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Core inverse and core partial order of Hilbert space operators ☆

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ABSTRACT

The core inverse of matrix is generalized inverse which is in some sense in-between the group and Moore–Penrose inverse. In this paper a generalization of core inverse and core partial order to Hilbert space operator case is presented. Some properties are generalized and some new ones are proved. Connections with other generalized inverses are obtained. The useful matrix representations of operator and its core inverse are given. It is shown that A is less than B under the core partial order if and only if they have specific kind of simultaneous diagonalization induced by appropriate decompositions of Hilbert space. The relation is also characterized by the inclusion of appropriate sets of generalized inverses. The spectral properties of core inverse are also obtained.

1. An introduction

The core inverse and core partial order for complex matrices of index one were recently introduced in [2] by Baksalary and Trenker. The core inverse is in some way in-between the group and Moore–Penrose inverse as well as the core partial order is in-between the sharp and star partial orders. A matrix $A^{\textcircled{\#}} \in \mathbb{C}^{n\times n}$ is core inverse of $A \in \mathbb{C}^{n\times n}$ if $AA^{\textcircled{\#}} = P_A$ and $\mathcal{R}(A^{\textcircled{\#}}) \subseteq \mathcal{R}(A)$, where $\mathcal{R}(A)$ is range of A and P_A is orthogonal projector onto $\mathcal{R}(A)$. We write $A <^{\textcircled{\#}} B$ if $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$ and $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$. It is showed in [2] that for every matrix $A \in \mathbb{C}^{n\times n}$ of index one and rank r there exist unitary matrix $U \in \mathbb{C}^{n\times n}$, diagonal matrix $\Sigma \in \mathbb{C}^{r\times r}$ of singular values of A and matrices $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ such that $KK^* + LL^* = I_r$ and

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^* \quad \text{and} \quad A^{\textcircled{\#}} = U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{1}$$

Also, $A \leq^{\textcircled{\#}} B$ if and only if

$$B = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix}U^*, \tag{2}$$

where $Z \in \mathbb{C}^{(n-r)\times(n-r)}$ is some matrix of index one. Using the above representations many properties of core inverse and core partial order are derived.

Our aim is to define an inverse of an Hilbert space bounded operator which coincides with core inverse in the finite dimensional case. In Theorem 3.1 we have shown that $X \in \mathbb{C}^{n\times n}$ is core inverse of $A \in \mathbb{C}^{n\times n}$ if and only if

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$AXA = A,\ \mathcal{R}(X) = \mathcal{R}(A)$ and $\mathcal{N}(X) = \mathcal{N}(A^*)$. This equivalent characterization serves us as definition of core inverse in Hilbert space settings. In Theorem 3.2 we have shown that $A \in \mathcal{L}(H)$ has core inverse if and only if index of A is less or equal one in which case $A_1 = A|_{\mathcal{R}(A)} : \mathcal{R}(A) \mapsto \mathcal{R}(A)$ is invertible and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \quad \text{and}$$
$$A^{\textcircled{\#}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Using these representations we give a number of properties of core inverse. In Theorem 3.5, we characterize the core inverse of $A \in \mathcal{L}(H)$ by the equations: $AXA = A,\ XAX = X,\ (AX)^* = AX,\ XA^2 = A$ and $AX^2 = X$. With assumption $\mathrm{ind}(A) \leqslant 1$ these equations reduce to $XAX = X,\ (AX)^* = AX$ and $XA^2 = A$ and the latter ones characterized core inverse in finite dimensional case. We have shown that A is EP if and only if any two elements of the set $\{A^\sharp, A^\dagger, A^{\textcircled{\#}}, A_{\textcircled{\#}}\}$ are equal.

In Theorem 5.3 it is proved that $A <^{\textcircled{\#}} B$ if and only if

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(BB^\sharp - AA^\sharp) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$
$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(BB^\sharp - AA^\sharp) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where A_1 and B_1 are invertible operators and $\mathcal{R}(B) = \mathcal{R}(A) \oplus^\perp \mathcal{R}(B-A)$.

In Theorem 5.5 it is shown that $A <^{\textcircled{\#}} B$ if and only if $(AX)^* = AX$ and $XA^2 = A$ for any X satisfying $(BX)^* = BX$ and $XB^2 = B$. Compared to representations (1) and (2), our representations have more zeros and all nonzero entries are invertible. Because of that our proofs are simpler.

It should be noted that, although we deal with Hilbert space operators, many of the presented results are new when they are considered in finite dimensional setting. As the finite dimensional linear algebra techniques are not suitable for our work, we use geometric approach instead, that is, we use decompositions of the space induced by the characteristic features of the core inverse and core partial order.

2. Preliminaries

Let H and K be Hilbert spaces, and let $\mathcal{L}(H,K)$ denote the set of all bounded linear operators from H to K; we abbreviate $\mathcal{L}(H,H) = \mathcal{L}(H)$. For $A \in \mathcal{L}(H,K)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A.

Throughout the paper, we will denote direct sum of subspaces by $\oplus$, and orthogonal direct sum by $\oplus^\perp$. Orthogonal direct sum $H_1 \oplus^\perp H_2 \oplus^\perp H_3$ means that $H_i \perp H_j$, for $i \neq j$. An operator $P \in \mathcal{L}(H)$ is projector if $P^2 = P$. A projector P is orthogonal if $P = P^*$. If $H = K \oplus L$ then $P_{K,L}$ denotes projector such that $\mathcal{R}(P_{K,L}) = K$ and $\mathcal{N}(P_{K,L}) = L$. If $H = K \oplus^\perp L$ then we write P_K instead of $P_{K,L}$.

An operator $B \in \mathcal{L}(K,H)$ is an inner inverse of $A \in \mathcal{L}(H,K)$, if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in K. If $BAB = B$ holds, then B is reflexive generalized inverse of A. If $ABA = A$ it is easy to see that $\mathcal{R}(A) = \mathcal{R}(AB)$ and $\mathcal{N}(A) = \mathcal{N}(BA)$ and we will often use these properties. The Moore–Penrose inverse of $A \in \mathcal{L}(H,K)$ is the operator $B \in \mathcal{L}(K,H)$ which satisfies the Penrose equations

$$(1)\ ABA = A, \quad (2)\ BAB = B, \quad (3)\ (AB)^* = AB, \quad (4)\ (BA)^* = BA.$$

The Moore–Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in K, and if it exists, then it is unique, and is denoted by $A^\dagger$.

The ascent and descent of linear operator $A : H \to H$ are defined by

$$asc(A) = \inf_{p \in N} \{\mathcal{N}(A^p) = \mathcal{N}(A^{p+1})\}, \quad dsc(A) = \inf_{p \in N} \{\mathcal{R}(A^p) = \mathcal{R}(A^{p+1})\}.$$

If they are finite, they are equal and their common value is $\mathrm{ind}(A)$, the index of A. Also, $H = \mathcal{R}(A^{\mathrm{ind}(A)}) \oplus \mathcal{N}(A^{\mathrm{ind}(A)})$ and $\mathcal{R}(A^{\mathrm{ind}(A)})$ is closed, see [6]. We will denote by $\mathcal{L}^1(H)$ the set of bounded operators on Hilbert space H with indices less or equal one,

$$\mathcal{L}^1(H) = \{A \in \mathcal{L}(H) : \mathrm{ind}(A) \leqslant 1\}.$$

The group inverse of an operator $A \in \mathcal{L}(H)$ is the operator $B \in \mathcal{L}(H)$ such that

$$(1)\ ABA = A, \quad (2)\ BAB = B, \quad (5)\ AB = BA.$$

The group inverse of A exists if and only if $\mathrm{ind}(A) \leqslant 1$. If the group inverse of A exists, then it is unique, and it is denoted by $A^\sharp$.

If X satisfies equations $i_1, i_2, \ldots, i_k$ then X is an $\{i_1, i_2, \ldots, i_k\}$ inverse of A. The set of all $\{i_1, i_2, \ldots, i_k\}$ inverses of A is denoted by $A\{i_1, i_2, \ldots, i_k\}$. If $\mathcal{R}(A)$ is closed, then $A\{1,2,3,4\} = \{A^\dagger\}$. The theory of generalized inverses on infinite dimensional Hilbert spaces can be found, for example, in [3,6,8].

Throughout this paper H will denote arbitrary Hilbert space. An operator $A \in \mathcal{L}(H)$ is Hermitian if $A = A^*$. Closed range operator A is EP ("equal-projection") if one of the following equivalent conditions holds $\mathcal{R}(A) = \mathcal{R}(A^*)$ or $\mathcal{N}(A) = \mathcal{N}(A^*)$ or $AA^\dagger = A^\dagger A$ or $A^\dagger = A^\sharp$. Closed range operator A is partial isometry if $A^* = A^\dagger$ or $AA^*A = A$.

In Section 3 we present the results related to the core inverse of Hilbert space operators with index less or equal one. Section 4 deals with spectral properties of core inverse and Section 5 is devoted to the study of core partial order. In Section 6 we collect some additional properties of core inverse and core partial order.

3. Core inverse and its properties

In this section we introduce the notion of core inverse for Hilbert space operators, as a generalization of core inverse for matrices.

In [2], Baksalary and Trenkler introduced new generalized inverse of complex matrix, so called "core inverse".

Definition 3.1 [2]. Let $A \in C^{n\times n}$. A matrix $A^{\oplus} \in C^{n\times n}$ satisfying $AA^{\oplus} = P_{\mathcal{R}(A)}$ and $\mathcal{R}(A^{\oplus}) \subseteq \mathcal{R}(A)$ is called core inverse of A.

They have shown that a complex matrix has core inverse if and only if its index is less or equal 1, and proved its uniqueness when it exists. If we define core inverse of an operator $A \in \mathcal{L}(H)$ in the same way as in matrix case, then we have the problem because the index of A need not be less or equal 1, as we will see later in Remark 3.1. This is the reason why we find another set of equivalent conditions which determines the core inverse.

Theorem 3.1. *Let $A, X \in C^{n\times n}$. The following conditions are equivalent:*

(i) $AX = P_{\mathcal{R}(A)}$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A)$*;*
(ii) $AXA = A,\ \mathcal{R}(X) = \mathcal{R}(A)$ *and* $\mathcal{N}(X) = \mathcal{N}(A^*)$*.*

Proof. (i) $\Rightarrow$ (ii): Suppose that (i) holds. By Theorem 1 from [2], $X = A^{\oplus}$, $\operatorname{ind}(A) \leqslant 1$, $X \in \{1,2\}$ and $XA = A^\sharp A$. Because of that $A = AA^\sharp A = A^\sharp AA = XA^2$, so $\mathcal{R}(A) \subseteq \mathcal{R}(X)$, which means $\mathcal{R}(X) = \mathcal{R}(A)$. From $AX = P_{\mathcal{R}(A)}$, we have $AX = (AX)^* = X^*A^*$, so

$$A = AXA \Rightarrow A^* = A^*X^*A^* = A^*AX \Rightarrow \mathcal{N}(X) \subseteq \mathcal{N}(A^*).$$

Also, we have $X = XAX = XX^*A^* \Rightarrow \mathcal{N}(A^*) \subseteq \mathcal{N}(X)$, therefore $\mathcal{N}(X) = \mathcal{N}(A^*)$.

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Now $\mathcal{R}(X) = \mathcal{R}(A)$ implies $\operatorname{rank}(X) = \operatorname{rank}(A)$. We already have $X \in A\{1\}$, so $X \in A\{1,2\}$ [1, p. 46]. It is evident that $(AX)^2 = AX$ and $\mathcal{R}(AX) = \mathcal{R}(A)$. It remains to prove that $\mathcal{N}(AX) = \mathcal{R}(A)^\perp = \mathcal{N}(A^*)$. But from $XAX = X$ it follows that $\mathcal{N}(AX) = \mathcal{N}(X) = \mathcal{N}(A^*)$, so we have the proof. $\square$

We use the condition (ii) from previous theorem as a definition of core inverse for Hilbert space operators.

Definition 3.2. Let H be arbitrary Hilbert space, and $A \in \mathcal{L}(H)$. An operator $A^{\oplus} \in \mathcal{L}(H)$ is core inverse of A if

$$AA^{\oplus}A = A, \quad \mathcal{R}(A^{\oplus}) = \mathcal{R}(A) \quad \text{and} \quad \mathcal{N}(A^{\oplus}) = \mathcal{N}(A^*).$$

From Theorem 3.1 it follows that the Definitions 3.1 and 3.2 are equivalent in complex matrix case. More characterizations of core inverse can be found later in Theorem 3.8.

The next theorem describes the bounded linear operators having core inverse and gives appropriate matrix forms.

Theorem 3.2. *Let $A \in \mathcal{L}(H)$. Then the core inverse of A exists if and only if* $\operatorname{ind}(A) \leqslant 1$ *in which case the following representations hold:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{3}$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}, \tag{4}$$

$$A^{\oplus} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}, \tag{5}$$

$$A^{\oplus} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{6}$$

$$A^{\sharp} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}, \tag{7}$$

$$A^{\sharp} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{8}$$

where $A_1 \in \mathcal{L}(\mathcal{R}(A))$ is invertible operator.

Proof. Suppose that $\mathrm{ind}(A) \leqslant 1$, it follows that $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and $\mathcal{R}(A)$ is closed. Also, we have $H = \mathcal{R}(A) \oplus^{\perp} \mathcal{N}(A^*)$, so A has the matrix forms (3) and (4), where $A_1 \in \mathcal{L}(\mathcal{R}(A))$ is invertible. Let us find the core inverse in the following form:

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

From the condition $\mathcal{R}(X) = \mathcal{R}(A)$ we have $X_3 = 0$ and $X_4 = 0$, and the condition $\mathcal{N}(X) = \mathcal{N}(A^*)$ implies $X_2 = 0$. From $AXA = A$ it follows $A_1 = A_1X_1A_1$, so $X_1 = A_1^{-1}$. Therefore,

$$X = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}. \tag{9}$$

On the other side, if X has the representation (9), then it obviously obeys $AXA = A,\ \mathcal{R}(X) = \mathcal{R}(A),\ \mathcal{N}(X) = \mathcal{N}(A^*)$. Hence, we proved the existence, and the uniqueness also, of core inverse. Since $\mathcal{R}(X) = \mathcal{R}(A)$ we also have the representation (6). The representations (7) and (8) can be derived in a same manner.

Suppose now that $A^{\textcircled{\#}}$ exists, we prove that $\mathrm{ind}(A) \leqslant 1$. From $AA^{\textcircled{\#}}A = A$ we conclude that $\mathcal{R}(A)$ is closed. From the conditions $\mathcal{R}(A^{\textcircled{\#}}) = \mathcal{R}(A)$ and $\mathcal{N}(A^{\textcircled{\#}}) = \mathcal{N}(A^*)$ it follows that

$$A^{\textcircled{\#}} = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $B_1 \in \mathcal{L}(\mathcal{R}(A))$ is invertible. It is clear that A has the following representation:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$$

for some $A_1 \in \mathcal{L}(\mathcal{R}(A))$ and $A_2 \in \mathcal{L}(\mathcal{N}(A^*), \mathcal{R}(A))$. From $AA^{\textcircled{\#}}A = A$ we obtain:

$$\begin{bmatrix} A_1B_1A_1 & A_1B_1A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix},$$

so $A_1 = A_1B_1A_1$ and $A_2 = A_1B_1A_2$. From the second equality we have $\mathcal{R}(A_2) \subseteq \mathcal{R}(A_1)$, so $\mathcal{R}(A) = \mathcal{R}(A_1)$, which implies A_1 is surjective. Since $\mathcal{R}(A)$ is Hilbert space, A_1 is right invertible so there exists $T \in \mathcal{L}(\mathcal{R}(A))$ such that $A_1T = I_{\mathcal{R}(A)} \in \mathcal{L}(\mathcal{R}(A))$. By post-multiplying $A_1B_1A_1 = A_1$ by T, we obtain $A_1B_1 = I_{\mathcal{R}(A)}$. Because of the invertibility of B_1, we conclude $A_1 = B_1^{-1}$, so $B_1 = A_1^{-1}$. From $A^2 = \begin{bmatrix} A_1^2 & A_1A_2 \\ 0 & 0 \end{bmatrix}$, and from the invertibility of A_1, it follows $\mathcal{R}(A) = \mathcal{R}(A_1) = \mathcal{R}(A_1^2) \subseteq \mathcal{R}(A^2)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(A^2)$.

Let us prove that $\mathcal{N}(A) = \mathcal{N}(A^2)$. It is obvious that $\mathcal{N}(A) \subseteq \mathcal{N}(A^2)$, so we must prove the opposite inclusion. Let $x \in \mathcal{N}(A^2), x = x_1 + x_2 \in \mathcal{R}(A) \oplus^{\perp} \mathcal{N}(A^*)$. We have $0 = A^2x = A_1^2x_1 + A_1A_2x_2 = A_1(A_1x_1 + A_2x_2)$, so $A_1x_1 + A_2x_2 \in \mathcal{N}(A_1)$. From the invertibility of A_1 it follows that $A_1x_1 + A_2x_2 = 0$, so $Ax = A_1x_1 + A_2x_2 = 0$, which means $x \in \mathcal{N}(A)$. Therefore, $\mathcal{N}(A) = \mathcal{N}(A^2)$, and we have proved $\mathrm{ind}(A) \leqslant 1$. $\square$

Remark 3.1. If we assume in Theorem 3.1 that $A, X \in \mathcal{L}(H)$, where H is arbitrary Hilbert space, then the condition (ii) implies the condition (i), but not vice versa.

For the first claim, suppose $AXA = A,\ \mathcal{R}(X) = \mathcal{R}(A)$ and $\mathcal{N}(X) = \mathcal{N}(A^*)$. It follows that AX is a projector with range $\mathcal{R}(A)$. It remains to prove that AX is Hermitian. Since $AXA = A$, from $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$ it follows that $X = XX^*A^* = X(AX)^*$ (cf. Lemma 2.1 from [14]); therefore $AX = AX(AX)^*$ is Hermitian.

To show that condition (i) does not imply condition (ii) in general, we give the following counterexample. Let $H = \ell^2(\mathbb{N})$ where $\ell^2(\mathbb{N})$ is the set of all complex sequences $x = (x_i)$ with property $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Recall that $\ell^2(\mathbb{N})$ is a Hilbert space with the inner product

$$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

Let A and X be the left and right shift operators on H respectively, defined by

$$A(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots), \quad X(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

It is easy to check the following well known properties of these operators:

1. A and X are bounded linear operators.
2. A is right invertible but not left invertible and its right inverse is X.
3. X is left invertible but not right invertible and its left inverse is A.
4. $A^* = X$ and $X^* = A$.

We obtain that $\mathcal{R}(A) = H$, $AX = I = P_{\mathcal{R}(A)}$, $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, so (i) is satisfied. But, it is evident that $\mathcal{R}(X) \neq H = \mathcal{R}(A)$ so (ii) is not satisfied. Note that for counterexample we could take any infinite dimensional Hilbert space H and any bounded operators A and X on H such that A is right but not left invertible and $AX = I$. This remark fully justifies Definition 3.2.

As in [2], dual core inverse of matrix A, denoted by $\widetilde{A}$ in [2] and here by $A_{\textcircled{\#}}$, can be defined in the following way.

Definition 3.3. Let H be arbitrary Hilbert space, and $A \in \mathcal{L}(H)$. An operator $A_{\textcircled{\#}} \in \mathcal{L}(H)$ is dual core inverse of A if

$$AA_{\textcircled{\#}}A = A, \quad \mathcal{R}(A_{\textcircled{\#}}) = \mathcal{R}(A^*) \quad \text{and} \quad \mathcal{N}(A_{\textcircled{\#}}) = \mathcal{N}(A).$$

Just as in Theorem 3.2, we can show the following result.

Theorem 3.3. *Let $A \in \mathcal{L}(H)$. There exists the dual core inverse of A if and only if $\operatorname{ind}(A) \leqslant 1$ in which case the following representations hold:*

$$A = \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}, \tag{10}$$

$$A = \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{11}$$

$$A_{\textcircled{\#}} = \begin{bmatrix} A_2^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}, \tag{12}$$

$$A_{\textcircled{\#}} = \begin{bmatrix} A_2^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{13}$$

$$A^{\dagger} = \begin{bmatrix} A_2^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}, \tag{14}$$

$$A^{\dagger} = \begin{bmatrix} A_2^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{15}$$

where $A_2 \in \mathcal{L}(\mathcal{R}(A))$ is invertible operator.

Proof. The proof is analogous to the proof of Theorem 3.2 and it is left to the reader. We only note that $\operatorname{ind}(A) \leqslant 1$ if and only if $\operatorname{ind}(A^*) \leqslant 1$. $\square$

In the next theorem we show that properties of core inverses from [2] are valid for operator case, too. We also give some new properties.

Theorem 3.4. *Let $A \in \mathcal{L}^1(H)$ and $m \in \mathbb{N}$. Then:*

(i) $A^{\textcircled{\#}} \in A\{1, 2\}$;
(ii) $AA^{\textcircled{\#}} = AA^{\dagger} = P_{\mathcal{R}(A)}$, *so* $(AA^{\textcircled{\#}})^* = AA^{\textcircled{\#}}$;
(iii) $A^{\textcircled{\#}}A^2 = A$;
(iv) $A(A^{\textcircled{\#}})^2 = A^{\textcircled{\#}}$;
(v) $A^{\textcircled{\#}} = A^{\#}P_{\mathcal{R}(A)}$;
(vi) $A^{\textcircled{\#}}A = A^{\#}A$;
(vii) $A^{\textcircled{\#}} = A^{\#}AA^{\dagger}$;
(viii) $A^{\textcircled{\#}}$ *is EP*;
(ix) $(A^{\textcircled{\#}})^{\dagger} = (A^{\textcircled{\#}})^{\#} = (A^{\textcircled{\#}})^{\textcircled{\#}} = AP_{\mathcal{R}(A)}$;
(x) $(A^{\textcircled{\#}})^2A = A^{\#}$;
(xi) $(A^{\textcircled{\#}})^m = (A^m)^{\textcircled{\#}}$;
(xii) $((A^{\textcircled{\#}})^{\textcircled{\#}})^{\textcircled{\#}} = A^{\textcircled{\#}}$.

Proof. In the proof we will use Definition 3.2, representations from Theorem 3.2, as well as the following unusual form for orthogonal projector on $\mathcal{R}(A)$:

$$P_{\mathcal{R}(A)} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}. \tag{16}$$

We must show the existence of various expressions appearing in the theorem. First, the condition $\mathrm{ind}(A) \leqslant 1$ provides the existence of the group inverse $A^\sharp$ and by Theorem 3.2 the existence of core inverse $A^{\circledsharp}$, too. Also, $\mathrm{ind}(A) \leqslant 1$ implies that $R(A)$ is closed and it ensures the existence of the Moore–Penrose invesre $A^\dagger$.

(i) This follows immediately from (3) and (5).
(ii) From (3), (5), (11) and (14) it follows

$$AA^{\circledsharp} = AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

(iii) By (4) we have

$$A^2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

so

$$A^2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}. \tag{17}$$

Now, the proof follows by (17), (5) and (4).
(iv) By (6) we get

$$(A^{\circledsharp})^2 = \begin{bmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

so we can conclude

$$(A^{\circledsharp})^2 = \begin{bmatrix} A_1^{-2} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}. \tag{18}$$

Applying (18), (4) and (5) we get the result.
(v) By using decomposition (7) for $A^\sharp$ and a form (16) for $P_{\mathcal{R}(A)}$, we have

$$A^\sharp P_{\mathcal{R}(A)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

which, by (5) and the uniqueness of core inverse, is equal to $A^{\circledsharp}$.
(vi) By (v), $A^{\circledsharp}A = A^\sharp P_{\mathcal{R}(A)}A = A^\sharp A$.
(vii) Follows by (v) since $P_{\mathcal{R}(A)} = AA^\dagger$.
(viii) By Definition 3.2, we have $\mathcal{R}(A^{\circledsharp}) = \mathcal{R}(A)$, so $\mathcal{R}(A^{\circledsharp})$ is closed and by the representation (6) we have $\mathcal{R}((A^{\circledsharp})^*) = (A_1^{-1})^*(\mathcal{R}(A)) = \mathcal{R}(A)$; therefore $\mathcal{R}(A^{\circledsharp}) = \mathcal{R}((A^{\circledsharp})^*)$, which means that $A^{\circledsharp}$ is the EP.
(ix) We show that $A^{\circledsharp}$ is closed so it is Moore–Penrose invertible. If we use the form (6), we have

$$A^{\circledsharp} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

now it is easy to obtain

$$(A^{\circledsharp})^\dagger = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

which is precisely equal to $AP_{\mathcal{R}(A)}$ when (3) and (16) are used.
By (viii) $A^{\circledsharp}$ is EP so $\mathrm{ind}(A^{\circledsharp}) \leqslant 1$, $(A^{\circledsharp})^\sharp = (A^{\circledsharp})^\dagger$ and

$$(A^{\circledsharp})^{\circledsharp} \overset{(v)}{=} (A^{\circledsharp})^\sharp P_{\mathcal{R}(A^{\circledsharp})} = (A^{\circledsharp})^\dagger P_{\mathcal{R}(A)} = AP_{\mathcal{R}(A)}^2 = AP_{\mathcal{R}(A)}.$$

(x) Using (v), we obtain $(A^{\circledsharp})^2 A = A^\sharp P_{\mathcal{R}(A)} A^\sharp P_{\mathcal{R}(A)} A = A^\sharp P_{\mathcal{R}(A)} A^\sharp A = A^\sharp P_{\mathcal{R}(A)} AA^\sharp = A^\sharp AA^\sharp = A^\sharp$.
(xi) By (4) we have

$$A^m = \begin{bmatrix} A_1^m & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

so

$$A^m = \begin{bmatrix} A_1^m & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

Since A_1^m is invertible it follows by (6) that

$$(A^m)^{\oplus} = \begin{bmatrix} (A_1^m)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

On the other hand, by (6) we obtain

$$(A^{\oplus})^m = \begin{bmatrix} (A_1^{-1})^m & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$$

(xii) By (ix),

$$((A^{\oplus})^{\oplus})^{\oplus} = A^{\oplus} P_{\mathcal{R}(A^{\oplus})} = A^{\oplus} P_{\mathcal{R}(A)} = A^{\oplus}. \qquad \square$$

As we saw in (vii) of preceding theorem, $A^{\oplus} = A^{\sharp} A A^{\dagger}$ so the core inverse is in-between the group and Moore–Penrose inverse in some way. Therefore, it is expected that the core inverse shares the properties of two inverses. Recall, that the group inverse of operator A is the unique operator X determined by equations

$$(1)\ AXA = A \quad (2)\ XAX = X \quad (5)\ AX = XA.$$

Note that these equations can be replaced by

$$\begin{aligned} &(1)\ AXA = A \qquad (2)\ XAX = X \qquad (6)\ XA^2 = A, \\ &(7)\ AX^2 = X \qquad (8)\ A^2X = A \qquad (9)\ X^2A = X. \end{aligned}$$

Namely, $AX = XA^2X = XA$. Of course, equation (7) follows by (2), (6) and (8). The Moore–Penrose inverse of operator A is defined by equations

$$(1)\ AXA = A \quad (2)\ XAX = X \quad (3)\ (AX)^* = AX \quad (4)\ (XA)^* = XA.$$

In the next theorem we give alternative definition of core inverse by the set of equations.

Theorem 3.5. *Let $A \in \mathcal{L}(H)$. Then $A\{1,2,3,6,7\} \neq \emptyset$ if and only if $\mathrm{ind}(A) \leqslant 1$. In this case $A\{1,2,3,6,7\} = \{A^{\oplus}\}$ i.e. the core inverse of A is the unique operator X satisfying the following equations:*

$$\begin{aligned} &(1)\ AXA = A \\ &(2)\ XAX = X \\ &(3)\ (AX)^* = AX \\ &(6)\ XA^2 = A \\ &(7)\ AX^2 = X. \end{aligned}$$

Proof. If $\mathrm{ind}(A) \leqslant 1$ then $A^{\oplus}$ exists and, by Theorem 3.2(i)–(iv), it satisfies above equations. Suppose now that there exists an operator $X \in \mathcal{L}(H)$ which satisfies the given equations. By (6) and (7), $\mathcal{R}(X) = \mathcal{R}(A)$. By (2) and (3), $X = XAX = XX^*A^*$ and hence $\mathcal{N}(A^*) \subseteq \mathcal{N}(X)$. Likewise, by (1) and (3),

$$A^* = A^*X^*A^* = A^*(AX)^* = A^*AX,$$

so $N(X) \subseteq \mathcal{N}(A^*)$. Now, $AXA = A$, $\mathcal{R}(X) = \mathcal{R}(A^*)$ and $\mathcal{N}(X) = \mathcal{N}(A^*)$, and therefore by definition, $X = A^{\oplus}$. Since $A^{\oplus}$ exist, $\mathrm{ind}(A) \leqslant 1$. $\square$

When H is a finite dimensional, i.e. when A is a complex matrix, then we have simpler situation.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$. Then $A\{2,3,6\} \neq \emptyset$ if and only if $\mathrm{ind}(A) \leqslant 1$. In this case $A\{2,3,6\} = \{A^{\oplus}\}$ i.e. the core inverse of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ which satisfies the following equations:*

(2) $XAX = X$
(3) $(AX)^* = AX$
(6) $XA^2 = A$.

Proof. If $\mathrm{ind}(A) \leqslant 1$ then $A^{\text{\textcircled{\#}}}$ exists and it satisfies given equations. Suppose now that there exists matrix X which satisfies Eqs. (2), (3) and (6). By (6), it follows $\mathcal{N}(A^2) \subseteq \mathcal{N}(A)$, thus $\mathcal{N}(A^2) = \mathcal{N}(A)$. Therefore, $\mathrm{ind}(A) \leqslant 1$ so the group inverse $A^\sharp$ exists. Now, $XA = XAA^\sharp A = XA^2A^\sharp = AA^\sharp$, so

$$AXA = A^2A^\sharp = A$$

and

$$AX^2 = AX(XAX) = AXAA^\sharp X = AA^\sharp X = XAX = X.$$

By Theorem 3.5 it follows that $X = A^{\text{\textcircled{\#}}}$.

Note that in the matrix case the condition $\mathcal{N}(A^2) = \mathcal{N}(A)$ is sufficient for $\mathrm{ind}(A) \leqslant 1$, but in the infinite dimensional case it is not. If we suppose that $A \in \mathcal{L}(H)$ has index less or equal one then the core inverse of operator is uniquely determined by Eqs. (2), (3) and (6). We emphasize that none of the equations in Theorem 3.5 can be removed. For instance we have following remark.

Remark 3.2. Let $H = \ell^2(\mathbb{N})$ and let A and X be right and left shift operators on H respectively, see Remark 3.1:

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots), \quad X(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Then $XA = I$ so Eqs. (1), (2) and (6) hold. Also, in view of Remark 3.1, we have $(AX)^* = X^*A^* = AX$ so (6) is satisfied. But,

$$AX^2(x_1, x_2, x_3, \ldots) = (0, x_3, x_4, \ldots),$$

so $AX^2 \neq X$ and (7) is not satisfied. Note that X is Moore–Penrose inverse of A.

The core inverse can be defined in many equivalent ways.

Theorem 3.7. *Let $A \in \mathcal{L}^1(H)$. An operator $X \in \mathcal{L}(H)$ is the core inverse of A if and only if $X = A^\sharp AA^\dagger$ if and only if $X \in \{1, 2, 3\}$ and $XA = A^\sharp A$.*

Proof. If X is the core inverse of A then by Theorem 3.2(vii), $X = A^\sharp AA^\dagger$. It is easy to show that $A^\sharp AA^\dagger \in \{1, 2, 3\}$ and $A^\sharp AA^\dagger A = A^\sharp A$. Suppose now that $X \in \{1, 2, 3\}$ and $XA = A^\sharp A$. Then

$$\mathcal{R}(X) = \mathcal{R}(XA) = \mathcal{R}(A^\sharp A) = \mathcal{R}(A)$$

and

$$\mathcal{N}(X) = \mathcal{N}(AX) = \mathcal{N}((AX)^*) = \mathcal{N}(X^*A^*) = \mathcal{N}(A^*),$$

so X is the core inverse of A. $\square$

We can summarize the results from Theorems 3.1, 3.5, 3.6 and 3.7 and obtain the following theorem which gives equivalent definitions of core inverse of matrix.

Theorem 3.8. *Let A and X be complex $n \times n$ matrices such that $\mathrm{ind}(A) \leqslant 1$. Then the following statements are equivalent:*

(i) *X is core inverse of A in a sense of Definition 3.1.*
(ii) *X is core inverse of A in a sense of Definition 3.2.*
(iii) *X is a least square g-inverse of A satisfying $XAX = X, XA^2 = A$ and $XA^2 = A$, i.e. $X \in \{1, 2, 3, 6, 7\}$.*
(vi) *X is a 2-inverse of A satisfying $AX = (AX)^*$ and $XA^2 = A$, i.e. $X \in \{2, 3, 6\}$.*
(v) *X is a least square g-inverse of A satisfying $XAX = X$ and $XA = XA^{\#}$.*

The next theorem deals with some special cases of core inverse.

Theorem 3.9. *Let $A \in \mathcal{L}^1(H)$. Then:*

(i) $A^{\text{\textcircled{\#}}} = 0 \Longleftrightarrow A = 0$;
(ii) $A^{\text{\textcircled{\#}}} = P_{\mathcal{R}(A)} \Longleftrightarrow A^2 = A$;
(iii) $A^{\text{\textcircled{\#}}} = A \Longleftrightarrow A^3 = A$ *and A is EP;*
(vi) $A^{\text{\textcircled{\#}}} = A^* \Longleftrightarrow$ *A is partial isometry and EP.*

Proof.

(i) It follows by $A^{\textcircled{\#}} \in A\{1,2\}$.
(ii) If $A^{\textcircled{\#}} = P_{\mathcal{R}(A)}$ then $A = AA^{\textcircled{\#}}A = AP_{\mathcal{R}(A)}A = A^2$. On the other hand, if $A^2 = A$ then $A^{\textcircled{\#}} = A^{\sharp}AA^{\dagger} = A^{\sharp}A^2A^{\dagger} = AA^{\dagger} = P_{\mathcal{R}(A)}$. Therefore, A is an idempotent if and only if $A^{\textcircled{\#}}$ is orthogonal projector.
(iii) From $A^{\textcircled{\#}} = A$ it follows $A = AA^{\textcircled{\#}}A = A^3$. From Theorem 3.4 (viii), we have $A^{\textcircled{\#}}$ is EP, so because of $A^{\textcircled{\#}} = A$ we have A is EP. Conversely, $A^{\textcircled{\#}} = A^{\sharp}AA^{\dagger} = A^{\sharp}A^3A^{\dagger} = A^2A^{\dagger} = A^2A^{\sharp} = A$.
(vi) If $A^{\textcircled{\#}} = A^*$, then $A^* = A^{\dagger}AA^* = A^{\dagger}AA^{\textcircled{\#}} = A^{\dagger}AA^{\sharp}AA^{\dagger} = A^{\dagger}$, so A is partial isometry. From $\mathcal{R}(A) = \mathcal{R}(A^{\textcircled{\#}}) = \mathcal{R}(A^*)$ we have A is EP. Conversely, from the EP-ness $(A^{\dagger} = A^{\sharp})$ and A being the partial isometry $(A^* = A^{\dagger})$, we have $A^{\textcircled{\#}} = A^{\sharp}AA^{\dagger} = A^{\dagger}AA^* = A^*$.

Next theorem further characterizes EP-ness of A via its core inverse.

Theorem 3.10. *Let $A \in \mathcal{L}^1(H)$. The following statements are equivalent:*

(i) *A is EP;*
(ii) *Any two elements of the set $\{A^{\sharp}, A^{\dagger}, A^{\textcircled{\#}}, A_{\textcircled{\#}}\}$ are equal;*
(iii) $(A^{\textcircled{\#}})^{\sharp} = A$;
(vi) $(A^{\textcircled{\#}})^{\dagger} = A$;
(v) $(A^{\textcircled{\#}})^{\textcircled{\#}} = A$;
(vi) $A^{\textcircled{\#}}A = AA^{\textcircled{\#}}$;
(vii) $(A^{\dagger})^{\textcircled{\#}} = A$;
(viii) $(A^{\textcircled{\#}})^{\dagger} = (A^{\dagger})^{\textcircled{\#}}$.

Proof. Let us show that (i) implies (ii)–(viii). Suppose that A is EP i.e. $A^{\sharp} = A^{\dagger}$. The proof of (ii) follows by $A^{\textcircled{\#}} = A^{\sharp}AA^{\dagger}$ and $A_{\textcircled{\#}} = A^{\dagger}AA^{\sharp}$. By Theorem 3.4 (ix),

$$(A^{\textcircled{\#}})^{\sharp} = (A^{\textcircled{\#}})^{\dagger} = (A^{\textcircled{\#}})^{\textcircled{\#}} = AP_{\mathcal{R}(A)} = AAA^{\dagger} = AAA^{\sharp} = A.$$

We just showed that EP-ness of A yields $A^{\textcircled{\#}} = A^{\sharp}$, so (vi) follows. By (v) of Theorem 3.4, $(A^{\dagger})^{\textcircled{\#}} = (A^{\dagger})^{\sharp}P_{\mathcal{R}(A^{\dagger})} = (A^{\sharp})^{\sharp}A^{\dagger}A = A$. Finally, $(A^{\textcircled{\#}})^{\dagger} = (A^{\dagger})^{\dagger} = A = (A^{\dagger})^{\textcircled{\#}}$, by (ii) and (vii).

Let us show that any of the conditions (ii)–(viii) implies that A is EP.

(ii) If $A^{\textcircled{\#}} = A^{\dagger}$ then $\mathcal{R}(A) = \mathcal{R}(A^{\textcircled{\#}}) = \mathcal{R}(A^{\dagger}) = \mathcal{R}(A^*)$, thus A is EP. In the same manner we can show the other cases.

(iii)–(v) By Theorem 3.4 (ix), any of the assumptions is equivalent to $AP_{\mathcal{R}(A)} = A$. Multiplying both sides by $(A^{\sharp})^2$ from the left, we obtain $A^{\sharp}AA^{\dagger} = A^{\sharp}$, i.e. $A^{\textcircled{\#}} = A^{\sharp}$, which is by (ii) equivalent to A is EP.

(vi) This is equivalent to $A^{\sharp}A = AA^{\dagger}$, which reduces to previous case.

(vii) We have $\mathcal{R}(A) = \mathcal{R}((A^{\dagger})^{\textcircled{\#}}) = \mathcal{R}(A^{\dagger}) = \mathcal{R}(A^*)$, so A is EP.

(viii) We have

$$\mathcal{R}(A^*) = \mathcal{R}(A^{\dagger}) = \mathcal{R}((A^{\dagger})^{\textcircled{\#}}) = \mathcal{R}((A^{\textcircled{\#}})^{\dagger}) = \mathcal{R}((A^{\textcircled{\#}})^*) = \mathcal{N}(A^{\textcircled{\#}})^{\perp} = \mathcal{N}(A^*)^{\perp} = \mathcal{R}(A),$$

so A is EP. □

The next theorem is also given in [2] for complex matrix case. Here we present much shorter and elementary proof for Hilbert space setting.

Theorem 3.11. *Let $A, B \in \mathcal{L}(H)$ be orthogonal projectors such that $\mathcal{R}(AB)$ is closed. Then $\mathcal{R}(ABA)$ is closed,* $\mathrm{ind}(AB) \leqslant 1$ *and*

$$(AB)^{\textcircled{\#}} = (ABA)^{\dagger}. \quad (19)$$

Proof. If $\mathcal{R}(AB)$ is closed, then $(AB)^{\dagger}$ exists and

$$AB = AB(AB)^*((AB)^\dagger)^* = ABA(BA)^\dagger$$
$$AB = ((AB)^\dagger)^*(AB)^*AB = (BA)^\dagger BAB$$

and similarly

$$BA = BAB(AB)^\dagger = (AB)^\dagger ABA.$$

It follows that $\mathcal{R}(AB) \subseteq \mathcal{R}(ABA)$ so $\mathcal{R}(ABA) = \mathcal{R}(AB)$ is closed. From above equations, it is easy to see that

$$AB = ABAB(AB)^\dagger(BA)^\dagger = (BA)^\dagger(AB)^\dagger ABAB.$$

It follows that $\mathcal{R}(AB) = \mathcal{R}(ABAB) = \mathcal{R}((AB)^2)$ and $\mathcal{N}(AB) = \mathcal{N}(ABAB) = \mathcal{N}((AB)^2)$, so $\mathrm{ind}(AB) \leqslant 1$.

We showed the existence of inverses in (39), so we can move to the proof of the formula. We will prove the equivalent statement: $((AB)^{\oplus})^\dagger = ABA$. According to Theorem 3.4 (ix), we have $((AB)^{\oplus})^\dagger = ABP_{\mathcal{R}(AB)} = (AB)^2(AB)^\dagger$. By using well-known formula $T^\dagger = T^*(TT^*)^\dagger$ for any closed range operator T, and the fact that A and B are orthogonal projectors, we have:

$$(AB)^\dagger = (AB)^*(AB(AB)^*)^\dagger = B^*A^*(ABB^*A^*)^\dagger = BA(ABA)^\dagger.$$

When we put this expression in formula above, we have

$$((AB)^{\oplus})^\dagger = (AB)^2BA(ABA)^\dagger = ABABA(ABA)^\dagger = ABAABA(ABA)^\dagger = (ABA)^*ABA(ABA)^\dagger = (ABA)^* = ABA. \qquad \square$$

4. Spectral properties

In this section we are dealing with so-called spectral properties of group and core inverses. Spectral properties of group inverse are well-known. Cline [4] has pointed out that square matrix A such that $\mathrm{ind}(A) = 1$ has $\{1,2,3\}$ – inverse whose range is $\mathcal{R}(A)$ and as "least-squares" inverse it has some spectral properties. We consider some spectral properties of group and core inverse of given operator A. Suppose that $A \in \mathcal{L}^1(H)$.

If $0 \in \sigma_p(A)$, and x is its associated eigenvector, then $x \in \mathcal{N}(A)$, so $\mathcal{N}(A) \neq \{0\}$. Since $H = \mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{R}(A) \oplus^\perp \mathcal{N}(A^*)$ it follows that $\mathcal{N}(A^{\oplus}) = \mathcal{N}(A^*) \neq \{0\}$. Therefore $0 \in \sigma_p(A^{\oplus})$. Moreover, $\mathcal{N}(A^\sharp) = \mathcal{N}(A)$, so

$$0 \in \sigma_p(A) \Longleftrightarrow 0 \in \sigma_p(A^\sharp) \Longleftrightarrow 0 \in \sigma_p(A^{\oplus})$$

only for the same eigenvector $x \in (\mathcal{N}(A) \cap \mathcal{N}(A^*)) \setminus \{0\}$. On the other side,

$$0 \in \sigma_p(A) \Longleftrightarrow 0 \in \sigma_p(A^\sharp)$$

always holds with an eigenvector $x \in \mathcal{N}(A) \setminus \{0\}$.

Suppose now that $0 \neq \lambda \in \sigma_p(A)$ with corresponding eigenvector $x = x_1 + x_2 \in \mathcal{R}(A) \oplus \mathcal{N}(A)$. Using representation (4) we obtain

$$0 = (A - \lambda I)x = \begin{bmatrix} A_1 - \lambda I_{\mathcal{R}(A)} & 0 \\ 0 & -\lambda I_{\mathcal{N}(A)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This is equivalent to $A_1x_1 = \lambda x_1$ and $-\lambda x_2 = 0$. Since $\lambda \neq 0$, we have $x_2 = 0$ and $\lambda \in \sigma_p(A_1)$. Thus $0 \neq \lambda \in \sigma_p(A)$ with eigenvector x if and only if $\lambda \in \sigma_p(A_1)$ corresponding to $x \in \mathcal{R}(A)$.

If $0 \neq \mu \in \sigma_p(A^\sharp)$ corresponding to eigenvector $y = y_1 + y_2 \in \mathcal{R}(A) \oplus \mathcal{N}(A)$ then using representation (7) we obtain

$$0 = (A^\sharp - \mu I)y = \begin{bmatrix} A_1^{-1} - \mu I_{\mathcal{R}(A)} & 0 \\ 0 & -\mu I_{\mathcal{N}(A)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

This gives $A_1^{-1}y_1 = \mu y_1$ and $-\mu y_2 = 0$ and this is equivalent with $\mu^{-1} \in \sigma_p(A_1)$ with corresponding eigenvector $y = y_1 \in \mathcal{R}(A)$.

Finally, if $0 \neq \nu \in \sigma_p(A^{\oplus})$ corresponding to eigenvector $z = z_1 + z_2 \in \mathcal{R}(A) \oplus^\perp \mathcal{N}(A^*)$ then using representation (6) we conclude

$$0 = (A^{\oplus} - \nu I)z = \begin{bmatrix} A_1^{-1} - \nu I_{\mathcal{R}(A)} & 0 \\ 0 & -\nu I_{\mathcal{N}(A^*)} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Therefore $A_1^{-1}z_1 = \nu z_1$ and $-\nu z_2 = 0$, so $\nu^{-1} \in \sigma_p(A_1)$ and $z_2 = 0$. Hence $0 \neq \nu \in \sigma_p(A^{\oplus})$ corresponding to eigenvector z if and only if $\nu^{-1} \in \sigma_p(A_1)$ corresponding to eigenvector $z = z_1 \in \mathcal{R}(A)$.

It follows that for $\lambda \neq 0$ we have

$$\lambda \in \sigma_p(A) \Longleftrightarrow \lambda^{-1} \in \sigma_p(A^\sharp) \Longleftrightarrow \lambda^{-1} \in \sigma_p(A^{\oplus})$$

corresponding to the same eigenvector x where $x \in \mathcal{R}(A)$ is also an eigenvector of A_1 corresponding to an eigenvalue $\lambda \in \sigma_p(A_1)$.

5. Core partial order

Using various generalized inverses we can define various partial orders. Let $A, B \in \mathcal{L}(H)$. Similar to the matrix case we can define minus, star and sharp partial order, respectively:

$$\begin{aligned} &A <^- B \iff AA^- = BA^- \quad \text{and} \quad A^-A = A^-B, \quad \text{for some } A^- \in A\{1\}, \\ &A <^* B \iff AA^\dagger = BA^\dagger \quad \text{and} \quad A^\dagger A = A^\dagger B, \\ &A <^\sharp B \iff AA^\sharp = BA^\sharp \quad \text{and} \quad A^\sharp A = A^\sharp B. \end{aligned} \tag{20}$$

For minus and star order we require that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and for sharp order we require that $\operatorname{ind}(A) \leqslant 1$ and $\operatorname{ind}(B) \leqslant 1$. In [15,7] the minus and the star partial orders are generalized for arbitrary $A, B \in \mathcal{L}(H)$. In [14] the authors defined the minus partial order for inner invertible Banach space operators as in (20) and showed that

$$A <^- B \iff \mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B - A) \iff B\{1\} \subseteq A\{1\}. \tag{21}$$

Also, when $A <^- B$ then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. In the same paper, it is shown that, when B is inner regular,

$$\mathcal{R}(A) \subseteq \mathcal{R}(B) \iff A = BB^-A \quad \text{for each } B^- \in B\{1\} \quad \text{and} \tag{22}$$
$$\mathcal{N}(B) \subseteq \mathcal{N}(A) \iff A = AB^-B \quad \text{for each } B^- \in B\{1\}. \tag{23}$$

It is not difficult to see that

$$A <^* B \iff AA^* = BA^* \quad \text{and} \quad A^*A = A^*B, \tag{24}$$
$$A <^\sharp B \iff A^2 = AB = BA. \tag{25}$$

The proof is the same as in the matrix case, see [13].

The core partial order for matrices was defined in [2] in the natural way. We use the same definition in the Hilbert space setting.

Definition 5.1. Let H be arbitrary Hilbert space and $A, B \in \mathcal{L}^1(H)$. We say that A is below B under the core partial order, denoted by $A <^{\circledsharp} B$, if $AA^{\circledsharp} = BA^{\circledsharp}$ and $A^{\circledsharp}A = A^{\circledsharp}B$.

To define core partial order it is enough to assume that $\operatorname{ind}(A) \leqslant 1$. We require that both operators have indices less or equal one because this is crucial for developing properties of core partial order. Since $A^{\circledsharp} \in A\{1\}$ we see that $A <^{\circledsharp} B$ implies $A <^- B$, so the core partial order satisfies all the properties of minus partial order.

Let us show that $A^{\circledsharp}A = A^{\circledsharp}B \iff A^*A = A^*B$. If $A^{\circledsharp}A = A^{\circledsharp}B$ then

$$A^*B = (AA^{\circledsharp}A)^*B = A^*AA^{\circledsharp}B = A^*AA^{\circledsharp}A = A^*A.$$

The proof for the opposite direction is similar. Also,

$$AA^{\circledsharp} = BA^{\circledsharp} \iff A^2 = BA.$$

Indeed, if $AA^{\circledsharp} = BA^{\circledsharp}$ then $BA = BA^{\circledsharp}A^2 = AA^{\circledsharp}A^2 = A^2$. If $A^2 = BA$ then $BA^{\circledsharp} = BA(A^{\circledsharp})^2 = A^2(A^{\circledsharp})^2 = AA^{\circledsharp}$. It follows that

$$\begin{aligned} A <^{\circledsharp} B &\iff A^*A = A^*B \quad \text{and} \quad A^2 = BA & (26)\\ &\iff A^\dagger A = A^\dagger B \quad \text{and} \quad AA^\sharp = BA^\sharp, & (27) \end{aligned}$$

so (see [10] for complex matrix case)

$$A <^{\circledsharp} B \iff A * < B \quad \text{and} \quad A < \sharp B, \tag{28}$$

where "$* <$" and "$< \sharp$" are left star and right sharp partial orders. Recall that $A * < B$ if $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $A < \sharp B$ if $A^2 = BA$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$, see [13]. We can conclude that the core partial order is in-between the star and the sharp partial orders. The condition $A <^{\circledsharp} B$ does not imply $B - A <^{\circledsharp} B$ even in the matrix case, see [2]. The conditions under which the property $B - A <^{\circledsharp} B$ is valid, for $A, B \in \mathbb{C}^{n \times n}$, are given in [11].

Theorem 5.1. *Let $A, B \in \mathcal{L}^1(H)$. If $A <^{\circledsharp} B$ then*

(i) $\mathcal{R}(B) = \mathcal{R}(A) \oplus^\perp \mathcal{R}(B - A)$;
(ii) $\mathcal{N}(A) = \mathcal{N}(B) \oplus \mathcal{R}(BB^\sharp - AA^\sharp)$;
(iii) $H = \mathcal{R}(A) \oplus^\perp \mathcal{R}(B - A) \oplus^\perp \mathcal{N}(B^*)$;
(iv) $H = \mathcal{R}(A) \oplus \mathcal{R}(BB^\sharp - AA^\sharp) \oplus \mathcal{N}(B)$.
(v) $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(BB^\sharp - AA^\sharp)$;

Proof.

(i) Suppose that $A <^{\textcircled{\#}} B$. By (26), we see that $\mathcal{R}(B-A) \subseteq \mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$. As $A <^{\textcircled{\#}} B$ implies $A <^{-} B$, by (21), $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A)$, so $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$.

(ii) If $A <^{\textcircled{\#}} B$ then $A = AA^{\textcircled{\#}}A = BA^{\textcircled{\#}}A = AA^{\textcircled{\#}}B$, hence $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. By (22) and (23), $A = BB^{-}A = AB^{-}B$, for every $A^{-} \in A\{1\}$. Because of

$$A(BB^{\sharp} - AA^{\sharp}) = ABB^{\sharp} - AAA^{\sharp} = AB^{\sharp}B - A = A - A = 0,$$

we have $\mathcal{R}(BB^{\sharp} - AA^{\sharp}) \subseteq \mathcal{N}(A)$. Let $x \in \mathcal{N}(B) \cap \mathcal{R}(BB^{\sharp} - AA^{\sharp})$, which means $Bx = 0$ and $x = (BB^{\sharp} - AA^{\sharp})z$ for some $z \in H$. We have

$$0 = Bx = B(BB^{\sharp} - AA^{\sharp})z = BBB^{\sharp}z - BAA^{\sharp}z = Bz - BA^{\sharp}Az \overset{(27)}{=} Bz - AA^{\sharp}Az = Bz - Az,$$

so $(B-A)z = 0$. From $A^2 = BA$ we see that $A^4 = B^3A$ and consequently

$$B^{\sharp}z = (B^{\sharp})^2 Bz = (B^{\sharp})^2 Az = (B^{\sharp})^2 A^4 (A^{\sharp})^3 z = (B^{\sharp})^2 B^3 A (A^{\sharp})^3 z = BA(A^{\sharp})^3 z = A^2 (A^{\sharp})^3 z = A^{\sharp}z.$$

It follows that

$$x = (BB^{\sharp} - AA^{\sharp})z = (BB^{\sharp} - BA^{\sharp})z = 0,$$

which means $\mathcal{N}(B) \cap \mathcal{R}(BB^{\sharp} - AA^{\sharp}) = \{0\}$ and we have $\mathcal{N}(B) \oplus \mathcal{R}(BB^{\sharp} - AA^{\sharp}) \subseteq \mathcal{N}(A)$.
Moreover, for any $x \in \mathcal{N}(A)$ we have $x = (BB^{\sharp} - AA^{\sharp})x + (x - (BB^{\sharp} - AA^{\sharp})x)$. The first term belongs to $\mathcal{R}(BB^{\sharp} - AA^{\sharp})$, and the second belongs to $\mathcal{N}(B)$ because $B(BB^{\sharp} - AA^{\sharp})x = Bx - BA^{\sharp}Ax = Bx$. We have proved that $\mathcal{N}(A) = \mathcal{N}(B) \oplus \mathcal{R}(BB^{\sharp} - AA^{\sharp})$.

(iii) The decomposition follows by (i) and the fact that $H = \mathcal{R}(B) \oplus^{\perp} \mathcal{N}(B^*)$.

(iv) The decomposition follows by (ii) and by the fact that $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

(v) Note that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and hence $\mathcal{R}(BB^{\sharp} - AA^{\sharp}) \subseteq \mathcal{R}(B)$. Since

$$H = \mathcal{R}(B) \oplus \mathcal{N}(B) = \mathcal{R}(A) \oplus \mathcal{R}(BB^{\sharp} - AA^{\sharp}) \oplus \mathcal{N}(B),$$

we conclude that $\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(BB^{\sharp} - AA^{\sharp})$. □

It will be nice to know if the condition (i) from Theorem 5.1 together with some additional condition imply $A <^{\textcircled{\#}} B$.

Theorem 5.2. *Let $A, B \in \mathcal{L}^1(H)$. Then $A <^{\textcircled{\#}} B$ if and only if $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$ and $BA^{\textcircled{\#}}B = A$.*

Proof. The only if part follows by (i) of Theorem 5.1 and the fact $BA^{\textcircled{\#}}B = AA^{\textcircled{\#}}A = A$. Conversely, suppose that $A, B \in \mathcal{L}^1(H)$ such that $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$ and $BA^{\textcircled{\#}}B = A$. It follows that $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and $H = \mathcal{R}(B) \oplus^{\perp} \mathcal{N}(B^*) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A) \oplus^{\perp} \mathcal{N}(B^*)$. Since the orthogonal complement of $\mathcal{R}(A)$ is unique subspace $\mathcal{N}(A^*)$ we conclude that $\mathcal{N}(A^*) = \mathcal{R}(B-A) \oplus^{\perp} \mathcal{N}(B^*)$. From (3) and (5) of Theorem 3.2 we obtain

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

$$A^{\textcircled{\#}} = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where $A_1 \in \mathcal{L}(\mathcal{R}(A))$ is invertible. Since $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$, the operator B has the following representation

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix}$$

for some operators B_i. Direct computation shows that condition $BA^{\textcircled{\#}}B = A$ is equivalent to

$$\begin{bmatrix} B_1A_1^{-1}B_1 & B_1A_1^{-1}B_2 \\ B_3A_1^{-1}B_1 & B_3A_1^{-1}B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From $B_1A_1^{-1}B_1 = A_1$ we obtain $A_1^{-1}B_1A_1^{-1}B_1 = B_1A_1^{-1}B_1A_1^{-1} = I_{\mathcal{R}(A)}$ so B_1 is invertible. Now, from $B_1A_1^{-1}B_2 = 0$ and $B_3A_1^{-1}B_1 = 0$ we obtain $B_2 = 0$ and $B_3 = 0$. It follows that

$$B - A = \begin{bmatrix} B_1 - A_1 & 0 \\ 0 & B_4 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

For any $x \in \mathcal{R}(A)$ we have $(B-A)x = (B_1 - A_1)x$. It follows that $(B_1 - A_1)x \in \mathcal{R}(B-A) \cap \mathcal{R}(A) = \{0\}$. Thus $B_1 = A_1$. An easy computation shows that $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$, i.e. $A <^{\textcircled{\dagger}} B$. □

Most of the matrix partial orders are characterized by some kind of simultaneous diagonalization. It is proven in [2] that $A <^{\textcircled{\dagger}} B$ if and only if

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^* \quad \text{and} \quad B = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix}U^*, \tag{29}$$

where U is unitary matrix, ΣK is invertible and Z is some matrix of index one. We do not know whether the matrices ΣL and Z are invertible or not. In the next theorem we consider infinite dimensional case and give better representations.

Theorem 5.3. *Let $A, B \in \mathcal{L}^1(H)$. The following conditions are equivalent:*

(i) $A <^{\textcircled{\dagger}} B$
(ii) $H = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A) \oplus^{\perp} \mathcal{N}(B^*)$,

$H = \mathcal{R}(A) \oplus \mathcal{R}(BB^\sharp - AA^\sharp) \oplus \mathcal{N}(B)$ and

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(BB^\sharp - AA^\sharp) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$
$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(BB^\sharp - AA^\sharp) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $A_1 \in \mathcal{L}(\mathcal{R}(A))$ and $B_1 \in \mathcal{L}(\mathcal{R}(BB^\sharp - AA^\sharp), \mathcal{R}(B-A))$ are invertible operators.

Proof. (i) ⇒ (ii): The decompositions of the space H follows from Theorem 5.1. Let $H_1 = \mathcal{R}(BB^\sharp - AA^\sharp)$. The matrix representation for the operator A is obvious, because $\mathcal{N}(A) = H_1 \oplus \mathcal{N}(B)$, by Theorem 5.1. Suppose that

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

The domain of operators B_{13}, B_{23} and B_{33} is $\mathcal{N}(B)$, so $B_{13} = 0$, $B_{23} = 0$ and $B_{33} = 0$. From $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$ we conclude $B_{31} = 0$ and $B_{32} = 0$. As $A <^{\textcircled{\dagger}} B$ we have $A^*A = A^*B$ and $A^2 = BA$. Suppose now that $x \in \mathcal{R}(A)$, which means $x = Az$ for some $z \in H$. We conclude that $Bx = B(Az) = A(Az) = Ax \in \mathcal{R}(A)$, which gives $B_{21} = 0$ and $B_{11} = A_1$. Suppose $x \in H_1$, which means $x = (BB^\sharp - AA^\sharp)z$ for some $z \in H$. We have:

$$Bx = B(BB^\sharp - AA^\sharp)z = Bz - A^2A^\sharp z = Bz - Az \in \mathcal{R}(B-A)$$

and therefore $B_{12} = 0$, so we have:

$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $B_1 = B_{22}$. From $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$ we conclude $B_1 \in \mathcal{L}(H_1, \mathcal{R}(B-A))$ is onto. Additionally, if $x \in H_1$ and $B_1x = 0$, we have $0 = B_1x = Bx$, so $x \in \mathcal{N}(B)$. From $x \in H_1 \cap \mathcal{N}(B) = \{0\}$ we conclude $x = 0$; hence B_1 is injective. Therefore, B_1 is invertible.

(ii) ⇒ (i): Let

$$C = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix}.$$

It is easy to see that $ACA = A$, $\mathcal{R}(C) = \mathcal{R}(A)$ and

$$\mathcal{N}(C) = \mathcal{R}(B-A) \oplus^{\perp} \mathcal{N}(B^*) = \mathcal{N}(A^*), \tag{30}$$

because $H = \mathcal{R}(A) \oplus^{\perp} \mathcal{N}(A^*)$ and $H = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A) \oplus^{\perp} \mathcal{N}(B^*)$. By definition, we conclude that $C = A^{\oplus}$. We check at once that:

$$AA^{\oplus} = BA^{\oplus} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix}$$

and

$$A^{\oplus}A = A^{\oplus}B = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix},$$

therefore $A \lt^{\oplus} B$. □

Remark 5.1. With the notation as in the proof of Theorem 5.3, we will show that

$$D = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & B_1^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix}$$

is the core inverse of B. It is obvious that $BDB = B$ and $\mathcal{N}(D) = \mathcal{N}(B^*)$. By (v) of Theorem 5.1, $\mathcal{R}(D) = \mathcal{R}(A) \oplus \mathcal{R}(BB^{\sharp} - AA^{\sharp}) = \mathcal{R}(B)$, so $D = B^{\oplus}$.

The following theorem, for complex matrix case, can be found in [11].

Theorem 5.4. *If $A, B \in \mathcal{L}^1(H)$ then $A \lt^{\oplus} B$ if and only if $A = BA^{\sharp}A = AA^{\dagger}B$.*

Proof. If $A \lt^{\oplus} B$ then $A = AA^{\oplus}A = BA^{\oplus}A = BA^{\sharp}A$ and $A = AA^{\oplus}A = AA^{\oplus}B = AA^{\dagger}B$. For converse implication suppose that $A = BA^{\sharp}A = AA^{\dagger}B$ and recall that by Theorem 3.4 $A = A^{\sharp}AA^{\dagger} = A^{\oplus}AA^{\dagger} = A^{\sharp}AA^{\oplus}$. We obtain that $BA^{\oplus} = BA^{\sharp}AA^{\oplus} = AA^{\oplus}$ and $A^{\oplus}B = A^{\oplus}AA^{\dagger}B = A^{\oplus}A$, so $A \lt^{\oplus} B$. □

Let us recall the equations from Theorem 3.5. It is known that the minus, sharp and star matrix partial orders can be characterized in the following way, see [13]:

$$\begin{aligned} &A \lt^{-} B \iff B\{1\} \subseteq A\{1\}, \\ &A \lt^{\sharp} B \iff B\{1,5\} \subseteq A\{1,5\}, \\ &A \lt^{*} B \iff B\{1,3,4\} \subseteq A\{1,3,4\}. \end{aligned}$$

Based on the properties of core inverse, it is natural to ask whether $A \lt^{\oplus} B$ is equivalent to $B\{1,3,6\} \subseteq A\{1,3,6\}$. For the proof of our hypothesis, we need the following lemma.

Lemma 5.1. *Let $B \in \mathcal{L}^1(H)$. Then B has the following representation*

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $B_1 \in \mathcal{L}(\mathcal{R}(B))$ is invertible. We also have following characterizations:

(i) $B\{1,3\} = \left\{ \begin{bmatrix} B_1^{-1} & 0 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} : X_3, X_4 \text{arbitrary} \right\};$

(ii) $B\{6\} = \left\{ \begin{bmatrix} B_1^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} : X_2, X_4 \text{arbitrary} \right\};$

(iii) $B\{3,6\} = \left\{ \begin{bmatrix} B_1^{-1} & 0 \\ 0 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} : X_4 \text{arbitrary} \right\}.$

Proof. The representation of B follows by Theorem 3.2. Note that if $\mathrm{ind}(B) \leqslant 1$ then

$$B\{3,6\} = B\{1,3\} \cap B\{6\}. \tag{31}$$

Indeed, suppose that $X \in B\{3,6\}$. Then $XB^2 = B$. Pre-multiplying this equation by B and post-multiplying by $B^{\#}$ (which exists as $\operatorname{ind}(B) \leqslant 1$), we have $BXB = B$, so $B\{3,6\} \subseteq B\{1,3\} \cap B\{6\}$. The converse inclusion is obvious. In view of this equality it is enough to show (i) and (ii). It is easy to see that the operator matrices from the right hand side belongs to the sets from the left hand side. Suppose that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

(i) The condition $BXB = B$ is equivalent to $B_1X_1B_1 = B_1$ so, by the invertibility of $B_1, X_1 = B_1^{-1}$. Direct computation shows that

$$BX = \begin{bmatrix} I_{\mathcal{R}(B)} & B_1X_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

Since $(BX)^* = BX$ and $\mathcal{R}(B) \perp \mathcal{N}(B^*)$ we obtain $B_1X_2 = 0$, so $X_2 = 0$.

(ii) As in (17) we have

$$B^2 = \begin{bmatrix} B_1^2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

so

$$\begin{bmatrix} X_1B_1^2 & 0 \\ X_3B_1^2 & 0 \end{bmatrix} = XB^2 = B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

We conclude that $X_1 = B_1^{-1}$ and $X_3 = 0$. $\square$

The equivalence of (i) and (ii) in the next theorem is proved for complex matrices in [12]. We note that the other equivalences have not been proved before, even in the case when A, B are complex matrices.

Theorem 5.5. *Let $A, B \in \mathcal{L}^1(H)$. Then the following conditions are equivalent*

(i) $A <^{\textcircled{\#}} B$;
(ii) $B\{1,3\} \subseteq A\{1,3\}$ *and* $B\{6\} \subseteq A\{6\}$;
(iii) $B\{3,6\} \subseteq A\{3,6\}$;
(iv) $B\{1\} \subseteq A\{1\}$ *and* $B\{3,6\} \subseteq A\{3,6\}$;
(v) $A *< B$ *and* $B\{6\} \subseteq A\{6\}$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $A <^{\textcircled{\#}} B$ and let $H_1 = \mathcal{R}(BB^{\sharp} - AA^{\sharp})$. By Theorem 5.1 it follows that

$$\begin{aligned} &\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B - A), \\ &\mathcal{R}(B) = \mathcal{R}(A) \oplus H_1, \\ &H = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B - A) \oplus^{\perp} \mathcal{N}(B^*), \\ &H = \mathcal{R}(A) \oplus H_1 \oplus \mathcal{N}(B). \end{aligned}$$

By Theorem 5.3 it follows that

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B - A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B - A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $A_1 \in \mathcal{L}(\mathcal{R}(A))$ and $B_1 \in \mathcal{L}(H_1, \mathcal{R}(B - A))$ are invertible operators. It follows that we can write

$$B = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where

$$C_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ H_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \end{bmatrix}.$$

Since A_1 and B_1 are invertible we conclude that C_1 is invertible and

$$C_1^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \end{bmatrix}.$$

By Lemma 5.1, we conclude that $X \in B\{1,3\}$ if and only if

$$X = \begin{bmatrix} C_1^{-1} & 0 \\ Y_1 & X_3 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix},$$

where $Y_1 \in \mathcal{L}(\mathcal{R}(B), \mathcal{N}(B))$ and $X_3 \in \mathcal{L}(\mathcal{N}(B^*), \mathcal{N}(B))$ are arbitrary. Since $\mathcal{R}(B) = \mathcal{R}(A) \oplus^{\perp} \mathcal{R}(B-A)$ we can write

$$Y_1 = [X_1 \quad X_2] : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \end{bmatrix} \to \mathcal{N}(B)$$

for some $X_1 \in \mathcal{L}(\mathcal{R}(A), \mathcal{N}(B))$ and $X_2 \in \mathcal{L}(\mathcal{R}(B-A), \mathcal{N}(B))$. Also, since $\mathcal{R}(B) = \mathcal{R}(A) \oplus H_1$, the null operator $0 \in \mathcal{L}(\mathcal{N}(B^*), \mathcal{R}(B))$ can be written in the form

$$0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} : \mathcal{N}(B^*) \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \end{bmatrix}.$$

It follows that $X \in B\{1,3\}$ if and only if

$$X = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & B_1^{-1} & 0 \\ X_1 & X_2 & X_3 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix} \tag{32}$$

for some X_i. By (30) we know that $\mathcal{N}(A^*) = \mathcal{R}(B-A) \oplus^{\perp} \mathcal{N}(B^*)$. Also, by Theorem 5.1, we have $\mathcal{N}(A) = H_1 \oplus \mathcal{N}(B)$. Now, in the same manner as in the proof of characterization (32), we can prove that $Y \in A\{1,3\}$ if and only if

$$Y = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ H_1 \\ \mathcal{N}(B) \end{bmatrix}$$

for some Y_i, $i = \overline{1,6}$. It is now clear that $B\{1,3\} \subseteq A\{1,3\}$. Using the same arguments, we can show that $B\{6\} \subseteq A\{6\}$ similarly.

(ii) $\Rightarrow$ (iii): We have proved that $B\{3,6\} = B\{1,3\} \cap B\{6\}$, see (31) in the proof of Lemma 5.1. It is now clear that $B\{1,3\} \subseteq A\{1,3\}$ and $B\{6\} \subseteq A\{6\}$ imply that $B\{3,6\} \subseteq A\{3,6\}$.

(iii) $\Rightarrow$ (i): Suppose now that $B\{3,6\} \subseteq A\{3,6\}$ and let us prove that $A <^{\textcircled{\dagger}} B$. As we know

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $B_1 \in \mathcal{L}(\mathcal{R}(B))$ is invertible. (This operator B_1 should not be confused with operator $B_1 \in \mathcal{L}(H_1, \mathcal{R}(B-A))$ which we used in the proof of the part (i) $\Rightarrow$ (ii).) Let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}$$

and let

$$X = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

Then $X \in B\{3,6\} \subseteq A\{3,6\}$. Since $AX = (AX)^*$ and $\mathcal{R}(B) \perp \mathcal{N}(B^*)$ it follows that $A_3 B_1^{-1} = 0$ and $A_1 B_1^{-1} = (A_1 B_1^{-1})^*$. Hence $A_3 = 0$ and

$$A_1^* B_1 = (A_1^* B_1)^*. \tag{33}$$

Since the subspaces $\mathcal{N}(B)$ and $\mathcal{N}(B^*)$ have the same complement subspace to H, namely $\mathcal{R}(B)$, it follows that there exists invertible operator $C \in \mathcal{L}(\mathcal{N}(B^*), \mathcal{N}(B))$. Taking $Z = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & C \end{bmatrix} \in B\{3,6\}$, we get

$$AZ = \begin{bmatrix} A_1B_1^{-1} & A_2C \\ 0 & A_4C \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

From $AZ = (AZ)^*$ wee see that $A_2C = 0$, so $A_2 = 0$. Now,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} \quad \text{and} \quad AXA = \begin{bmatrix} A_1B_1^{-1}A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and since $A = AXA$ we conclude that $A_4 = 0$. It follows that $\mathcal{R}(A) = \mathcal{R}(A_1) \subseteq \mathcal{R}(B)$ so

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

Therefore,

$$A^2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ W \end{bmatrix},$$

where W can be either $\mathcal{N}(B)$ or $\mathcal{N}(B^*)$. Now,

$$\begin{bmatrix} B_1^{-1}A_1^2 & 0 \\ 0 & 0 \end{bmatrix} = XA^2 = A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix},$$

which implies

$$B_1A_1 = A_1^2. \tag{34}$$

Since $\operatorname{ind}(A) \leqslant 1$, we have $\operatorname{ind}(A_1) \leqslant 1$ so $A_1^{\textcircled{\#}}$ exists and

$$A^{\textcircled{\#}} = \begin{bmatrix} A_1^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B) \end{bmatrix}.$$

Let us show that $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$ and $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$. This is equivalent to $A_1^{\textcircled{\#}}A_1 = A_1^{\textcircled{\#}}B_1$ and $A_1A_1^{\textcircled{\#}} = B_1A_1^{\textcircled{\#}}$. Using (33) and (34) and the basic properties of core inverse we deduce:

$$\begin{aligned} A_1^{\textcircled{\#}}B_1 &= A_1^{\textcircled{\#}}(A_1A_1^{\textcircled{\#}})^*B_1 = A_1^{\textcircled{\#}}(A_1^{\textcircled{\#}})^*A_1^*B_1 = A_1^{\textcircled{\#}}(A_1^{\textcircled{\#}})^*(A_1^*B_1)^* \quad (\text{by } (33)) = A_1^{\textcircled{\#}}(A_1^*B_1A_1^{\textcircled{\#}})^* \\ &= A_1^{\textcircled{\#}}\left(A_1^*B_1A_1(A_1^{\textcircled{\#}})^2\right)^* \quad \left(\text{by } A_1^{\textcircled{\#}} = A_1(A_1^{\textcircled{\#}})^2\right) = A_1^{\textcircled{\#}}\left(A_1^*A_1^2(A_1^{\textcircled{\#}})^2\right)^* \quad (\text{by } (34)) = A_1^{\textcircled{\#}}(A_1^*A_1A_1^{\textcircled{\#}})^* \\ &= A_1^{\textcircled{\#}}(A_1A_1^{\textcircled{\#}})^*A_1 = A_1^{\textcircled{\#}}A_1 \end{aligned}$$

and

$$B_1A_1^{\textcircled{\#}} = B_1A_1(A_1^{\textcircled{\#}})^2 = A_1^2(A_1^{\textcircled{\#}})^2 = A_1A_1^{\textcircled{\#}}.$$

It follows that $A \overset{\textcircled{\#}}{<} B$.

(i) $\Rightarrow$ (iv) In view of (i) $\Rightarrow$ (iii) it is enough to show that $B\{1\} \subseteq A\{1\}$. But $A \overset{\textcircled{\#}}{<} B$ implies that $A \overset{-}{<} B$ and as we now $A \overset{-}{<} B$ implies that $B\{1\} \subseteq A\{1\}$.

(iv) $\Rightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (v) We have shown in (28) that $A \overset{\textcircled{\#}}{<} B$ implies $A \overset{*}{<} B$. Also, we have shown in the part (i) $\Rightarrow$ (ii) that $A \overset{\textcircled{\#}}{<} B$ implies $B\{6\} \subseteq A\{6\}$.

(v) $\Rightarrow$ (iii) Suppose that $A \overset{*}{<} B$ and $B\{6\} \subseteq A\{6\}$. Let $X \in B\{3,6\}$. As we know $B\{3,6\} = B\{1,3\} \cap B\{6\}$ so $BXB = B$. By assumption it follows that $X \in A\{6\}$. It remains to show that $X \in A\{3\}$. As $A \overset{*}{<} B$ we have $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. By (22), we have $A = BB^-A$ for each $B^- \in B\{1\}$. It follows that

$$AX = AA^\dagger AX = (A^\dagger)^*A^*AX = (A^\dagger)^*A^*BX = (AA^\dagger)^*(BX)^* = (BXAA^\dagger)^* = (BXBB^-AA^\dagger)^* = (BB^-AA^\dagger)^* = (AA^\dagger)^*,$$

so $(AX)^* = AX$. $\square$

It remains to show that the core partial order is actually partial order on the set of bounded Hilbert space operators with indices less or equal one.

Theorem 5.6. *The relation "$<^{\textcircled{\#}}$" is a partial order on the set $\mathcal{L}^1(H)$.*

Proof. The reflexivity and transitivity follows by Theorem 5.5. Since $A <^{\textcircled{\#}} B$ implies $A <^{-} B$ and $<^{-}$ is a partial order, the anti-symmetry of $<^{\textcircled{\#}}$ follows. Thus, $<^{\textcircled{\#}}$ is a partial order on $\mathcal{L}^1(H)$. $\square$

Theorem 5.6 can be proved without using Theorem 5.5.

6. Some remarks

1. Any $A \in \mathcal{L}(H)$ can be written in the form

$$A = \begin{bmatrix} A_3 & A_4 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}. \tag{35}$$

It is shown in [5] that if $\mathrm{ind}(A) \leqslant 1$ then A_3 is invertible. It is easy to check that

$$A^{\textcircled{\#}} = \begin{bmatrix} A_3^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}. \tag{36}$$

The representations (35) and (36) are analogous to representations (1) given by Baksalary and Trenkler in [2]. Using these decompositions one can obtain the characterization of the core partial order analogous to (29). Similarly, if $\mathrm{ind}(A) \leqslant 1$ then

$$A = \begin{bmatrix} A_3 & 0 \\ A_4 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix},$$
$$A_{\textcircled{\#}} = \begin{bmatrix} A_3^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

2. We should emphasize the following. Although for the dual core inverse $A_{\textcircled{\#}}$ all properties analogous to those of core inverse $A^{\textcircled{\#}}$ are valid, the proofs of some properties requires additional caution. Namely, we often use the following trick in the proofs regarding the core inverse. Let $C \in \mathcal{L}(H), H = H_1 \oplus H_2 = H_3 \oplus H_4 = H_3 \oplus H_5$ such that $H_1,\ H_2,\ H_3,\ H_4,\ H_5$ are closed subspaces of H and $H_2 \subseteq \mathcal{N}(C), \mathcal{R}(C) \subseteq H_3$. The operator C has the representation ($C_1 = C|_{H_1} : H_1 \mapsto H_3$)

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \to \begin{bmatrix} H_3 \\ H_4 \end{bmatrix}.$$

We can also write the following

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \to \begin{bmatrix} H_3 \\ H_5 \end{bmatrix}. \tag{37}$$

Observe that a similar method can not be applied to the domain. Namely, if $H = H_6 \oplus H_2$, where H_6 is closed, the only thing we can write is

$$C = \begin{bmatrix} C_2 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} H_6 \\ H_2 \end{bmatrix} \to \begin{bmatrix} H_3 \\ H_4 \end{bmatrix}, \tag{38}$$

where $C_2 = C|_{H_6} : H_1 \to H_3$. In the proofs for some properties of dual core inverse (for example $(A_{\textcircled{\#}})^2 A = A_{\textcircled{\#}}$ or $A^{<}{}_{\textcircled{\#}}B \iff B\{4,8\} \subseteq A\{4,8\}$) the representation (37) is of no use, but the representation (38) is convenient. As a drawback we have some new operator C_2, but this fact appears not to be problem. The proofs of all properties of dual core inverse have the same idea as those of core inverse, with the previously described observation.

3. For dual core inverse there is the following theorem analogous to the Theorem 3.4:

Theorem 6.1. *Let $A \in \mathcal{L}^1(H)$ and $m \in \mathbb{N}$. Then:*

(i) $A_{\textcircled{\#}} \in A\{1,2\}$; (ii) $A_{\textcircled{\#}}A = A^\dagger A = P_{\mathcal{R}(A^*)}$, *so* $(A_{\textcircled{\#}}A)^* = A_{\textcircled{\#}}A$;

(iii) $A^2 A_{\textcircled{\#}} = A$; (iv) $(A_{\textcircled{\#}})^2 A = A_{\textcircled{\#}}$; (v) $A_{\textcircled{\#}} = P_{\mathcal{R}(A^*)} A^\sharp$;

(vi) $AA_{\textcircled{\#}} = AA^\sharp$; (vii) $A_{\textcircled{\#}} = A^\dagger A A^\sharp$; (viii) $A_{\textcircled{\#}}$ *is EP*;

(ix) $(A_{\textcircled{\#}})^\dagger = (A_{\textcircled{\#}})^\sharp = (A_{\textcircled{\#}})_{\textcircled{\#}} = P_{\mathcal{R}(A^*)}A$; (x) $A(A_{\textcircled{\#}})^2 = A^\sharp$;

(xi) $(A_{\textcircled{\#}})^m = (A^m)_{\textcircled{\#}}$; (xii) $((A_{\textcircled{\#}})_{\textcircled{\#}})_{\textcircled{\#}} = A_{\textcircled{\#}}$.

4. Similarly as in Theorems 3.5 and 3.7, one can show that $X = A_{\textcircled{\#}}$ if and only if one of the following equivalent conditions hold:
   (i) (1) $AXA = A$ (2) $XAX = X$ (4) $(XA)^* = XA$ (8) $A^2X = A$ (9) $X^2A = X$;
   (ii) $X = A^\dagger A A^\sharp$;
   (iii) $X \in A\{1,2,4\}$ and $AX = AA^\sharp$.
5. The following theorem for dual core inverse is analogous to the Theorem 3.9.

Theorem 6.2. *Let $A \in \mathcal{L}^1(H)$. Then:*
   (i) $A_{\textcircled{\#}} = 0 \iff A = 0$; (ii) $A_{\textcircled{\#}} = P_{\mathcal{R}(A^*)} \iff A^2 = A$;
   (iii) $A_{\textcircled{\#}} = A \iff A^3 = A$ *and A is EP;*
   (iv) $A_{\textcircled{\#}} = A^* \iff A$ *is partial isometry and EP.*
6. Theorem 3.11 also has equivalent form for dual core inverse.

Theorem 6.3. *Let $A, B \in \mathcal{L}(H)$ be orthogonal projectors such that $\mathcal{R}(AB)$ is closed. Then $\mathcal{R}(BAB)$ is closed,* $\operatorname{ind}(AB) \leqslant 1$ *and*

$$(AB)_{\textcircled{\#}} = (BAB)^\dagger. \tag{39}$$

Proof.

$$(AB)_{\textcircled{\#}} = (AB)^\dagger (AB)^{\textcircled{\#}} AB = (AB)^\dagger (ABA)^\dagger AB = (AB)^\dagger (ABBA)^\dagger AB = (AB)^\dagger (ABB^*A^*)^\dagger AB = (AB)^\dagger (B^*A^*)^\dagger = (AB)^\dagger ((AB)^\dagger)^*$$
$$= (B^*A^*AB)^\dagger = (BAAB)^\dagger = (BAB)^\dagger. \quad \square$$

7. Using dual core inverse $A_{\textcircled{\#}}$ we can define another partial order. For $A, B \in \mathcal{L}^1(H)$ we write $A <_{\textcircled{\#}} B$ if $AA_{\textcircled{\#}} = BA_{\textcircled{\#}}$ and $A_{\textcircled{\#}}A = A_{\textcircled{\#}}B$. As in Theorem 5.3, it can be shown that $A <_{\textcircled{\#}} B$ if and only if

$$H = \mathcal{R}(A^*) \oplus^\perp \mathcal{R}((B-A)^*) \oplus^\perp \mathcal{N}(B),$$
$$H = \mathcal{R}(A) \oplus \mathcal{R}(B-A) \oplus \mathcal{N}(B) \quad \text{and}$$
$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{R}((B-A)^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B) \end{bmatrix},$$
$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{R}((B-A)^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B) \end{bmatrix},$$

where A_1 and B_1 are invertible operators.

8. As in Theorem 5.5, one can show that $A <_{\textcircled{\#}} B$ if and only if $B\{1,4\} \subseteq A\{1,4\}$ and $B\{8\} \subseteq A\{8\}$ if and only if $B\{4,8\} \subseteq A\{4,8\}$ when $A, B \in \mathcal{L}^1(H)$. Also, the relation "$<_{\textcircled{\#}}$" is a partial order on $\mathcal{L}^1(H)$.
9. $A^{\textcircled{\#}}$ is reflexive generalized inverse, and $A^{\textcircled{\#}} = A^{(1,2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$; recall $A^\dagger = A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$, $A^\sharp = A^{(1,2)}_{\mathcal{R}(A),\mathcal{N}(A)}$. Also, $A_{\textcircled{\#}} = A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}$.
10. $AA^{\textcircled{\#}}$, $A_{\textcircled{\#}}A$ are orthogonal, and $A^{\textcircled{\#}}A$, $AA_{\textcircled{\#}}$ oblique projectors;
11. From Theorem 3.4 (ix), $A^{\textcircled{\#}} = (AP_{\mathcal{R}(A)})^\dagger = (A^2A^\dagger)^\dagger = (A^2A^\dagger)^\sharp$;
12. From Theorem 3.4 (vii), $A^{\textcircled{\#}} = A^\sharp AA^- AA^\dagger = P_{\mathcal{R}(A),\mathcal{N}(A)} A^- P_{\mathcal{R}(A)}$; for arbitrary $A^- \in A\{1\}$.
13. An easy computation shows that $A^{\textcircled{\#}}A - AA^{\textcircled{\#}}$, $AA_{\textcircled{\#}} - A_{\textcircled{\#}}A$ are nilpotent of order 2.
14. $(A^{\textcircled{\#}})^m = (A^\sharp)^{m-1}A^\dagger$, $(A_{\textcircled{\#}})^m = A^\dagger (A^\sharp)^{m-1}$, $m \geqslant 2$. The proof is by induction on m.
15. $(A^{\textcircled{\#}})^m = (A^\sharp)^{m-1}A^{\textcircled{\#}}$, $(A_{\textcircled{\#}})^m = A^{\textcircled{\#}}(A^\sharp)^{m-1}$, $m \geqslant 1$. It follows by previous one.
16. Let $p(t) = \sum_{k=0}^n a_k t^k$ be some polynomial. Then:

$$p(A^{\textcircled{\#}}) = \sum_{k=0}^n a_k (A^{\textcircled{\#}})^k = \sum_{k=1}^n a_k (A^\sharp)^{k-1} A^{\textcircled{\#}} + a_0 I = a_0 I + q(A^\sharp) A^{\textcircled{\#}},$$

where $q(t) = \frac{p(t)-a_0}{t}$. Another way is

$$p(A^{\textcircled{\#}}) = p(A^\sharp)AA^\dagger + a_0 I - a_0 AA^\dagger = a_0(I - AA^\dagger) + p(A^\sharp)AA^\dagger.$$

For dual core inverse we have $p(A_{\textcircled{\#}}) = a_0 I + A_{\textcircled{\#}} q(A^\sharp)$ and

$$p(A_{\textcircled{\#}}) = a_0(I - A^\dagger A) + A^\dagger A p(A^\sharp).$$

17. Recall the definition of Bott–Duffin inverse. Let $A \in \mathcal{L}(H)$ and let L be closed subspace of H. The Bott–Duffin inverse of A with respect to L is defined by

$$A_L^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1},$$

where P_L denotes an orthogonal projector on L. The Bott–Duffin inverse arises in electrical network theory, see for example [1, Section 2.10]. The Bott–Duffin and core inverse are related in the following way:

$$A_{\mathcal{R}(A)}^{(-1)} = P_{\mathcal{R}(A)}[AP_{\mathcal{R}(A)} + P_{\mathcal{N}(A^*)}]^{-1} = A^{\text{\textcircled{\#}}}$$

provided that $\mathrm{ind}(A) \leqslant 1$. It follows by representations (16), (3) and (5) and

$$P_{\mathcal{N}(A^*)} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{N}(A^*)} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

On analogous way one can prove that $A_{\text{\textcircled{\#}}} = A_{\mathcal{R}(A^*)}^{(-1)}$.

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