APPLICATIONS OF THE GROETCH THEOREM

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Abstract. In this paper we investigate general representations of various classes of generalized inverses of bounded operators over Hilbert spaces, based on the full-rank factorization of operators. Using these general representations we introduce a generalization of the Groetch representation theorem for the Moore-Penrose inverse. As corollaries, we derive a few iterative methods for computing reflexive g-inverses. In a particular case we get the main result from [9]. The present method is compared with [6].

1. Introduction

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote arbitrary Banach spaces and $B(\mathcal{X}_1, \mathcal{X}_2)$ denote the set of all bounded operators from $\mathcal{X}_1$ into $\mathcal{X}_2$. For an arbitrary operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$, we use $\mathcal{N}(A)$ to denote its kernel, and $\mathcal{R}(A)$ to denote its image. An operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ is $g$-invertible, provided that there exists some $X \in B(\mathcal{X}_2, \mathcal{X}_1)$, such that $AXA = A$. In this case $X$ is called a $g$-inverse of $A$. If $X$ satisfies both of the equations $AXA = A$ and $XAX = X$, then $X$ is called a reflexive $g$-inverse of $A$. It is well-known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has a $g$-inverse if and only if $\mathcal{R}(A)$ is closed, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are complemented subspaces of $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. An arbitrary right inverse and an arbitrary left inverse of $A$ are denoted by $A_r^{-1}$ and $A_l^{-1}$, respectively.

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We say that \( A \in B(\mathcal{X}) \) has the Drazin inverse, if there exists an operator \( A^D \in B(\mathcal{X}) \), such that \( A^D \) satisfies the equation (2) and the equations

\[
(1^k) \quad A^{k+1}A^D = A^k, \quad (5) \quad A^D A = AA^D,
\]

for some non-negative integer \( k \). Let us mention that the Drazin inverse, if it exists, is unique. The smallest \( k \) in the previous definition is called the index of \( A \) and denoted by \( \text{ind}(A) \). In the case \( \text{ind}(A) = 1 \) the Drazin inverse is known as the group inverse of \( A \), denoted by \( A^\# \).

The full rank factorization of matrices is well-known and frequently used in representations of pseudoinverses [1, 7, 8, 10]. The following analogy of the full rank factorization for matrices is established in [2], [3]:

Let \( A \in B(\mathcal{X}_1, \mathcal{X}_2) \). If there exist a Banach space \( \mathcal{X}_3 \) and operators \( Q \in B(\mathcal{X}_1, \mathcal{X}_3) \) and \( P \in B(\mathcal{X}_3, \mathcal{X}_2) \), such that \( P \) is left invertible, \( Q \) is right invertible and

\[
(1.1) \quad A = PQ,
\]

then we say that (1.1) is the full-rank decomposition of \( A \).

It is well-known that an operator \( A \in B(\mathcal{X}_1, \mathcal{X}_2) \) has the full-rank decomposition, if and only if \( A \) is \( g \)-invertible. In this case \( \mathcal{X}_3 \) is isomorphic to \( \mathcal{R}(A) \), and \( \mathcal{R}(A) = \mathcal{R}(P) \) [3].

In the case when \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are Hilbert spaces, it is well-known that an operator \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) has a \( g \)-inverse if and only if \( \mathcal{R}(A) \) is closed. We consider the following equations in \( X \):

\[
(1) \quad AXA = A, \quad (2) \quad XAX = X, \quad (3) \quad (AX)^* = AX, \quad (4) \quad (XA)^* = XA.
\]

For a subset \( S \) of the set \{1, 2, 3, 4\}, the set of operators obeying the conditions contained in \( S \) is denoted by \( A\{S\} \). An operator in \( A\{S\} \) is called an \( S \)-inverse of \( A \) and is denoted by \( A^{(S)} \). If \( \mathcal{R}(A) \) is closed, the set \( A\{1, 2, 3, 4\} \) consists of a single element, the Moore-Penrose inverse of \( A \), denoted by \( A^\dagger \).
A basic tool used in this paper is the following general representation theorem for the Moore-Penrose inverse of a bounded linear operator [3], [4], [5]:

**Theorem 1.1.** Let \( T \in B(\mathcal{H}_1, \mathcal{H}_2) \) has closed range. Then [5, p. 45]

\[
T^\dagger = \tilde{T}^{-1}T^*, \quad \text{where} \quad \tilde{T} = T^*T|_{\mathcal{R}(T^*)}.
\]

Moreover, if \( \Omega \) is an open set with \( \sigma(\tilde{T}) \subset \Omega \subset (0, \infty) \), and \( \{S_\beta(x)\}_{\beta} \) is a family of continuous real valued functions on \( \Omega \), with \( \lim_{\beta} S_\beta(x) = \frac{1}{x} \) uniformly on \( \sigma(\tilde{T}) \), then [3, p. 42], [4], [5, p. 57]

\[
T^\dagger = \lim_{\beta} S_\beta(\tilde{T})T^*,
\]

where the convergence is in the uniform topology for \( B(\mathcal{H}_2, \mathcal{H}_1) \). Furthermore,

\[
\|S_\beta(\tilde{T})T^* - T^\dagger\| \leq \sup_{x \in \sigma(\tilde{T})} |xS_\beta(x) - 1| : \|T^\dagger\|.
\]

We investigate general representations of bounded operators on Hilbert spaces, based on the full-rank factorization (1.1). These representations are extensions of known results from [2], [7], [8] and [10].

Using these general representations together with the Groetch representation theorem for the Moore-Penrose inverse of a bounded operator on Hilbert spaces, we introduce representations for various subsets of the set of all reflexive \( g \)-inverses of a bounded operator. Using this extension of the Groetch representation theorem, as particular cases, we derive a few iterative methods for computing \( g \)-inverses. As a partial result we get an improvement of the hyper-power iterative method, which is investigated in [9] for operators acting on finite dimensional complex Hilbert spaces. This method is not known for matrices before.
2. Results

Firstly we state the following general representations based on the full-rank factorization of operators. These representations are known for matrices (see [7], [8] and [10]). For bounded operators between Hilbert spaces it is known a representation of the Moore-Penrose inverse, introduced in [2].

Lemma 2.1. Let $A = PQ$ be a full-rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to (1.1). Then:

(a) $X \in A\{1, 2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ and $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that $QW_1$ and $W_2 P$ are invertible in $B(\mathcal{H}_3)$. In such a case, $X$ possesses the following general representation

\begin{equation}
X = Q_r^{-1} P_l^{-1}, \quad Q_r^{-1} = W_1 (QW_1)^{-1}, \quad P_l^{-1} = (W_2 P)^{-1} W_2.
\end{equation}

(b) $X \in A\{1, 2, 3\}$ if and only if there exists an operator $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, such that $QW_1$ is invertible in $B(\mathcal{H}_3)$. In the case when it exists, a general representation for $X$ is as follows:

\begin{equation}
X = W_1 (QW_1)^{-1} (P^* P)^{-1} P^*.
\end{equation}

(c) $X \in A\{1, 2, 4\}$ if and only if there exists an operator $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that $W_2 P$ is invertible in $B(\mathcal{H}_3)$. In this case

\begin{equation}
X = Q^* (QQ^*)^{-1} (W_2 P)^{-1} W_2.
\end{equation}

(d) $A^\dagger = Q^\dagger P^\dagger = Q^* (QQ^*)^{-1} (P^* P)^{-1} P^* = Q^* (P^* A Q)^{-1} P^*$ [2].

(e) Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $X : \mathcal{H}_2 \rightarrow \mathcal{H}_1$. Then $X \in A\{2\}$ if and only if there exist operators

$C \in B(\mathcal{H}_4, \mathcal{H}_1), \ D \in B(\mathcal{H}_2, \mathcal{H}_3), \ W_1 \in B(\mathcal{H}_5, \mathcal{H}_4), \ W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$

such that $DAC$ is $q$-invertible and $W_2 DAC W_1$ is invertible and $X$ posses the following general form:

\begin{equation}
X = CW_1 (W_2 DAC W_1)^{-1} W_2 D.
\end{equation}
Proof. (a) This statement can be proved as in [8, Theorem 2.1.1 and Lemma 2.5.2].

(b) If \( X \) has the form (2.2), then it is easy to verify \( X \in A\{1, 2, 3\} \). We need to prove that the form (2.2) holds for all \( \{1, 2, 3\} \) inverses of \( A \). Indeed, if \( X \in A\{1, 2, 3\} \), then \( X = Q_r^{-1}P_l^{-1} \), and from the equation (3) it follows that \( (PP_l^{-1})^* = PP_l^{-1} \). Thus \( P^*PP_l^{-1} = P^* \). The operator \( P^*P \) is invertible, so that \( P_l^{-1} = (P^*P)^{-1}P^* \). The right inverse of \( Q \) retains the general form \( Q_r^{-1} = W_1(QW_1)^{-1} \) given in (2.1). Consequently,

\[
X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*.
\]

(c) This part of the proof can be proved in the same way as (b).

(d) Follows from (b) and (c) (also, this fact is proved in [2]).

(e) If \( X \) possesses the form (2.3), it is not difficult to verify \( X \in A\{2\} \). On the other hand, using the method from [8, Theorem 3.4.1], it is easy to verify that \( X \in A\{2\} \) if and only if there exist operators \( C \) and \( D \), such that \( DAC \) is \( g \)-invertible and

\[
X = C(DAC)^{(1,2)}D, \quad C \in B(\mathcal{H}_4, \mathcal{H}_1), \quad D \in B(\mathcal{H}_2, \mathcal{H}_3).
\]

According to part (a), \( X \in A\{2\} \) if and only if there exist operators \( W_1 \in B(\mathcal{H}_5, \mathcal{H}_4) \) and \( W_2 \in B(\mathcal{H}_3, \mathcal{H}_5) \), such that \( W_2DACW_1 \) is invertible, and \( X \) possesses the form (2.3). \( \square \)

**Lemma 2.2.** Let \( X \) be a Banach space. If \( A \in B(X) \), \( l \geq k = \text{asc}(A) = \text{des}(A) < \infty \) and \( A^l = PA^lQ_A^l \) is the full-rank decomposition of \( A^l \), then

\[
A^D = P_A^l(Q_A^lAP_A^l)^{-1}Q_A^l.
\]

Proof. If \( \text{asc}(A) = \text{des}(A) = k < \infty \), then it is well-known that \( N(A^l) = \mathcal{N}(A^k) \) and \( \mathcal{R}(A^l) = \mathcal{R}(A^k) \) for all \( l \geq k \),

\[
X = X_1 \oplus X_2.
\]
where $X_1 = \mathcal{N}(A^i)$ and $X_2 = \mathcal{R}(A^i)$, $A(X_i) \subset X_i$ for $i = 1, 2$, $A_1 = A|_{X_1}$ is nilpotent and $A_2 = A|_{X_2}$ is invertible ($A$ is not nilpotent) [3], [4]. We can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix}$$

with respect to the decomposition (2.4) (see [3]). Since $\mathcal{N}(A^i)$ and $\mathcal{R}(A^i)$ are complementary and closed subspaces of $X$, it follows that $A^i$ is $g$-invertible, so there exists the full-rank decomposition $A^i = P_{A^i}Q_{A^i}$, where $P_{A^i} \in B(\mathcal{Z}, X)$ is left invertible and $Q_{A^i} \in B(X, \mathcal{Z})$ is right invertible, for some Banach space $\mathcal{Z}$. By the isomorphism theorem [3], we can take that $\mathcal{Z} = X_2$. We conclude that $P_{A^i}$ and $Q_{A^i}$ have the following representations with respect to (2.4):

$$P_{A^i} = \begin{bmatrix} M' \\ \tilde{P} \end{bmatrix}, \quad Q_{A^i} = \begin{bmatrix} N & \tilde{Q} \end{bmatrix},$$

where $\tilde{P}, \tilde{Q} \in B(X_2)$, $M \in B(X_2, X_1)$, $N \in B(X_1, X_2)$. Now, $P_{A^i}$ is left invertible and $Q_{A^i}$ is right invertible, so $P_{A^i}$ and $Q_{A^i}$ are $g$-invertible operators, $\mathcal{N}(P_{A^i}) = \{0\}$ and $\mathcal{R}(Q_{A^i}) = X_2$. It follows that $\mathcal{R}(P_{A^i}) = \mathcal{R}(A^i) = X_2$ and $\mathcal{N}(Q_{A^i}) = \mathcal{N}(A^i) = X_1$, so $M = 0$, $N = 0$ and

$$P_{A^i} = \begin{bmatrix} 0 \\ \tilde{P} \end{bmatrix}, \quad Q_{A^i} = \begin{bmatrix} 0 & \tilde{Q} \end{bmatrix}.$$  

It is easy to verify that $\tilde{P}$ is left invertible and $\tilde{Q}$ is right invertible in $B(X_2)$. But

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2^i \end{bmatrix} = A^i = P_{A^i}Q_{A^i} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}\tilde{Q} \end{bmatrix},$$

so $A_2^i = \tilde{P}\tilde{Q}$. Since $A_2^i$ is invertible, it follows that $\tilde{P}$ and $\tilde{Q}$ are invertible in $B(X_2)$.

Now, $Q_{A^i}AP_{A^i} = \tilde{Q}A_2\tilde{P}$ is invertible in $B(X_2)$, so

$$A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}(\tilde{Q}A_2\tilde{P})^{-1}\tilde{Q} \end{bmatrix} = P_{A^i}(Q_{A^i}AP_{A^i})^{-1}Q.$$

□
Remark 2.1. The result of part (e) of Lemma 2.1 is an extension of the analogous result, introduced in [10, Theorem 2.1], stated for the set of complex matrices. Also, the result of Lemma 2.2 is an extension of an analogous result [10, Theorem 2.2], which is derived for complex matrices.

Our main aim is an application of considered general representations in a generalization of the Groetch representation theorem.

We begin with the result which enable us to get various reflexive generalized inverses of the considered operator, changing initial operators $W_1$ and $W_2$.

**Theorem 2.1.** Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ be the full-rank decomposition of $A$ and $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$. Suppose that $QW_1$ is right invertible, $W_2P$ is left invertible, $W = WAW_1$ and $\tilde{W} = W^*W|_{\mathcal{R}(W^*)}$. If $\Omega$ is an open set with $\sigma(\tilde{W}) \subset \Omega \subset (0, \infty)$, and $\{S_\beta(x)\}_\beta$ is a family of continuous real valued functions on $\Omega$, with $\lim_\beta S_\beta(x) = \frac{1}{x}$ uniformly on $\sigma(\tilde{W})$, then:

$$X = \lim_\beta W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 \in A\{1, 2\},$$

where the convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$.

Furthermore,

$$\|W_1S_\beta(\tilde{W})W^*W_2 - X\| \leq \|W_1\| \sup_{x \in \sigma(\tilde{W})} |xS_\beta(x) - 1| \cdot \|W^*\|\|W_2\|.$$ 

**Proof.** Since $W = (W_2P)(QW_1)$, $QW_1$ is onto, $W_2P$ is one-to-one and $\mathcal{R}(W_2P)$ is closed, it follows that $\mathcal{R}(W) = \mathcal{R}(W_2P)$, so we may apply Theorem 1.1 for $W$ instead of $T$. We conclude

$$X = \lim_\beta W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 = W_1(W_2AW_1)^\dagger W_2 = W_1((W_2P)(QW_1))^\dagger W_2.$$ 

Operators $W_2P$ and $QW_1$ form the full-rank decomposition for $W$, and applying the part (d) of Lemma 2.1 we immediately obtain $((W_2P)(QW_1))^\dagger = W_2P(W_1)^\dagger = W_2PQ^{-1}W_1^\dagger = W_2PQ^{-1}W_1^\dagger$.
(QW_1)^\dagger(W_2P)^\dagger. Since (QW_1)^\dagger is the right inverse of QW_1 and (W_2P)^\dagger is the left inverse for W_2P, we easily conclude that

\[ X = W_1(QW_1)^\dagger(W_2P)^\dagger W_2 \in A\{1, 2\}. \]

\[ \square \]

Using Lemma 2.1, similar results can be stated for \{i, j, k\} generalized inverses. For example, if \( W_1 = Q^* \) then \( X \in A\{1, 2, 3\} \). Also, if \( W_2 = P^* \) then \( X \in A\{1, 2, 4\} \). To avoid repetition we omit the proof.

Applying Lemma 2.1, Lemma 2.2 and the method from Theorem 2.1, we get the following representations of \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\} inverses, the Moore-Penrose inverse and the Drazin inverse.

**Corollary 2.1.** Let \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) has closed range and \( A = PQ \) be the full–rank decomposition of \( A \) according to (1.1). Let \( \{S_\beta(x)\}_\beta \) be a family of continuous real valued functions on \((0, +\infty)\), with \( \lim_\beta S_\beta(x) = \frac{1}{x} \) uniformly on all compact subsets of \((0, +\infty)\). Then:

(a) \( X \in A\{1, 2\} \) if and only if there exist operators \( W_1 \in B(\mathcal{H}_3, \mathcal{H}_1) \), \( W_2 \in B(\mathcal{H}_2, \mathcal{H}_3) \), such that \( QW_1 \) and \( W_2P \) are invertible, and

\[ X = \lim_\beta W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 = W_1\tilde{W}^{-1}W^*W_2, \quad W = W_2AW_1. \]

(b) \( X \in A\{1, 2, 3\} \) if and only if there exists \( W_2 \in B(\mathcal{H}_2, \mathcal{H}_3) \) such that \( W_2P \) is invertible and

\[ X = \lim_\beta Q^* \left[ S_\beta(W_2AQ^*) \right] (W_2AQ^*)^* W_2 = Q^*(W_2AQ^*)^{-1}(W_2AQ^*)^* W_2. \]

(c) \( X \in A\{1, 2, 4\} \) if and only if there exists \( W_1 \in B(\mathcal{H}_3, \mathcal{H}_1) \) such that \( QW_1 \) is left invertible and

\[ X = \lim_\beta W_1 \left[ S_\beta(P^*AW_1) \right] (P^*AW_1)^*P^* = W_1(P^*AW_1)^{-1}(P^*AW_1)^*P^*. \]

(d) \( A^\dagger = \lim_\beta Q^* \left[ S_\beta(P^*AQ^*) \right] (P^*AQ^*)^*P^* = Q^*(P^*AQ^*)^{-1}(P^*AQ^*)^*P^*. \)
(e) $X \in A\{2\}$ if and only if there exist operators $C \in B(\mathcal{H}_4, \mathcal{H}_1)$, $D \in B(\mathcal{H}_2, \mathcal{H}_3)$, $W_1 \in B(\mathcal{H}_5, \mathcal{H}_4)$, $W_2 \in B(\mathcal{H}_3, \mathcal{H}_5)$, such that $DAC$ is $g$-invertible, $W_2 DACW_1$ is invertible and $X = \lim_{\beta} CW_1 \left[ S_\beta(W_2 \widehat{DAC} W_1) \right] (W_2 DACW_1)^* W_2 D$.

(f) If $l \geq k = \text{ind}(A)$ and $Q_A^t A^t P_A^t$ is nonsingular, then $A^D = \lim_{\beta} P_A^t \left[ S_\beta(Q_A^t A^t P_A^t) \right] (Q_A^t A^t P_A^t)^* Q$.

The convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$.

Our aim is to use various initial conditions for $W_1$ and $W_2$, so we need the next result.

**Theorem 2.2.** Let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, let $\Omega$ be an open set with $\sigma(T^*T|_{R(T^*)}) \cup \sigma(TT^*|_{R(T)}) \subset \Omega \subset (0, \infty)$, and let $\{S_\beta(x)\}_{\beta}$ be a family of continuous real valued functions on $\Omega$, with $\lim_{\beta} S_\beta(x) = \frac{1}{x}$ uniformly on $\sigma(T^*T|_{R(T^*)}) \cup \sigma(TT^*|_{R(T)})$. Then

$$\lim_{\beta} T^* \left[ S_\beta(TT^*|_{R(T)}) \right] = \lim_{\beta} \left[ S_\beta(T^*T|_{R(T^*)}) \right] T^* = T^\dagger.$$  

**Proof.** Using the Weierstrass Approximation Theorem, we get that the operator $S_\beta(T^*T|_{R(T^*)})$ is selfadjoint on $R(T^*)$ and $S_\beta(TT^*|_{R(T)})$ is selfadjoint on $R(T)$. By Theorem 1.1 we get

$$\lim_{\beta} T^* \left[ S_\beta(TT^*|_{R(T)}) \right] = \lim_{\beta} \left( [S_\beta(TT^*|_{R(T)})] T^* \right)^* = (T^*)^\dagger = T^\dagger$$

$$= \lim_{\beta} [S_\beta(T^*T|_{R(T^*)})] T^*.$$

In the following theorem we obtain a few additional initial conditions for the operators $W_1$ and $W_2$, which produce various subsets of $\{i,j,k\}$ generalized inverses.
Theorem 2.3. Let $A \in B(H_1, H_2)$ has closed range, $A = PQ$ be a full-rank decomposition of $A$, $W_1 \in B(H_3, H_1)$, $W_2 \in B(H_2, H_3)$ and $W = W_2AW_1 \in B(H_3)$.

(a) If $W_2$ is unitary, $QW_1$ is right invertible and $S_\beta$ is a family possessing the properties from Theorem 1.1 with $T = AW_1$, then

$$
\lim_{\beta} W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 = W_1(AW_1)^\dagger \in A\{1, 2, 3\}.
$$

(b) If $W_1$ is unitary, $W_2P$ is left invertible and $S_\beta$ is a family which satisfies conditions of Theorem 1.1 for the operator $T = A^*W_2^*$, then

$$
\lim_{\beta} W_1W^* \left[ S_\beta(\tilde{W}) \right] W_2 = (W_2A)^\dagger W_2 \in A\{1, 2, 4\}.
$$

(c) If both $W_1$ and $W_2$ are unitary and $S_\beta$ has the properties from (a) and (b), then

$$
A^\dagger = \lim_{\beta} W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 = W_1(AW_1)^\dagger = \lim_{\beta} W_1W^* \left[ S_\beta(\tilde{W}) \right] W_2 = (W_2A)^\dagger W_2.
$$

(d) If (a) is valid and $W_1 = Q^*$, then

$$
\lim_{\beta} W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 = Q^*(AQ^*)^\dagger = A^\dagger.
$$

(e) If (b) is valid and $W_2 = P^*$, then

$$
\lim_{\beta} W_1W^* \left[ S_\beta(\tilde{W}) \right] W_2 = (P^*A)^\dagger P^* = A^\dagger.
$$

Proof. (a) The operator $W_2$ is unitary, which implies

$$
W^*W = (AW_1)^*AW_1, \quad W^*W_2 = (AW_1)^*.
$$

Since $W^* = (AW_1)^*W_2^*$ and $W_2$ is invertible, it follows that $\mathcal{R}(W^*) = \mathcal{R}((AW_1)^*)$. Using Theorem 1.1 we obtain

$$
X = \lim_{\beta} W_1 \left[ S_\beta(\tilde{W}) \right] W^*W_2 = W_1(AW_1)^\dagger \in A\{2, 3\}.
$$
We need to prove $W_1(AW_1) \dagger \in A\{1\}$. Note that $X = W_1[P(QW_1)] \dagger$. Now, $P$ is left invertible, and $QW_1$ is right invertible, so $P(QW_1)$ is given as the full-rank factorization. Using the result from Lemma 2.1 (d) or [2], we get $X = W_1(QW_1) \dagger P \dagger$. Now, the equation (1) can be easily verified.

(b) We use (a) with: $A^*$ instead of $A$, $W_2^*$ instead of $W_1$ and $W_1^*$ instead of $W_2$. Note that $W_1^*$ is unitary and $(W_2P)^* = P^*W_2^*$ is right invertible. In this case we have

$$W_1^*W^* = A^*W_2^*, \quad WW^* = W_2A(W_2A)^*, \quad R(W_2A) = R(W)$$

which implies $\tilde{W}^* = (\tilde{W}^*)_2^*$. Using the Weierstrass Approximation Theorem, we get that $S_\beta(W_1A(W_2A)^*|_{R(W_2A)})$ is selfadjoint, so

$$\lim_{\beta}(A^*W_2^*)^* \left[S_\beta(W_2A(W_2A)^*|_{R(W_2A)})\right] W_2 =$$

$$= \lim_{\beta} \left\{W_2^* \left[S_\beta((A^*W_2^*)^*A^*W_2^*|_{R(W_2A)})\right] (A^*W_2^*)^* \right\} = (W_2^*(A^*W_2^*)^*)^*.$$  

By (a) we know that $W_2^*(A^*W_2^*)^* \in A\{1, 2, 3\}$, so

$$(W_2^*(A^*W_2^*)^*)^* = (W_2A)^\dagger W_2 \in A\{1, 2, 4\}.$$  

(c) It is enough to prove that the limits from (a) and (b) are equal. If $W_2$ is unitary, from the proof of (a) we get $\tilde{W} = AW_1$. If $W_1$ is unitary, from the proof of (b) we get $\tilde{W}^* = (\tilde{W}_2A)^*$. Now, by Theorem 2.4, and using the parts (a) and (b) of this proof, we get:

$$W_1(AW_1) \dagger = \lim_{\beta} \left[S_\beta((AW_1)^*AW_1|_{R(AW_1^*)})\right] (AW_1)^*W_2^*W_2 = \lim_{\beta} W_1 \left[S_\beta(W^*W|_{R(W^*)})\right] W^*W_2 = W_1W^\dagger W_2 = \lim_{\beta} W_1W^* \left[S_\beta(W^*W|_{R(W^*)})\right] W_2 = \lim_{\beta} A^*W_2^* \left[S_\beta(W^*W|_{R(W^*)})\right] W_2 =$$

$$= \lim_{\beta} A^*W_1^* S_\beta(W_2A(W_2A)^*|_{R(W_2A)})W_2 = (W_2A)^\dagger W_2. \quad \square$$

Finally, as corollaries, we introduce a few iterative methods for computing reflexive $g$-inverses.
Corollary 2.2. Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ is the full-rank decomposition of $A$ and let $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ be two operators, such that $QW_1$ is right invertible and $W_2P$ is left invertible. Let $W = W_2AW_1$ and $\tilde{W} = W^*W|_{\mathcal{R}(W^*)}$. Then the following representations of the reflexive $g$-inverses are convergent in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$:

(a) $A^{(1,2)} = W_1 \left[ \int_{0}^{\infty} e^{-W^*Wu} W^* W_2 \right]$;

(b) $A^{(1,2)} = \alpha W_1 \sum_{k=0}^{\infty} (I - \alpha W^*W)^k W^*W_2$, where $0 < \alpha < 2\|W\|^{-2}$;

(c) $A^{(1,2)} = W_1 \lim_{t \to 0^+} (tI + W^*W)^{-1} W^*W_2$;

(d) $A^{(1,2)} = W_1 \lim_{t \to 0^+} \sum_{k=0}^{\infty} \frac{1}{k+1} \left( \prod_{j=0}^{k-1} \left( I - \frac{1}{j+1} W^*W \right) \right) W^*W_2$;

(e) $A^{(1,2)} = W_1 \lim_{t \to 0^+} \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+tk)} [I - W^*W]^k W^*W_2$;

(f) $A^{(1,2)} = W_1 \left( W^* + \lim_{t \to 0^+} \sum_{k=1}^{\infty} e^{-tk \log k} [I - W^*W]^k W^* \right) W_2$;

(g) $A^{(1,2)} = W_1 \lim_{t \to 0^+} \sum_{k=0}^{\infty} \frac{\Gamma(1 + (1-t)k)}{\Gamma(1+k)} [I - W^*W]^k W^*W_2$.

Also, as a corollary, we get the next generalization of the main result from [9].

Corollary 2.3 ([9, Lemma 2.1, Theorem 2.1]). Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ is the full-rank decomposition of $A$ and let $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ be two operators, such that $QW_1$ is invertible and $W_2P$ is invertible. Let $W = W_2AW_1$. Then the class of $\{1, 2\}$ inverses of $A$ can be generated by changing the values of the operators $W_1, W_2$ in the following two iterative processes:

$$Y_0 = Y'_0 = \alpha (W_2AW_1)^*, \quad 0 < \alpha \leq 2\|W\|^{-2},$$
\begin{align*}
T_k &= I_X - Y_k W, \\
Y_{k+1} &= (I_X + T_k + \cdots + T_{k+1}^q)Y_k, \\
X_{k+1} &= W_1 Y_{k+1} W_2,
\end{align*}
\begin{align*}
T'_k &= I_Y - WY'_k, \\
Y'_{k+1} &= Y'_k (I_Y + T'_k + \cdots + T'_{k+1}^q), \\
X'_{k+1} &= W_1 Y'_{k+1} W_2, \quad k = 0, 1, \ldots
\end{align*}

Moreover, the following statements are valid:

(a) If $W_2$ is unitary, then $X_k \to X = W_1 (AW_1)^\dagger \in A\{1, 2, 3\}$ as $k \to \infty$.

(b) If $W_1$ is unitary then $X'_k \to X = (W_2 A)^\dagger W_2 \in A\{1, 2, 4\}$ as $k \to \infty$.

(c) If (a) and (b) are valid, then $X_k \to A^\dagger$.

(d) If (a) is valid and $W_1 = Q^*$, then $X_k \to X = A^\dagger$.

(e) If (b) is valid and $W_2 = P^*$, then $X'_k \to X = A^\dagger$.

Remark 2.1. In [6] it is also introduced a modification of the hyper-power method, which generates the class of all $\{1, 2\}$-inverses for operators on Banach spaces. Using the method from [6] for Hilbert space operators, it is not clear how to choose the initial values to get $\{1, 2, 3\}, \{1, 2, 4\}$-inverses. Also, our method is applicable for various classes of $\{S_\beta\}$ families.

References


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