and (xi), it is enough to apply the equivalence between statements (i) a (x) to $a^{\dagger}_{e,h} = c^{\dagger}_{f,h} b^{\dagger}_{e,f}$.

Clearly, statement (vi) and (xii) are equivalent.

Observe that statement (i) is equivalent to $N(R_{aa^{\dagger}_{e,h}}) = N(R_{a^{\dagger}_{e,h}a})$ and $R(R_{aa^{\dagger}_{e,h}}) = R(R_{a^{\dagger}_{e,h}a})$. In addition, it is not difficult to prove that $R(R_{aa^{\dagger}_{e,h}}) = R(R_{bb^{\dagger}_{e,f}}) = Ab^{\dagger}_{e,f}$ and $R(R_{a^{\dagger}_{e,h}a}) = R(R_{c^{\dagger}_{f,h}c}) = Ac$; similarly, $(aa^{\dagger}_{e,h})_{-1}(0) = (bb^{\dagger}_{e,f})_{-1}(0) = b_{-1}(0)$ and $(a^{\dagger}_{e,h}a)_{-1}(0) = (c^{\dagger}_{f,h}c)_{-1}(0) = (c^{\dagger}_{f,h})_{-1}(0)$ (use the fact that $a^{\dagger}_{e,h}$ (respectively $b^{\dagger}_{e,f}$, $c^{\dagger}_{f,h}$) is a normalized generalized inverse of a (respectively b, c). Therefore, statement (i) and (xiii) are equivalent. □

3. Factorization $a^{\dagger}_{e,f} = sa$

In the following theorem, given a unital Banach algebra A, elements of the form $a^{\dagger}_{e,f} = sa$ will be characterized (a, $s \in A$, e, $f \in A$ invertible and positive, a weighted Moore-Penrose invertible with weights e and f). Recall that according to [5, Remark 2.6], $a^{\dagger}_{e,f}A = a^{\dagger}_{e,f}aA$, $aa^{\dagger}_{e,f}A = aA$, $(aa^{\dagger}_{e,f})^{-1}(0) = (a^{\dagger}_{e,f})^{-1}(0)$, $(a^{\dagger}_{e,f}a)^{-1}(0) = a^{-1}(0)$ and $A = a^{\dagger}_{e,f}A \oplus a^{-1}(0) = aA \oplus (a^{\dagger}_{e,f})^{-1}(0)$.

Theorem 3.1. *Let A be a unital Banach algebra and consider two invertible positive elements $e, f \in A$. Let $a \in A$ such that $a^{\dagger}_{e,f}$ exists. Then the following statements are equivalent.*

(i) *The element a is weighted EP with weights e and f.*
(ii) *There exist $s, t \in A$ such that $s^{-1}(0) = \{0\}$, $tA = A$ and $a^{\dagger}_{e,f} = sa = at$.*
(iii) *There exist $s_1, t_1 \in A$ such that $a^{\dagger}_{e,f} = s_1a = at_1$.*
(iv) *There exist $u, v, u_1, v_1 \in A$ such that $a^{\dagger}_{e,f}a = ua^{\dagger}_{e,f} = av$ and $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}u_1 = v_1a$.*
(v) *There exist $u_2, v_2, u_3, v_3 \in A$ such that $a^{\dagger}_{e,f}a = u_2a^{\dagger}_{e,f}$, $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}u_3$ and $a^{\dagger}_{e,f} = av_2 = v_3a$.*
(vi) *There exist $x, y \in A^{-1}$ such that $a^{\dagger}_{e,f}a = xaa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}y$.*
(vii) *There exist $x_1, y_1 \in A$ such that $x_1^{-1}(0) = \{0\}$, $y_1A = A$ and $a^{\dagger}_{e,f}a = x_1aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}y_1$.*
(viii) *There exist $x_2, y_2 \in A$ such that $x_2^{-1}(0) = \{0\}$ and $a^{\dagger}_{e,f}a = x_2aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}y_2$.*
(ix) *There exist z_1, $z_2 \in A$ such that $a^{\dagger}_{e,f}a = az_1a^{\dagger}_{e,f}$ and $aa^{\dagger}_{e,f} = a^{\dagger}_{e,f}z_2a$.*

Proof. If statement (i) holds, according to [5, Theorem 3.7(xiv)], $a \in a^{\dagger}_{e,f}A^{-1} \cap A^{-1}a^{\dagger}_{e,f}$. In particular, there exist $s, t \in A^{-1}$ such that $a^{\dagger}_{e,f} = sa = at$, which implies statement (ii).

Note that statement (ii) clearly implies statement (iii). On the other hand, if statement (iii) holds, then $a^{\dagger}_{e,f}A \subset aA$ and $a^{-1}(0) \subset (a^{\dagger}_{e,f})^{-1}(0)$. However, since $A = a^{\dagger}_{e,f}A \oplus a^{-1}(0) = aA \oplus (a^{\dagger}_{e,f})^{-1}(0)$ ([5, Remark 2.6(iv)]), it is not difficult to prove that $a^{\dagger}_{e,f}A = aA$ and $a^{-1}(0) = (a^{\dagger}_{e,f})^{-1}(0)$. Therefore, according to [5, Theorem 3.7(ii)], statement (i) holds.

Clearly, statement (i) implies statement (iv). On the other hand, if statement (iv) holds, a straightforward calculation, using in particular [5, Remark 2.6], proves that $aA = a^{\dagger}_{e,f}A$ and $a^{-1}(0) = (a^{\dagger}_{e,f})^{-1}(0)$. In particular, $R(L_{aa^{\dagger}_{e,f}}) = R(L_{a^{\dagger}_{e,f}a})$ and $N(L_{aa^{\dagger}_{e,f}}) = N(L_{a^{\dagger}_{e,f}a})$. Since, $aa^{\dagger}_{e,f}$ and $a^{\dagger}_{e,f}a$ are idempotents, a is weighted EP with weights e and h.

A similar argument proves the equivalence between statements (i) and (v).

Clearly, (i) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii). Suppose that (viii) holds. Then, according to [5, Remark 2.6], $a^{\dagger}_{e,f}A \subseteq aA$ and $a^{-1}(0) \subseteq (a^{\dagger}_{e,f})^{-1}(0)$. However, using an argument similar to the one in the proof of Theorem 2.5 (statement (x) implies statement (i)), it is easy to prove that statement (i) holds.

It is clear that statement (i) implies statement (ix). On the other hand, if statement (ix) holds, then $a^{\dagger}_{e,f}A = aA$ and $(a^{\dagger}_{e,f})^{-1}(0) = a^{-1}(0)$. To conclude the proof, procced as before (statement (iv) implies statement (i)). ☐

4. Factorization of the form $a = ucu^{-1}$

Let H be a Hilbert space and consider $T \in L(H)$. It is well known that necessary and sufficient for T to be EP is that there exists an orthogonal idempotent $P \in L(H)$ such that if $H_1 = R(P)$, $H_2 = N(P)$, $H = H_1 \oplus H_2$ and $T_1 = T\mid_{H_1}^{H_1}\colon H_1 \to H_1$, then $T_1 \in L(H_1)$ is an isomorphism and with respect to the aforementioned decomposition, T can be presented as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix},$$

see for example [8, Theorem 2.1]. The case of EP Banach space operators was considered in [3, Section 5]. Next this kind of factorization will be studied for weighted EP bounded and linear maps. However, in first place some preparation is needed.

In the following proposition, given two Banach spaces X_1 and X_2, $X_1 \oplus_1 X_2$ will denote the Banach space $X_1 \oplus X_2$ with the 1-norm, i.e., $\parallel x_1 \oplus x_2 \parallel = \parallel x_1 \parallel + \parallel x_2 \parallel$.

Proposition 4.1. *Let X_1 and X_2 be two Banach spaces and consider $T_1 \in L(X_1)$ an isomorphic operator. Let X be a Banach space and consider E, F, $T \in L(X)$ such that E and F are two invertible positive operators and there exists a linear and bounded isomorphism $J\colon X_1 \oplus_1 X_2 \to X$ with the property $T = J(T_1 \oplus 0)J^{-1}$. Consider $T' = J(T_1^{-1} \oplus 0)J^{-1} \in L(X)$. Then, the following statements are equivalent.*
(i) *T is weighted Moore-Penrose invertible with weights E and F and $T^\dagger_{E,F} = T'$.*
(ii) *T is weighted EP with weights E and F and $T^\dagger_{E,F} = T'$.*
(iii) *$Q_1 = J(I_{X_1} \oplus 0)J^{-1} \in H(L(X)^E)$ and $Q_2 = J(0 \oplus I_{X_2})J^{-1} \in H(L(X)^F)$.*

Proof. It is not difficult to prove that T' is a normalized generalized inverse of T and that Q_1 and Q_2 are the projections onto the closed and complemented subspaces $J(X_1 \oplus 0)$ and $J(0 \oplus X_2)$ respectively. Furthermore, according to [9, Theorem 4.4(i)], since $TT' = T'T = Q_1 = I - Q_2$, statements (i)-(iii) are equivalent. □

In the following theorem, weighted EP bounded and linear maps will be characterized.

Theorem 4.2. *Let X be a Banach space and consider E, $F \in L(X)$ two invertible positive operators. Let $T \in L(X)$. Then, the following statements are equivalent.*
(i) *T is weighted EP with weights E and F,*
(ii) *There exist two Banach spaces X_1 and X_2, $T_1 \in L(X_1)$ an isomorphic operator, and $J\colon X_1 \oplus_1 X_2 \to X$ a linear and bounded isomorphism such that $T = J(T_1 \oplus 0)J^{-1}$, $T^\dagger_{E,F} = J(T_1^{-1} \oplus 0)J^{-1}$, $J(I_{X_1} \oplus 0)J^{-1} \in H(L(X)^E)$ and $J(0 \oplus I_{X_2})J^{-1} \in H(L(X)^F)$.*

Proof. According to Proposition 4.1, statement (ii) implies that T is weighted EP with weights E and F. On the other hand, if statement (i) holds, according to [5, Theorem 3.4(a)], there exist $P \in H(L(X)^E) \cap H(L(X)^F)$ such that $N(P) = N(T)$ and $R(P) = R(T)$. Denote then $X_1 = R(P)$, $X_2 = N(P)$ and $T_1 =$

$T\mid_{X_1}^{X_1}: X_1 \to X_1$. It is clear that $T_1 \in L(X_1)$ is an isomorphism. Moreover, $J: X_1 \oplus_1 X_2 \to X$ is the map $J(x_1 \oplus x_2) = x_1 + x_2$. Since $J^{-1}: X \to X_1 \oplus_1 X_2$ is such that $J^{-1} = P \oplus (I_X - P)$, $T = J(T_1 \oplus 0)J^{-1}$, $T^{\dagger}_{E,F} = J(T_1^{-1} \oplus 0)J^{-1}$, $J(I_{X_1} \oplus 0)J^{-1} = P \in H(L(X)^E)$ and $J(0 \oplus I_{X_2})J^{-1} = I - P \in H(L(X)^F)$ ([9, Theorem 4.4(i)]). □

Next weighted EP Banach space operators will be characterized in terms of injective and surjective bounded and linear maps.

Theorem 4.3. *Let X be a Banach space and consider E and F two invertible positive operators. Let $T \in L(X)$. Then, the following statements are equivalent.*
(i) *T is weighted EP with weights E and F.*
(ii) *There exist Banach spaces X_1 and X_2, $T_1 \in L(X_1)$ an isomorphism, $S \in L(X_1 \oplus_1 X_2, X)$ injective, $U \in L(X, X_1 \oplus_1 X_2)$ surjective, and an idempotent $P \in H(L(X)^E) \cap H(L(X)^F)$ such that $T = S(T_1 \oplus 0)U$, $R(P) = S(X_1 \oplus 0)$ and $N(P) = U^{-1}(0 \oplus X_2)$.*

Proof. If T is weighted EP with weights E and F, then let X_1, X_2 and $T_1 \in L(X_1)$ be as in Theorem 4.2 and define $S = J \in L(X_1 \oplus_1 X_2, X)$ and $U = J^{-1} \in L(X, X_1 \oplus_1 X_2)$, J as in Theorem 4.2. Moreover, define $P = J(I_{X_1} \oplus 0)J^{-1} \in H(L(X)^E) \cap H(L(X)^F)$. Since $R(P) = R(T) = S(X_1 \oplus 0)$ and $N(P) = N(T) = U^{-1}(0 \oplus X_2)$, statement (ii) holds.

On the other hand, if statement (ii) holds, then P is an idempotent such that $P \in H(L(X)^E) \cap H(L(X)^F)$, $R(P) = R(T)$ and $N(P) = N(T)$. Therefore, according to [5, Theorem 3.4(a)], T is weighted EP with weights E and F. □

As a corollary of Theorem 4.2, more characterizations of weighted EP Banach space operators will be given.

Corollary 4.4. *Let X be a Banach space and consider E and F two invertible positive operators. Let $T \in L(X)$. Then, the following statements are equivalent.*
(i) *T is weighted EP with weights E and F.*
(ii) *T is weighted EP with weights F and E.*
(iii) *T is weighted EP both with weights E and E and with weights F and F.*

Proof. According to Theorem 4.2, statement (ii) is equivalent to the fact that there exist two Banach spaces X_1 and X_2, $T_1 \in L(X_1)$ an isomorphic operator, and $J: X_1 \oplus_1 X_2 \to X$ a linear and bounded isomorphism such that $T = J(T_1 \oplus 0)J^{-1}$, $T^{\dagger}_{E,F} = J(T_1^{-1} \oplus 0)J^{-1}$, $J(I_{X_1} \oplus 0)J^{-1} \in H(L(X)^E)$ and $J(0 \oplus I_{X_2})J^{-1} \in H(L(X)^F)$. Now well, according to [9, Theorem 4.4(i)], $J(I_{X_1} \oplus 0)J^{-1} \in H(L(X)^E)$ if and only if $J(0 \oplus I_{X_2})J^{-1} = I - J(I_{X_1} \oplus 0)J^{-1} \in H(L(X)^E)$. Similarly, $J(0 \oplus I_{X_2})J^{-1} \in H(L(X)^F)$ if and only if $J(I_{X_1} \oplus 0)J^{-1} = I - J(0 \oplus I_{X_2})J^{-1} \in H(L(X)^F)$. As a result, statement (i) and (ii) are equivalent.

Clearly, statements (i) and (ii) imply statement (iii), and statement (iii) implies both statements (i) and (ii). □

When Hilbert space are considered, Theorem 4.2 reduces to the following corollary.

Corollary 4.5. *Let H be a Hilbert space and consider T, E and $F \in L(H)$ such that E and F are invertible positive. Then, the following statement are equivalent.*
(i) *T is weighted EP with weights E and F.*
(ii) *There exists an idempotent $P \in L(H)$ such that $E^{-1}P^*E = P$, $F^{-1}P^*F = P$, and if $H_1 = R(P)$, $H_2 = N(P)$ and $T_1 = T\mid_{H_1}^{H_1}\colon H_1 \to H_1$, then $H = H_1 \oplus H_2$ and with respect to the aforementioned decomposition, T and $T^\dagger_{E,F}$ can be presented as*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad\qquad T^\dagger_{E,F} = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

respectively.

Proof. Apply Theorem 4.2 to the case under consideration and recall that according to [6, Proposition 20, Chapter I, section 12], $P \in H(L(H)^E)$ (respectively $P \in H(L(H)^F)$) if and only if $E^{-1}P^*E = P$ (respectively $F^{-1}P^*F = P$), see the discussion before [5, Definition 2.3]. □

Next the Banach algebra case will be studied. Recall that according to [8, Theorem 1.4], if A is a C^*-algebra, then $a \in A$ is EP if and only if there exists a hermitian idempotent $q \in A$ such that $qa = a = aq$ and $a = qaq \in (qAq)^{-1}$, where given a Banach algebra B and $p \in B$ such that $p^2 = p$, then pBp is a Banach algebra with unit p.

Theorem 4.6. *Let A be a unital Banach algebra and consider e, $f \in A$ two invertible positive elements. Let $a \in A$. Then the following statements are equivalent.*
(i) *The element a is weighted EP with weights e and f.*
(ii) *There exist $c \in A^{-1}$ and $p \in A$ such that $p = p^2$, $cpc^{-1} \in H(A^e) \cap H(A^f)$, $pap \in (pAp)^{-1}$ and $a = cpapc^{-1}$.*
(iii) *The element a is weighted EP with weights f and e.*
(iv) *The element a is weighted EP both with weights e and e and with weights f and f.*

Proof. If $a \in A$ is weighted EP with weights e and f, then $p = aa^\dagger_{e,f} = a^\dagger_{e,f}a$ and $c = 1 = p + (1-p)$ satisfy the conditions in statement (ii). Note that in this case $a = pap$ and $pa^\dagger_{e,f}p = a^\dagger_{e,f} \in pAp$ is the inverse of $a = pap$ in the subalgebra pAp.

On the other hand, suppose that statement (ii) holds. Since $pap \in (pAp)^{-1}$, there exists $b \in A$ such that $papbp = pbpap = p$. As a result, $d = cpbpc^{-1}$ is a normalized generalized inverse of a such that $ad = da = cpc^{-1}$. In fact,

$$\begin{aligned} ad &= acpbpc^{-1} = cpapc^{-1}cpbpc^{-1} = cpc^{-1}, \\ da &= cpbpc^{-1}cpapc^{-1} = cpc^{-1}, \\ ada &= acpc^{-1} = cpapc^{-1}cpc^{-1} = cpapc^{-1} = a \\ dad &= cpc^{-1}cpbpc^{-1} = cpbpc^{-1} = d. \end{aligned}$$

Since $cpc^{-1} \in H(A^e) \cap H(A^f)$, $a^\dagger_{e,f}$ exists and a is weighted EP with weights e and f. In fact, $a^\dagger_{e,f} = d$.

To prove the equivalences among statement (i) and statements (iii) and (iv), use an argument similar to the one in the proof of Corollary 4.4. □

Under the same hypothesis of Theorem 4.6, when $e = f = 1$, the following corollary presents the case $a \in A$, a an EP element, see [3, section 5].

Theorem 4.7. *Let A be a unital Banach algebra and consider $a \in A$. Then the following statements are equivalent.*
(i) *The element a is EP.*
(ii) *There exist $c \in A^{-1}$ and $p \in A$ such that $p = p^2$, $cpc^{-1} \in H(A)$, $pap \in (pAp)^{-1}$ and $a = cpapc^{-1}$.*

Proof. Apply Theorem 4.6 to the case under consideration. □

As a corollary of Theorem 4.6, weighted EP C^*-algebra elements will be characterized.

Corollary 4.8. *Let A be a C^*-algebra and consider e, $f \in A$ two invertible positive elements. Let $a \in A$. Then the following statements are equivalent.*
(i) *The element a is weighted EP with weights e and f.*
(ii) *There exist $c \in A^{-1}$ and $p \in A$ such that $p = p^2$, $e(cpc^{-1})^*e^{-1} = cpc^{-1}$, $f(cpc^{-1})^*f^{-1} = cpc^{-1}$ and $a = cpapc^{-1}$.*

Proof. Adapt the proof of Corollary 4.5 to the case under consideration applying Theorem 4.6 instead of Theorem 4.2. □

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References

1. J. Benítez, *Moore-Penrose inverses and commuting elements of C^*-algebras*, J. Math. Anal. Appl. **345** (2008), 766-770.
2. E. Boasso, *On the Moore-Penrose inverse, EP Banach space operators, and EP Banach algebra elements*, J. Math. Anal. Appl. **339** (2008), 1003-1014.
3. E. Boasso, *Factorizations of EP Banach space operators and EP Banach algebra elements*, J. Math. Anal. Appl. **379** (2011), 245-255.
4. E. Boasso and V. Rakočević, *Characterizations of EP and normal Banach algebra elements and Banach space operators*, Linear Algebra Appl. **435** (2011), 342-353.
5. E. Boasso, D. Mosić and D.S. Djordjević, *Weighted Moore-Penrose invertible and weighted EP Banach algebra elements*, J. Korean Math. Soc. **50** (2013), 1349-1367.
6. F.F. Bonsall and J. Duncan, *Complete normed algebras* Springer Verlag, Berlin, Heidelberg, New York, 1973.
7. J.B. Conway, *A Course in Functional Analysis* Springer Verlag, New York, Berlin, Heidelberg, Tokio, 1985.
8. D.S. Djordjević, J.J. Koliha and I. Straskaba, *Factorizations of EP elements of C^*-algebras*, Linear and Multilinear Algebra **57** (2009), 587-594.
9. H.R. Dowson, *Spectral Theory of Linear operators* London, New York, San Francisco, 1978.
10. L.T. Gardner, *Square roots in Banach algebras*, Proc. Amer. Math. Soc **17** (1966), 132-134.
11. R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. **103** (1992), 71-77.

12. R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras II*, Studia Math. **106** (1993), 129-138.
13. J. J. Koliha, *Elements of C^*-algebras commuting with their Moore-Penrose inverse*, Studia Math. **139** (2000), 81-90.
14. T. Palmer, *Unbounded normal operators on Banach spaces*, Trans. Amer. Math. Soc. **133** (1968), 385-414.
15. R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406-413.
16. V. Rakočević, *Moore-Penrose inverse in Banach algebras*, Proc. Roy. Irish Acad. Sect. A **88** (1988), 57-60.
17. Y. Tian and H. Wang, *Characterizations of EP matrices ad weighted-EP matrices*, Linear Algebra Appl. **434** (2011), 1295-1318.

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