Inner image-kernel (p, q)-inverses in rings

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Abstract

We define and study the inner image-kernel inverse as natural algebraic extension of the inner inverse with prescribed idempotents of elements in rings and the inner inverse of linear operators with prescribed range and kernel. Also, we give applications to perturbation bounds and reverse order laws.

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1 Introduction

Let $\mathcal{R}$ be an associative ring with unit 1. We use $\mathcal{R}^\bullet$ to denote the set of all idempotents of $\mathcal{R}$. For $a \in \mathcal{R}$, we define the following kernel ideals $a^\circ = \{x \in \mathcal{R} : ax = 0\}$, ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$, and image ideals $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$ and $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$.

Let $a \in \mathcal{R}$. An element $b \in \mathcal{R}$ is an inner inverse of a if $aba = a$, and we write $b \in a\{1\}$. In this case a is called inner regular (or relatively regular). If there exists $b \in \mathcal{R}$ such that $0 \neq b = bab$ holds, then we say that b is an outer inverse for a, and write $b \in a\{2\}$. For such an a we say that it is outer regular. If b is both inner and outer inverse of a, then it is a reflexive generalized inverse of a. If b is an inner generalized inverse of a, then bab is a reflexive generalized inverse of a. So, inner regularity implies outer regularity of a.

Recall that inner or outer inverses of a given $a \in \mathcal{R}$ do not necessarily exist. Also, neither inner, outer nor reflexive inverses are unique in general.

The outer inverse is unique if we fix the idempotents ab and ba ([3]): let $p, q \in \mathcal{R}^\bullet$, the (p, q)-outer generalized inverse of a is the unique element

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$b \in \mathcal{R}$ (in the case when it exists) satisfying

$$bab = b, \quad ba = p, \quad 1 - ab = q,$$

In this case, we write $b = a^{(2)}_{p,q}$.

The inner inverse with prescribed idempotents was defined and investigated in [6]. If an inner inverse with prescribed idempotents exists, it is not necessarily unique.

For outer inverses, instead of prescribing the idempotents ab and ba, Kantún-Montiel ([5]) prescribed certain kernel and image ideals related to these idempotents: let $p, q \in \mathcal{R}^\bullet$, an element $b \in \mathcal{R}$ is the image-kernel (p, q)-inverse of a if

$$bab = b, \quad ba\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (1 - ab)\mathcal{R} = q\mathcal{R}.$$

The image-kernel (p, q)-inverse b is unique if it exists ([5]), and it will be denoted by $a^{(2,i,k)}_{p,q}$. Observe that the image-kernel (p, q)-inverse of Kantún-Montiel ([5]) coincides with the (p, q, l)-outer generalized inverse of Cao and Xue ([1]), but his approach is different.

In this paper, we define and study the inner inverse prescribing certain kernel and image ideals in a ring. Beside this, in Section 2, we consider the reverse order rule for new inner inverses of elements of a ring. In Section 3, some properties of new inner inverses are investigated in a ring with involution. In Section 4, a generalization of the condition number in a normed algebra is given and some perturbation bounds are obtained.

Several generalizations of invertibility, such as the Moore-Penrose and the group inverse, are special types of outer inverses or inner inverses. An element $a \in \mathcal{R}$ is group invertible if there is $b \in \mathcal{R}$ such that

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (5)\ ab = ba.$$

Recall that b is uniquely determined by previous equations and it is called the group inverse of a. The group inverse of a will be denoted by $a^\#$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is, $(a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^*$. An element $a \in \mathcal{R}$ is self-adjoint (or Hermitian) if $a^* = a$.

Let $\mathcal{R}$ be a ring with involution. The Moore–Penrose inverse of $a \in \mathcal{R}$ is the element $b \in \mathcal{R}$, if the following equations hold [7]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [7]), and such b is denoted by $a^\dagger$. In [4] it was proved that each inner regular element a in a

C^*-algebra has a Moore-Penrose inverse. In rings with involution the inner regularity is not enough to ensure the existence of a Moore–Penrose inverse.

If $\delta \subset \{1, 2, 3, 4, 5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denoted by $a\{\delta\}$. Obviously, $a\{1, 2, 5\} = \{a^{\#}\}$ and $a\{1, 2, 3, 4\} = \{a^{\dagger}\}$.

For $u, v \in \mathcal{R}^{\bullet}$, notice that $u^{\circ} = (1-u)\mathcal{R}$ and ${}^{\circ}u = \mathcal{R}(1-u)$. Also, we have

$$u\mathcal{R} = v\mathcal{R} \;\Leftrightarrow\; {}^{\circ}u =^{\circ} v$$

and

$$\mathcal{R}u = \mathcal{R}v \;\Leftrightarrow\; u^{\circ} = v^{\circ}.$$

We will use the following auxiliary result.

Lemma 1.1. [8] *Let $c, s \in \mathcal{R}$ satisfy $cs = sc$ and $s \in \mathcal{R}^{\bullet}$. Then c is invertible in $\mathcal{R}$ if and only if cs is invertible in $s\mathcal{R}s$ and $c(1-s)$ is invertible in $(1-s)\mathcal{R}(1-s)$. In this case*

$$c^{-1} = [cs]^{-1}_{s\mathcal{R}s} + [c(1-s)]^{-1}_{(1-s)\mathcal{R}(1-s)}.$$

2 Inner image-kernel (p, q)-inverses in rings

In [6], the authors presented the definition of the inner inverse with prescribed idempotents in rings: let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}^{\bullet}$. An element $b \in \mathcal{R}$ satisfying

$$aba = a, \quad ba = p \quad \text{and} \quad 1 - ab = q \tag{1}$$

is called a (p, q)-inner inverse of a (or an inner inverse of a with prescribed idempotents p and q). If an inner inverse of a with prescribed idempotents exists, it is not necessarily unique [2, 6].

We prescribing the image ideal $ba\mathcal{R}$ and the kernel ideal $(ab)^{\circ} = (1-ab)\mathcal{R}$ in the following sense:

Definition 2.1. Let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}^{\bullet}$. An element $b \in \mathcal{R}$ satisfying

$$(1)\ aba = a, \quad (3p)\ ba\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (4q)\ (1-ab)\mathcal{R} = q\mathcal{R}$$

is called an inner image-kernel (p, q)-inverse of a.

Notice that condition $(1-ab)\mathcal{R} = q\mathcal{R}$ is equivalent to $\mathcal{R}ab = \mathcal{R}(1-q)$.

Since the above conditions involve ideals, we present an equivalent condition for the existence of the inner image-kernel (p, q)-inverse without explicit reference to ideals.

Theorem 2.1. *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *the inner image-kernel (p, q)-inverse $a_{p,q}^{(1,i,k)}$ exists,*

(ii) *there exists some $b \in \mathcal{R}$ such that*

$$aba = a, \quad pba = ba, \quad bap = p, \quad abq = 0, \quad 1 - q = (1 - q)ab.$$

Proof. (i) $\Rightarrow$ (ii): Denote by $b = a_{p,q}^{(1,i,k)}$. From $ba\mathcal{R} = p\mathcal{R}$, we obtain that $ba = pba$ and $p = bap$. Also, by $\mathcal{R}ab = \mathcal{R}(1 - q)$, we deduce that $(1 - q) = (1 - q)ab$ and $ab = ab(1 - q)$ implying $abq = 0$.

(ii) $\Rightarrow$ (i): Observe that

$$ba\mathcal{R} = pba\mathcal{R} \subset p\mathcal{R} = bap\mathcal{R} \subset ba\mathcal{R}$$

and

$$\mathcal{R}(1 - q) = \mathcal{R}(1 - q)ab \subset \mathcal{R}ab = \mathcal{R}ab(1 - q) \subset \mathcal{R}(1 - q).$$

give $ba\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}(1 - q) = \mathcal{R}ab$. Thus, b is the inner image-kernel (p, q)-inverse of a. $\square$

Obviously, if b satisfies (1), then $b \in \ a\{1, 3p, 4q\}$. Conversely, if b is the inner image-kernel (p, q)-inverse of a, then $ba = pba$ and $1 - ab = q(1 - ab)$ implying b is the $(pba, q(1 - ab)$–inner inverse of a.

The following example shows that the image-kernel (p, q)-inner inverse is not the same as the (p, q)-inner inverse.

Example 2.1 Let $M_2(\mathbf{R})$ be the ring of 2×2 matrices with real entries. Suppose that $\alpha, \beta, \gamma \in \mathbf{R} \setminus \{0\}$ and $a, b, p, q \in M_2(\mathbf{R})$ defined by

$$a = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & \alpha^{-1} \\ 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix}.$$

By elementary computations, we prove that $aba = a$, $pba = ba$, $bap = p$, $abq = 0$ and $(1-q)ab = 1-q$. Thus, b is the inner image-kernel (p, q)-inverse of a, but it is not the (p, q)-inner inverse of a, since $ba \neq p$ and $1 - ab \neq q$ $(\beta \neq 0 \neq \gamma)$.

In order to establish the uniqueness, following [6], we require an extra condition: if $r \in \mathcal{R}$, then we consider the equation

$$(5r) \ r = b - bab.$$

Theorem 2.2. *There exists at most one element in the set $a\{1, 3p, 4q, 5r\}$.*

Proof. Let $a', a'' \in a\{1, 3p, 4q, 5r\}$. Using Theorem 2.1, we show that

$$a'a = pa'a = a''apa'a = a''aa'a = a''a$$

and

$$aa' = aa'(1-q) = aa'(1-q)aa'' = aa'aa'' = aa''$$

which yield $a'aa' = a''aa' = a''aa''$. Therefore,

$$a' - a'' = r + a'aa' - (r + a''aa'') = 0.$$

□

We will denote by $a_{p,q,r}^{(1,i,k)}$ the unique element of the set $a\{1, 3p, 4q, 5r\}$, in the case when this set is non-empty.

Lemma 2.1. *Let $a, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a_{p,q,r}^{(1,i,k)}$ exists. Then*

$$r = (1 - a_{p,q,r}^{(1,i,k)}a)a_{p,q,r}^{(1,i,k)}(1 - aa_{p,q,r}^{(1,i,k)}).$$

We will show that if we know one inner image-kernel (p, q)-inverse $a_{p,q,r}^{(1,i,k)}$ of a, then we can describe all of them.

Lemma 2.2. *Let $a, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a_{p,q,r}^{(1,i,k)}$ exists. Then*

$$\begin{aligned} a\{1, 3p, 4q\} &= \{a_{p,q,r}^{(1,i,k)} - r + s \ : \ s = (1 - a_{p,q,r}^{(1,i,k)}a)u(1 - aa_{p,q,r}^{(1,i,k)}), u \in \mathcal{R}\} \\ &= \{a_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)} + s \ : \ s = (1 - a_{p,q,r}^{(1,i,k)}a)u(1 - aa_{p,q,r}^{(1,i,k)}), u \in \mathcal{R}\}. \end{aligned}$$

Proof. If $b \in a\{1, 3p, 4q\}$ and $s = b - bab$, by Theorem 2.1,

$$a_{p,q,r}^{(1,i,k)}abaa_{p,q,r}^{(1,i,k)} = a_{p,q,r}^{(1,i,k)}apbaa_{p,q,r}^{(1,i,k)} = pbaa_{p,q,r}^{(1,i,k)} = baa_{p,q,r}^{(1,i,k)}.$$

So,

$$\begin{aligned} (1 - a_{p,q,r}^{(1,i,k)}a)b(1 - aa_{p,q,r}^{(1,i,k)}) &= b - baa_{p,q,r}^{(1,i,k)} - a_{p,q,r}^{(1,i,k)}ab + a_{p,q,r}^{(1,i,k)}abaa_{p,q,r}^{(1,i,k)} \\ &= b - pa_{p,q,r}^{(1,i,k)}ab = b - bapa_{p,q,r}^{(1,i,k)}ab \\ &= b - bab = s \end{aligned}$$

and

$$a_{p,q,r}^{(1,i,k)} - r + s = a_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)} + s = cac + s = c.$$

Converse, since $s = u - uaa^{(1,i,k)}_{p,q,r} - a^{(1,i,k)}_{p,q,r}au + a^{(1,i,k)}_{p,q,r}auaa^{(1,i,k)}_{p,q,r}$, set $b = a^{(1,i,k)}_{p,q,r} - r + s = a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r} + u - uaa^{(1,i,k)}_{p,q,r} - a^{(1,i,k)}_{p,q,r}au + a^{(1,i,k)}_{p,q,r}auaa^{(1,i,k)}_{p,q,r}$. It follows $ba = a^{(1,i,k)}_{p,q,r}a$ which gives $aba = a$, $bap = a^{(1,i,k)}_{p,q,r}ap = p$ and $pba = pa^{(1,i,k)}_{p,q,r}a = a^{(1,i,k)}_{p,q,r}a = ba$. Also, we get $ab = aa^{(1,i,k)}_{p,q,r}$ implying $(1-q)ab = (1-q)aa^{(1,i,k)}_{p,q,r} = 1-q$ and $ab(1-q) = aa^{(1,i,k)}_{p,q,r}(1-q) = aa^{(1,i,k)}_{p,q,r} = ab$. We can check that $b - bab = a^{(1,i,k)}_{p,q,r}ab - b = u - uaa^{(1,i,k)}_{p,q,r} - a^{(1,i,k)}_{p,q,r}au + a^{(1,i,k)}_{p,q,r}auaa^{(1,i,k)}_{p,q,r} = s$. □

Now we prove one important result.

Theorem 2.3. *Let $a, b, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a^{(1,i,k)}_{p,q,r}$ and $b^{(1,i,k)}_{p,q,r}$ exist. Then*

$$1 + a^{(1,i,k)}_{p,q,r}(b-a),\ 1 + (b-a)a^{(1,i,k)}_{p,q,r} \in \mathcal{R}^{-1}$$

and

$$b^{(1,i,k)}_{p,q,r} = (1 + a^{(1,i,k)}_{p,q,r}(b-a))^{-1}a^{(1,i,k)}_{p,q,r} = a^{(1,i,k)}_{p,q,r}(1 + (b-a)a^{(1,i,k)}_{p,q,r})^{-1}.$$

Proof. Denote by $x = 1 + a^{(1,i,k)}_{p,q,r}b - a^{(1,i,k)}_{p,q,r}a$ and $y = 1 + b^{(1,i,k)}_{p,q,r}a - b^{(1,i,k)}_{p,q,r}b$. Notice that $ra = (a^{(1,i,k)}_{p,q,r} - a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r})a = 0$, $br = b(b^{(1,i,k)}_{p,q,r} - b^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r}) = 0$,

$$\begin{aligned} a^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r} &= (r + a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r})bb^{(1,i,k)}_{p,q,r} = a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}(1-q)bb^{(1,i,k)}_{p,q,r} \\ &= a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}(1-q) = a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}, \end{aligned} \tag{2}$$

$$\begin{aligned} a^{(1,i,k)}_{p,q,r}ab^{(1,i,k)}_{p,q,r} &= a^{(1,i,k)}_{p,q,r}a(r + b^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r}) = a^{(1,i,k)}_{p,q,r}apb^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r} \\ &= pb^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r} = b^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r} \end{aligned} \tag{3}$$

and

$$\begin{aligned} (1 - a^{(1,i,k)}_{p,q,r}a)b^{(1,i,k)}_{p,q,r}a &= (1 - a^{(1,i,k)}_{p,q,r}a)(r + b^{(1,i,k)}_{p,q,r}bb^{(1,i,k)}_{p,q,r})a \\ &= (b^{(1,i,k)}_{p,q,r}b - a^{(1,i,k)}_{p,q,r}ab^{(1,i,k)}_{p,q,r}b)b^{(1,i,k)}_{p,q,r}a \\ &= (b^{(1,i,k)}_{p,q,r}b - a^{(1,i,k)}_{p,q,r}apb^{(1,i,k)}_{p,q,r}b)b^{(1,i,k)}_{p,q,r}a \\ &= (b^{(1,i,k)}_{p,q,r}b - pb^{(1,i,k)}_{p,q,r}b)b^{(1,i,k)}_{p,q,r}a \\ &= (b^{(1,i,k)}_{p,q,r}b - b^{(1,i,k)}_{p,q,r}b)b^{(1,i,k)}_{p,q,r}a = 0, \end{aligned}$$

which imply

$$\begin{aligned} xy &= 1 + b_{p,q,r}^{(1,i,k)}a - b_{p,q,r}^{(1,i,k)}b + a_{p,q,r}^{(1,i,k)}b + a_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)}a - a_{p,q,r}^{(1,i,k)}b \\ &- a_{p,q,r}^{(1,i,k)}a - a_{p,q,r}^{(1,i,k)}ab_{p,q,r}^{(1,i,k)}a + a_{p,q,r}^{(1,i,k)}ab_{p,q,r}^{(1,i,k)}b \\ &= 1 + (1 - a_{p,q,r}^{(1,i,k)}a)b_{p,q,r}^{(1,i,k)}a - b_{p,q,r}^{(1,i,k)}b + a_{p,q,r}^{(1,i,k)}a - a_{p,q,r}^{(1,i,k)}a + b_{p,q,r}^{(1,i,k)}b \\ &= 1. \end{aligned}$$

In the same way, we obtain $yx = 1$. Hence, $x = 1 + a_{p,q,r}^{(1,i,k)}(b - a) \in \mathcal{R}^{-1}$ and $x^{-1} = y$. As the equalities (2) and (3), we can prove $b_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)} = b_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)}$ and $b_{p,q,r}^{(1,i,k)}ba_{p,q,r}^{(1,i,k)} = a_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)}$ which give

$$\begin{aligned} x^{-1}a_{p,q,r}^{(1,i,k)} &= (1 + b_{p,q,r}^{(1,i,k)}a - b_{p,q,r}^{(1,i,k)}b)a_{p,q,r}^{(1,i,k)} \\ &= a_{p,q,r}^{(1,i,k)} + b_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)} - b_{p,q,r}^{(1,i,k)}ba_{p,q,r}^{(1,i,k)} \\ &= a_{p,q,r}^{(1,i,k)} + b_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)} - a_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)} \\ &= r + b_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)} = b_{p,q,r}^{(1,i,k)}. \end{aligned}$$

Similarly, we verify that $(1 + (b - a)a_{p,q,r}^{(1,i,k)})^{-1} = 1 + (a - b)b_{p,q,r}^{(1,i,k)}$ and $b_{p,q,r}^{(1,i,k)} = a_{p,q,r}^{(1,i,k)}(1 + (b - a)a_{p,q,r}^{(1,i,k)})^{-1}$. □

We also prove the following result.

Theorem 2.4. *Let $a, b, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a_{p,q,r}^{(1,i,k)}$ and $b_{p,q,r}^{(1,i,k)}$ exist. Then*

$$1 - p + a_{p,q,r}^{(1,i,k)}bp,\ q + (1 - q)ba_{p,q,r}^{(1,i,k)} \in \mathcal{R}^{-1},$$

$$(1 - p + a_{p,q,r}^{(1,i,k)}bp)^{-1} = 1 - p + b_{p,q,r}^{(1,i,k)}ap$$

and

$$(q + (1 - q)ba_{p,q,r}^{(1,i,k)})^{-1} = q + (1 - q)ab_{p,q,r}^{(1,i,k)}.$$

Proof. Set $x = 1 - p + a_{p,q,r}^{(1,i,k)}bp$ and $y = 1 - p + b_{p,q,r}^{(1,i,k)}ap$. From

$$\begin{aligned} pb_{p,q,r}^{(1,i,k)}a &= p(r + b_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)})a = pb_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)}a \\ &= b_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)}a = (b_{p,q,r}^{(1,i,k)} - r)a = b_{p,q,r}^{(1,i,k)}a \end{aligned} \tag{4}$$

and (2), we get

$$\begin{aligned} xy &= 1 - p + b_{p,q,r}^{(1,i,k)}ap - pb_{p,q,r}^{(1,i,k)}ap + a_{p,q,r}^{(1,i,k)}bpb_{p,q,r}^{(1,i,k)}ap \\ &= 1 - p + a_{p,q,r}^{(1,i,k)}bb_{p,q,r}^{(1,i,k)}ap \\ &= 1 - p + a_{p,q,r}^{(1,i,k)}aa_{p,q,r}^{(1,i,k)}ap \\ &= 1. \end{aligned}$$

In the similar manner, we show that $yx = 1$. So, $x \in \mathcal{R}^{-1}$ and $x^{-1} = y$. The rest of the proof follows as the first part of proof. $\square$

Observe that we do not assume that $b_{p,q,r}^{(1,i,k)}$ exists in the next theorem.

Theorem 2.5. *Let $a, b, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a_{p,q,r}^{(1,i,k)}$ exists. If $1 - p + a_{p,q,r}^{(1,i,k)} bp \in \mathcal{R}^{-1}$ and $rb = 0 = br$, then*

(i) $r + (1 - p + a_{p,q,r}^{(1,i,k)} bp)^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} \in b\{3p, 4q, 5r\}$,

(ii) $(bp)_{p,q}^{(1,2,i,k)} = (1 - p + a_{p,q,r}^{(1,i,k)} bp)^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} = (a a_{p,q,r}^{(1,i,k)} b)_{p,q}^{(1,2,i,k)}$,

(iii) $(a_{p,q,r}^{(1,i,k)} b)_{p,1-a_{p,q,r}^{(1,i,k)} a}^{(1,2,i,k)} = (1 - p + a_{p,q,r}^{(1,i,k)} bp)^{-1} a_{p,q,r}^{(1,i,k)} a$.

Proof. We can prove that $p a_{p,q,r}^{(1,i,k)} b = a_{p,q,r}^{(1,i,k)} b$ similarly as (4). By Lemma 1.1, since $x = 1 - p + a_{p,q,r}^{(1,i,k)} bp = 1 - p + p a_{p,q,r}^{(1,i,k)} bp$ is invertible and $px = a_{p,q,r}^{(1,i,k)} bp = xp$, we deduce that

$$x^{-1} = 1 - p + (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1}.$$

(i) Denote by $y = r + x^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} = r + (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)}$. From $(a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} = p (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1}$, we get

$$\begin{aligned} yb &= (r + (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)}) b = (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} (a_{p,q,r}^{(1,i,k)} - r) b \\ &= (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} b = p (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} b \end{aligned}$$

which gives $yb\mathcal{R} \subset p\mathcal{R}$. By $p = (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} bp = ybp$, we have $p\mathcal{R} \subset yb\mathcal{R}$. So, $yb\mathcal{R} = p\mathcal{R}$.

The equalities

$$by = b (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} = b (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} (1 - q)$$

and

$$\begin{aligned} (1 - q) &= (1 - q) a a_{p,q,r}^{(1,i,k)} = (1 - q) a p a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} \\ &= (1 - q) a a_{p,q,r}^{(1,i,k)} bp (a_{p,q,r}^{(1,i,k)} bp)_{p\mathcal{R}p}^{-1} a_{p,q,r}^{(1,i,k)} a a_{p,q,r}^{(1,i,k)} \\ &= (1 - q) by \end{aligned}$$

yield $\mathcal{R}by = \mathcal{R}(1 - q)$.

Also,

$$\begin{aligned} yby &= (a^{(1,i,k)}_{p,q,r}bp)^{-1}_{p\mathcal{R}p}a^{(1,i,k)}_{p,q,r}by \\ &= (a^{(1,i,k)}_{p,q,r}bp)^{-1}_{p\mathcal{R}p}a^{(1,i,k)}_{p,q,r}b(r+(a^{(1,i,k)}_{p,q,r}bp)^{-1}_{p\mathcal{R}p}a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}) \\ &= (a^{(1,i,k)}_{p,q,r}bp)^{-1}_{p\mathcal{R}p}a^{(1,i,k)}_{p,q,r}bp(a^{(1,i,k)}_{p,q,r}bp)^{-1}_{p\mathcal{R}p}a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r} \\ &= (a^{(1,i,k)}_{p,q,r}bp)^{-1}_{p\mathcal{R}p}a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r} \end{aligned}$$

implies $y - yby = r$.

In the same manner as (i), we can verify parts (ii) and (iii). □

Similarly as Theorem 2.5, we can check the next result.

Theorem 2.6. *Let $a, b, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a^{(1,i,k)}_{p,q,r}$ exists. If $q + (1-q)ba^{(1,i,k)}_{p,q,r} \in \mathcal{R}^{-1}$ and $rb = 0 = br$, then*

(i) $r + a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}(q + (1-q)ba^{(1,i,k)}_{p,q,r})^{-1} \in b\{3p, 4q, 5r\}$,

(ii) $[(1-q)b]^{(1,2,i,k)}_{p,q} = a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}(q+(1-q)ba^{(1,i,k)}_{p,q,r})^{-1} = (ba^{(1,i,k)}_{p,q,r}a)^{(1,2,i,k)}_{p,q}$,

(iii) $(ba^{(1,i,k)}_{p,q,r})^{(1,2,i,k)}_{aa^{(1,i,k)}_{p,q,r},q} = aa^{(1,i,k)}_{p,q,r}(q + (1-q)ba^{(1,i,k)}_{p,q,r})^{-1}$.

Applying Theorem 2.4, Theorem 2.5 and Theorem 2.6, we get the following result concerning some reverse order laws for inner image-kernel inverses as a consequence.

Corollary 2.1. *Let $a, b, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a^{(1,i,k)}_{p,q,r}$ and $b^{(1,i,k)}_{p,q,r}$ exist. Then*

(i) $(bp)^{(1,2,i,k)}_{p,q} = b^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r} = (aa^{(1,i,k)}_{p,q,r}b)^{(1,2,i,k)}_{p,q}$,

(ii) $(a^{(1,i,k)}_{p,q,r}b)^{(1,2,i,k)}_{p,1-a^{(1,i,k)}_{p,q,r}a} = b^{(1,i,k)}_{p,q,r}a$.

(iii) $[(1-q)b]^{(1,2,i,k)}_{p,q} = a^{(1,i,k)}_{p,q,r}ab^{(1,i,k)}_{p,q,r} = (ba^{(1,i,k)}_{p,q,r}a)^{(1,2,i,k)}_{p,q}$,

(iv) $(ba^{(1,i,k)}_{p,q,r})^{(1,2,i,k)}_{aa^{(1,i,k)}_{p,q,r},q} = ab^{(1,i,k)}_{p,q,r}$.

Proof. We only prove the part (i), because parts (ii)–(iv) follow similarly.

By Theorem 2.5 and Theorem 2.4, we obtain

$$\begin{aligned} (bp)^{(1,2,i,k)}_{p,q} &= (1-p+a^{(1,i,k)}_{p,q,r}bp)^{-1}a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r} \\ &= (1-p+b^{(1,i,k)}_{p,q,r}ap)a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r} = b^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}. \end{aligned}$$

Also, we conclude that $(aa^{(1,i,k)}_{p,q,r}b)^{(1,2,i,k)}_{p,q} = b^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r}$.

□

More results related to the reverse order law in a ring are presented in the next theorems.

Theorem 2.7. *Let $a, b \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a\{1, 3p\} \neq \emptyset$ and $b\{1, 4q\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $b'a' \in (ab)\{1\}$ for some $a' \in a\{1, 3p\}$ and for some $b' \in b\{1, 4q\}$,

(ii) $b'a' \in (ab)\{1\}$ for all $a' \in a\{1, 3p\}$ and for all $b' \in b\{1, 4q\}$.

Proof. (i) $\Rightarrow$ (ii): Assume that $abb'a'ab = ab$ for some $a' \in a\{1, 3p\}$ and for some $b' \in b\{1, 4q\}$. Let $a'' \in a\{1, 3p\}$ and $b'' \in b\{1, 4q\}$ be arbitrary. Then

$$bb'' = bb''(1 - q) = bb''(1 - q)bb' = bb''bb' = bb' \tag{5}$$

and

$$a''a = pa''a = a'apa''a = a'aa''a = a'a. \tag{6}$$

Therefore,

$$abb''a''ab = abb'a'ab = ab,$$

i.e. $b''a'' \in (ab)\{1\}$.

(ii) $\Rightarrow$ (i): It follows obviously. $\square$

Theorem 2.8. *Let $a, b \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $a\{1, 2, 3p\} \neq \emptyset$ and $b\{1, 2, 4q\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $b'a' \in (ab)\{1, 2\}$ for some $a' \in a\{1, 2, 3p\}$ and for some $b' \in b\{1, 2, 4q\}$,

(ii) $b'a' \in (ab)\{1, 2\}$ for all $a' \in a\{1, 2, 3p\}$ and for all $b' \in b\{1, 2, 4q\}$.

Proof. (i) $\Rightarrow$ (ii): Let $abb'a'ab = ab$ and $b'a'abb'a' = b'a'$ for some $a' \in a\{1, 2, 3p\}$ and for some $b' \in b\{1, 2, 4q\}$. If $a'' \in a\{1, 2, 3p\}$ and $b'' \in b\{1, 2, 4q\}$ are arbitrary, then the equalities (5) and (6) hold. So, $b'' = b''bb'' = b''bb'$ and $a'' = a''aa'' = a'aa''$. Now, we get

$$b''a''abb''a'' = b''bb'a'aa''abb''bb'a'aa'' = b''bb'a'abb'a'aa'' = b''bb'a'aa'' = b''a''.$$

Hence, $b''a'' \in (ab)\{2\}$. By Theorem 2.7, $b''a'' \in (ab)\{1\}$.

(ii) $\Rightarrow$ (i): This is trivial. $\square$

Theorem 2.9. *Let $a, b \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that there exist $\bar{a} \in a\{1, 3p\}$ and $\bar{b} \in b\{1, 4q\}$. Then the following statements are equivalent:*

(i) $b'a' \in (ab)\{1, 2\}$ for some $a' \in a\{2, 3p\}$ and for some $b' \in b\{2, 4q\}$,

(ii) $b'a' \in (ab)\{1,2\}$ *for all* $a' \in a\{2,3p\}$ *and for all* $b' \in b\{2,4q\}$.

Proof. (i) $\Rightarrow$ (ii): Suppose that, for some $a' \in a\{2,3p\}$ and for some $b' \in b\{2,4q\}$, $abb'a'ab = ab$ and $b'a'abb'a' = b'a'$. Notice that $b\bar{b} = b\bar{b}(1-q) = b\bar{b}(1-q)bb' = b\bar{b}bb' = bb'$ and $\bar{a}a = p\bar{a}a = a'ap\bar{a}a = a'a\bar{a}a = a'a$. Also, for arbitrary $a'' \in a\{2,3p\}$ and $b'' \in b\{2,4q\}$, we have $b\bar{b} = bb''$, $\bar{a}a = a''a$, $b'' = b''bb'' = b''b\bar{b}$ and $a'' = a''aa'' = \bar{a}aa''$. Then, we obtain

$$abb''a''a = ab\bar{b}\bar{a}ab = abb'a'ab = ab$$

and

$$b''a''abb''a'' = b''b\bar{b}\bar{a}abb''b\bar{b}\bar{a}aa'' = b''bb'a'abb'a'aa'' = b''bb'a'aa'' = b''a''.$$

So, $b''a'' \in (ab)\{1,2\}$.

(ii) $\Rightarrow$ (i): Obviously. $\square$

3 Inner image-kernel (p,q)-inverses in rings with involution

Let $\mathcal{R}$ be a ring with involution. An element $p \in \mathcal{R}^\bullet$ is called projection if $p^* = p$.

An element $a \in \mathcal{R}$ is $*$-cancellable if

$$a^*ax = 0 \Rightarrow ax = 0 \quad \text{and} \quad xaa^* = 0 \Rightarrow xa = 0. \tag{7}$$

Applying the involution to (7), we observe that a is $*$-cancellable if and only if a^* is $*$-cancellable. In C^*-algebras all elements are $*$-cancellable.

Lemma 3.1. *Let* $a, r \in \mathcal{R}$ *and* $p, q \in \mathcal{R}$ *be projections such that* $a^{(1,i,k)}_{p,q,r}$ *exists. Then* $(a^{(1,i,k)}_{p,q,r})^* \in a^*\{1, 3(1-q), 4(1-p), 5r^*\}$, *that is* $(a^{(1,i,k)}_{p,q,r})^* = (a^*)^{(1,i,k)}_{1-q,1-p,r^*}$.

Proof. First, observe that $a^* = (aa^{(1,i,k)}_{p,q,r}a)^* = a^*(a^{(1,i,k)}_{p,q,r})^*a^*$ and $r^* = (a^{(1,i,k)}_{p,q,r} - a^{(1,i,k)}_{p,q,r}aa^{(1,i,k)}_{p,q,r})^* = (a^{(1,i,k)}_{p,q,r})^* - (a^{(1,i,k)}_{p,q,r})^*a^*(a^{(1,i,k)}_{p,q,r})^*$. Further, we deduce that $a^{(1,i,k)}_{p,q,r}a\mathcal{R} = p\mathcal{R}$ is equivalent to $\mathcal{R}a^*(a^{(1,i,k)}_{p,q,r})^* = \mathcal{R}p^* = \mathcal{R}p$. Also, $\mathcal{R}aa^{(1,i,k)}_{p,q,r} = \mathcal{R}(1-q)$ is equivalent to $(a^{(1,i,k)}_{p,q,r})^*a^*\mathcal{R} = (1-q)\mathcal{R}$. Thus, $(a^{(1,i,k)}_{p,q,r})^* = (a^*)^{(1,i,k)}_{1-q,1-p,r^*}$. $\square$

The following result holds.

Theorem 3.1. *Let $a, a' \in \mathcal{R}$ and $p \in \mathcal{R}$ be projection such that $a'ap = p$. Then $(a')^*pa' \in (apa^*)\{1,2\}$. Moreover, $apa' = (apa')^*$ if and only if $(apa^*)^\# = (a')^*pa' = (apa^*)^\dagger$.*

Proof. We have $(a')^*pa' \in (apa^*)\{1,2\}$, by

$$apa^*(a')^*pa'apa^* = a(a'ap)^*p^2a^* = ap^*pa^* = apa^*$$

and

$$(a')^*pa'apa^*(a')^*pa' = (a')^*p(a'ap)^*pa' = (a')^*pa'.$$

Since $apa^*(a')^*pa' = apa'$ and $(a')^*pa'apa^* = (a')^*pa^* = (apa')^*$, we conclude that $(a')^*pa' \in (apa^*)\{3,4,5\}$ if and only if $apa' = (apa')^*$. $\square$

Similarly to Theorem 3.1, we can prove the next result.

Theorem 3.2. *Let $a, a' \in \mathcal{R}$ and $q \in \mathcal{R}$ be projection such that $(1-q)aa' = 1-q$. Then $a'(1-q)(a')^* \in [a^*(1-q)a]\{1,2\}$. Moreover, $a'(1-q)a = [a'(1-q)a]^*$ if and only if $[a^*(1-q)a]^\# = a'(1-q)(a')^* = [a^*(1-q)a]^\dagger$.*

Using Theorem 3.1 and Theorem 3.2, we obtain the following corollary.

Corollary 3.1. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be projections such that $a_{p,q}^{(1,i,k)}$ exists. Then*

$$apa_{p,q}^{(1,i,k)} = (apa_{p,q}^{(1,i,k)})^* \quad \Leftrightarrow \quad (apa^*)^\# = (a_{p,q}^{(1,i,k)})^*pa_{p,q}^{(1,i,k)} = (apa^*)^\dagger$$

and

$$a_{p,q}^{(1,i,k)}(1-q)a = [a_{p,q}^{(1,i,k)}(1-q)a]^* \quad \Leftrightarrow \quad [a^*(1-q)a]^\# = a_{p,q}^{(1,i,k)}(1-q)(a_{p,q}^{(1,i,k)})^* = [a^*(1-q)a]^\dagger.$$

In the following theorem, we consider expressions for the inner image-kernel inverses of a^*a and aa^* under corresponding conditions.

Theorem 3.3. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be projections such that $a_{p,q}^{(1,i,k)}$ exists. Then the following statements hold:*

(i) if $(aa_{p,q}^{(1,i,k)})^* = aa_{p,q}^{(1,i,k)}$, then $(a^*a)_{p,1-p}^{(1,i,k)} = a_{p,q}^{(1,i,k)}(a_{p,q}^{(1,i,k)})^*$;

(ii) if $(a_{p,q}^{(1,i,k)}a)^* = a_{p,q}^{(1,i,k)}a$, then $(aa^*)_{1-q,q}^{(1,i,k)} = (a_{p,q}^{(1,i,k)})^*a_{p,q}^{(1,i,k)}$;

(iii) if $(aa_{p,q}^{(1,i,k)})^* = aa_{p,q}^{(1,i,k)}$ and $(a_{p,q}^{(1,i,k)}a)^* = a_{p,q}^{(1,i,k)}a$, then a is $*$-cancelable.

Proof. (i) The condition $a_{p,q}^{(1,i,k)}a\mathcal{R} = p\mathcal{R}$ is equivalent to $\mathcal{R}(a_{p,q}^{(1,i,k)}a)^* = \mathcal{R}p^* = \mathcal{R}p$. Since $(aa_{p,q}^{(1,i,k)})^* = aa_{p,q}^{(1,i,k)}$, then

$$a^*aa_{p,q}^{(1,i,k)}(a_{p,q}^{(1,i,k)})^*a^*a = a^*aa_{p,q}^{(1,i,k)}(aa_{p,q}^{(1,i,k)})^*a = a^*aa_{p,q}^{(1,i,k)}aa_{p,q}^{(1,i,k)}a = a^*a,$$

$$a_{p,q}^{(1,i,k)}(a_{p,q}^{(1,i,k)})^*a^*a\mathcal{R} = a_{p,q}^{(1,i,k)}(aa_{p,q}^{(1,i,k)})^*a\mathcal{R} = a_{p,q}^{(1,i,k)}a\mathcal{R} = p\mathcal{R},$$

$$\begin{aligned}\mathcal{R}a^*aa_{p,q}^{(1,i,k)}(a_{p,q}^{(1,i,k)})^* &= \mathcal{R}a^*(aa_{p,q}^{(1,i,k)})^*(a_{p,q}^{(1,i,k)})^* = \mathcal{R}(a_{p,q}^{(1,i,k)}aa_{p,q}^{(1,i,k)}a)^* \\ &= \mathcal{R}(a_{p,q}^{(1,i,k)}a)^* = \mathcal{R}p.\end{aligned}$$

Thus, we conclude that $(a^*a)_{p,1-p}^{(1,i,k)} = a_{p,q}^{(1,i,k)}(a_{p,q}^{(1,i,k)})^*$.

(ii) Similarly as part (i), we can prove this part.

(iii) Suppose that $a^*ax = 0$. Then

$$ax = aa_{p,q}^{(1,i,k)}ax = (aa_{p,q}^{(1,i,k)})^*ax = (a_{p,q}^{(1,i,k)})^*a^*ax = 0.$$

If $xaa^* = 0$, we get

$$xa = xaa_{p,q}^{(1,i,k)}a = xa(a_{p,q}^{(1,i,k)}a)^* = xaa^*(a_{p,q}^{(1,i,k)})^* = 0.$$

So, a is $*$-cancelable. □

In some way, the next result is converse to Theorem 3.3.

Theorem 3.4. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be projections. If a is $*$-cancelable and there exist the inner image-kernel $(p, 1-p)$-inverse of a^*a and the inner image-kernel $(1-q, q)$-inverse of aa^*, then there exists an inner image-kernel (p, q)-inverse of a. In addition, any inner image-kernel (p, q)-inverse of a is given with*

$$a_{p,q}^{(1,i,k)} = ba^*aa^*c,$$

where b and c are the inner image-kernel $(p, 1-p)$-inverse of a^*a and the inner image-kernel $(1-q, q)$-inverse of aa^*, respectively.

Proof. Assume that $b \in (a^*a)\{1, 3p, 4(1-p)\}$ and $c \in (aa^*)\{1, 3(1-q), 4q\}$. Set $z = ba^*aa^*c$ and $x = (za-1)a^*$. Since a is $*$-cancelable, from

$$a^*ax = a^*aba^*aa^*caa^* - a^*aa^* = a^*aa^*caa^* - a^*aa^* = a^*aa^* - a^*aa^* = 0,$$

we get $ax = 0$. Hence, $(az-1)aa^* = 0$ implying $(az-1)a = 0$, that is $aza = a$.

We have

$$(z-ba^*)aa^* = ba^*aa^*caa^* - ba^*aa^* = ba^*aa^* - ba^*aa^* = 0$$

which gives $(z-ba^*)a = 0$, i.e. $za = ba^*a$. Thus, $za\mathcal{R} = ba^*a\mathcal{R} = p\mathcal{R}$.

Observe that

$$a^*a(z-a^*c) = a^*aa^*c - a^*aa^*c = 0$$

yields $a(z-a^*c) = 0$ implying $\mathcal{R}az = \mathcal{R}aa^*c = \mathcal{R}q$. So, $z = a^{(1,i,k)}_{p,q}$. □

4 Inner image-kernel (p,q)-inverses in Banach algebras

Let $\mathcal{R}$ ba a normed algebra. If $a^{(1,i,k)}_{p,q,r}$ and $b^{(1,i,k)}_{p,q,r}$ exist, then $a = b-u = b-(b-a)$ is called the (p,q,r)-splitting of a.

The condition number of a is related to the sensitivity of the equation $ax = b$ to perturbation of a. If a is invertible, the condition number of a is defined by $k(a) = \|a\|\|a^{-1}\|$. If a is not invertible, we use the generalized condition number.

The generalized condition number $k_{p,q,r}(a)$ of a is defined with $k_{p,q,r}(a) = \|a\|\|a^{(1,i,k)}_{p,q,r}\|$.

Now, we can prove the following result.

Theorem 4.1. *Let $a,b,r\in\mathcal{R}$ and $p,q\in\mathcal{R}^\bullet$ be such that $a^{(1,i,k)}_{p,q,r}$ and $b^{(1,i,k)}_{p,q,r}$ exist. Then the following statements hold:*

(i) $a^{(1,i,k)}_{p,q,r} - b^{(1,i,k)}_{p,q,r} = b^{(1,i,k)}_{p,q,r}(b-a)a^{(1,i,k)}_{p,q,r} = a^{(1,i,k)}_{p,q,r}(b-a)b^{(1,i,k)}_{p,q,r}$,

(ii) If $\mathcal{R}$ is a Banach algebra and $\|a^{(1,i,k)}_{p,q,r}(b-a)\| < 1$, then

$$\begin{aligned}\frac{\|a^{(1,i,k)}_{p,q,r}(b-a)a^{(1,i,k)}_{p,q,r}a\|}{k_{p,q,r}(a)(1+\|a^{(1,i,k)}_{p,q,r}\|\|b-a\|)} &\le \frac{\|b^{(1,i,k)}_{p,q,r} - a^{(1,i,k)}_{p,q,r}\|}{\|a^{(1,i,k)}_{p,q,r}\|}\\ &\le \frac{\|a^{(1,i,k)}_{p,q,r}(b-a)\|}{1-\|a^{(1,i,k)}_{p,q,r}(b-a)\|} \le \frac{k_{p,q,r}(a)\|b-a\|/\|a\|}{1-k_{p,q,r}(a)\|b-a\|/\|a\|}.\end{aligned}$$

(iii) If $\mathcal{R}$ is a normed algebra and $\|a^{(1,i,k)}_{p,q,r}(b-a)\| < 1$, then

$$\frac{\|a^{(1,i,k)}_{p,q,r}\|}{1+\|a^{(1,i,k)}_{p,q,r}(b-a)\|} \le \|b^{(1,i,k)}_{p,q,r}\| \le \frac{\|a^{(1,i,k)}_{p,q,r}\|}{1-\|a^{(1,i,k)}_{p,q,r}(b-a)\|}.$$

Proof. (i) Using Theorem 2.3, we check this part.

(ii) The part (i) implies

$$\begin{aligned} a_{p,q,r}^{(1,i,k)}(b-a)a_{p,q,r}^{(1,i,k)}a &= a_{p,q,r}^{(1,i,k)}(b-a)a_{p,q,r}^{(1,i,k)}(1+(b-a)a_{p,q,r}^{(1,i,k)})^{-1} \\ &\times (1+(b-a)a_{p,q,r}^{(1,i,k)})a \\ &= a_{p,q,r}^{(1,i,k)}(b-a)b_{p,q,r}^{(1,i,k)}(1+(b-a)a_{p,q,r}^{(1,i,k)})a \\ &= (a_{p,q,r}^{(1,i,k)}-b_{p,q,r}^{(1,i,k)})(1+(b-a)a_{p,q,r}^{(1,i,k)})a. \end{aligned}$$

Consequently, the first inequality holds.

By Theorem 2.3, we obtain

$$\begin{aligned} b_{p,q,r}^{(1,i,k)}-a_{p,q,r}^{(1,i,k)} &= (1+a_{p,q,r}^{(1,i,k)}(b-a))^{-1}a_{p,q,r}^{(1,i,k)}-a_{p,q,r}^{(1,i,k)} \\ &= \left(\sum_{n=0}^{\infty}(-1)^n(a_{p,q,r}^{(1,i,k)}(b-a))^n-1\right)a_{p,q,r}^{(1,i,k)} \\ &= \sum_{n=1}^{\infty}(-1)^n(a_{p,q,r}^{(1,i,k)}(b-a))^n a_{p,q,r}^{(1,i,k)} \end{aligned}$$

implying the second and the third inequalities.

(iii) Obviously. □

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References

[1] J. Cao, Y. Xue, *The characterizations and representations for the generalized inverses with prescribed idempotents in Banach algebras*, Filomat 27(5) (2013) 851-863.

[2] D. S. Djordjević, P.S. Stanimirović, *Splittings of operators and generalized inverses*, Publ. Math Debrecen 59 (2001), 147159.

[3] D. S. Djordjević, Y. Wei, *Outer generalized inverses in rings*, Comm. Algebra 33 (2005), 3051–3060.

[4] R.E. Harte, M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. 103 (1992), 71-77.

[5] G. Kantun-Montiel, *Outer generalized inverses with prescribed ideals*, Linear and Multilinear Algebra, DOI:10.1080/03081087.2013.816302.

[6] B. Načevska, D. S. Djordjević, *Inner Generalized Inverses with Prescribed Idempotents*, Communications in Algebra 39:2 (2011), 634-646.

[7] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

[8] V. Rakočević, *Functional analysis*, (in Serbian), Naučna knjiga, Belgrade, 1994.

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