On closed upper and lower semi-Browder operators

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Abstract

We give several necessary and sufficient conditions for a closed operator to be upper (lower) semi-Browder. We also apply these results to give some characterizations of upper (lower) semi-Browder spectrum.

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1 Introduction and preliminaries

Let X be an infinite dimensional Banach space. We denote by $\mathcal{L}(X)$ the set of all linear operators on X. The class $\mathcal{C}(X)$ ($\mathcal{BL}(X)$) consists of all closed (linear bounded) operators on X. As usual, $\mathcal{K}(X)$ ($\mathcal{F}(X)$) is the set of all compact (finite rank) operators on X. Let $T \in \mathcal{C}(X)$. We use $\mathcal{D}(T)$ to denote the domain of the operator T and, in general, $\mathcal{D}(T) \neq X$. The null space of T, denoted by $\mathcal{N}(T)$, is the set $\mathcal{N}(T) = \{x \in \mathcal{D}(T) : Tx = 0\}$. The set $\mathcal{R}(T) = \{Tx : x \in \mathcal{D}(T)\}$ is the range of T. Let $\alpha(T) = \dim \mathcal{N}(T)$ if $\mathcal{N}(T)$ is finite dimensional, and let $\alpha(T) = \infty$ if $\mathcal{N}(T)$ is infinite dimensional. Similarly, let $\beta(T) = \dim X/\mathcal{R}(T) = \operatorname{codim} \mathcal{R}(T)$ if $X/\mathcal{R}(T)$ is finite dimensional, and let $\beta(T) = \infty$ if $X/\mathcal{R}(T)$ is infinite dimensional.

Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of all positive (non-negative) integers, and let $\mathbb{C}$ denote the set of all complex numbers. For $T \in \mathcal{C}(X)$ we consider iterates

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$T^2, T^3, \ldots$ of T. If $n > 1$, then

$$\mathcal{D}(T^n) = \{x \in X : x, Tx, \ldots, T^{n-1}x \in \mathcal{D}(T)\},$$

and $T^n x = T(T^{n-1}x)$. It is well known that $\mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n)$ if $n \in \mathbb{N}_0$. Let $T^0 = I$ (the identity operator on X, with $\mathcal{D}(I) = X$). Thus $\mathcal{N}(T^0) = \{0\}$ and $\mathcal{R}(T^0) = X$. It is also well known that if $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$, then $\mathcal{N}(T^k) = \mathcal{N}(T^n)$ for $k \geq n$. In this case the ascent of T, denoted by $\operatorname{asc}(T)$, is the smallest $n \in \mathbb{N}_0$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$. If such an n does not exist, then $\operatorname{asc}(T) = \infty$. Similarly, if $\mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)$, then $\mathcal{R}(T^k) = \mathcal{R}(T^n)$ for $k \geq n$. In this case the descent of T, denoted by $\operatorname{dsc}(T)$, is the smallest $n \in \mathbb{N}_0$ such that $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$. If such an n does not exist, then $\operatorname{dsc}(T) = \infty$.

For $T \in \mathcal{C}(X)$ we can also define the generalized kernel of T by $\mathcal{N}^\infty(T) = \cup_{n=1}^\infty \mathcal{N}(T^n)$ and the generalized range of T by $\mathcal{R}^\infty(T) = \bigcap_{n=1}^\infty \mathcal{R}(T^n)$.

We need the following auxiliary result (see [16, Lemma 3.4] and [3, Lemma 2.1])

Lemma 1.1. *Let $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$, be a linear operator.*
(i) *If* $\operatorname{asc}(T) < \infty$*, then* $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$*.*
(ii) *If* $\alpha(T) < \infty$ *and* $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$*, then* $\operatorname{asc}(T) < \infty$*.*

Let X' be the space of bounded linear functionals on X. The adjoint operator T' of the densely defined closed operator T is defined by

$$\mathcal{D}(T') = \{y' \in X' : y'T \text{ is bounded on } \mathcal{D}(T)\},$$

and for $y' \in \mathcal{D}(T')$, $T'y' = \overline{y'T}$, where $\overline{y'T}$ is the unique continuous linear extension of $y'T$ to all of X.

We prove the following result.

Lemma 1.2. *Let $T \in \mathcal{C}(X)$ be a densely defined operator and $S \in \mathcal{BL}(X)$. Then $(T - S)' = T' - S'$.*

Proof. The operator $T - S$ is densely defined because $\mathcal{D}(T - S) = \mathcal{D}(T)$ and thus $(T - S)'$ exists. For $y' \in X'$, $y'(T - S)$ is bounded on $\mathcal{D}(T)$ if and only if $y'T$ is bounded on $\mathcal{D}(T)$, and hence, $\mathcal{D}((T - S)') = \mathcal{D}(T') = \mathcal{D}(T' - S')$. For $y' \in \mathcal{D}((T - S)') = \mathcal{D}(T' - S')$ it follows that

$$\begin{aligned} (T - S)'y' &= \overline{y'(T - S)} = \overline{y'T - y'S}, \\ (T' - S')y' &= T'y' - S'y' = \overline{y'T} - y'S. \end{aligned}$$

Since the functionals $\overline{y'T - y'S}$ and $\overline{y'T} - y'S$ coincide on $\mathcal{D}(T)$, they coincide on X. Therefore, $(T - S)' = T' - S'$. $\square$

An operator $T \in \mathcal{C}(X)$ is *bounded below* if there exists $c > 0$ such that

$$c\|x\| \leq \|Tx\| \quad \text{for every} \quad x \in \mathcal{D}(T).$$

Recall that $T \in \mathcal{C}(X)$ is bounded below if and only if T is injective with closed range [15, Theorem 5.1, p. 70].

Let consider following subsets of $\mathcal{C}(X)$:

$$\begin{aligned}
\Phi_+(X) &= \{T \in \mathcal{C}(X) : \alpha(T) < \infty \ \text{ and } \ \mathcal{R}(T) \ \text{ is closed}\}, \\
\Phi_-(X) &= \{T \in \mathcal{C}(X) : \beta(T) < \infty\}, \\
\Phi_\pm(X) &= \Phi_+(X) \cup \Phi_-(X), \\
\Phi(X) &= \Phi_+(X) \cap \Phi_-(X), \\
\mathcal{B}_+(X) &= \{T \in \mathcal{C}(X) : T \in \Phi_+(X) \ \text{ and } \ \mathrm{asc}(T) < \infty\}, \\
\mathcal{B}_-(X) &= \{T \in \mathcal{C}(X) : T \in \Phi_-(X) \ \text{ and } \ \mathrm{dsc}(T) < \infty\}, \\
\mathcal{B}(X) &= \mathcal{B}_+(X) \cap \mathcal{B}_-(X).
\end{aligned}$$

The classes $\Phi_+(X), \Phi_-(X), \Phi_\pm(X), \Phi(X), \mathcal{B}_+(X), \mathcal{B}_-(X)$ and $\mathcal{B}(X)$, respectively, consist of all upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm, Fredholm, upper semi-Browder, lower semi-Browder and Browder operators. For upper and lower semi-Fredholm operators the index is defined by $i(A) = \alpha(A) - \beta(A)$. If $A \in \Phi_+(X) \backslash \Phi_-(X)$, then $i(A) = -\infty$, and if $A \in \Phi_-(X) \backslash \Phi_+(X)$, then $i(A) = +\infty$. Corresponding spectra of $T \in \mathcal{C}(X)$ are defined as:

$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not bounded below$\}$-the approximate point spectrum,

$\sigma_d(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not surjective$\}$-the defect spectrum,

$\sigma_{\Phi_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+(X)\}$-the upper semi-Fredholm spectrum,

$\sigma_{\Phi_-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_-(X)\}$-the lower semi-Fredholm spectrum,

$\sigma_{\mathcal{B}_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}_+(X)\}$-the upper semi-Browder spectrum,

$\sigma_{\mathcal{B}_-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}_-(X)\}$-the lower semi-Browder spectrum.

For $T \in \mathcal{C}(X)$, set $\rho(T)$ for the resolvent set of T and $\rho_\Phi(T)$ for the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda \in \Phi(X)$.

In general, let us consider a linear operator A such that its domain $\mathcal{D}(A)$ is contained in a linear space X, and its range $\mathcal{R}(A)$ is contained in a linear space Y, and $\mathcal{D}(A)$ need not be the whole space X. For the convenience, we express this by saying that A is a linear operator $A : X \to Y$. If X_0, Y_0 are linear subspaces of X, Y respectively, we can define a linear operator $A_0 : X_0 \to Y_0$ by setting $A_0 x = Ax$ for every $x \in X_0$ such that $x \in \mathcal{D}(A)$ and $Ax \in Y_0$. We shall say that A_0 is induced by A in the pair X_0, Y_0.

A linear operator T, $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$, is *semi regular* if $\mathcal{R}(T)$ is closed

$$\mathcal{N}(T) \subset \mathcal{R}(T^m) \text{ for each } m \in \mathbb{N}. \tag{1.1}$$

We remark that condition (1.1) is equivalent to each of the following conditions:

$$\mathcal{N}(T^n) \subset \mathcal{R}(T) \text{ for each } n \in \mathbb{N}; \tag{1.2}$$

$$\mathcal{N}(T^n) \subset \mathcal{R}(T^m) \text{ for each } n \in \mathbb{N} \text{ and each } m \in \mathbb{N}. \tag{1.3}$$

We shall use the Kato decomposition theorem (see [3, Proposition 2.3], [11, Theorem 4]):

Theorem 1.1. *Let X be a Banach space and $T \in \Phi_{\pm}(X)$. Then there exists $d \in \mathbb{N}$ such that T has a Kato decomposition of degree d, i.e. there exists a pair (M, N) of two closed subspaces of X such that:*

(i) $X = M \oplus N$,

(ii) $T = T_M \oplus T_N$,

(iii) $T(M \cap \mathcal{D}(T)) \subset M$, $T_M : M \cap \mathcal{D}(T) \to M$ *is a closed and semi regular operator,*

(iv) $N \subset \mathcal{D}(T)$, $\dim N < \infty$, $T(N) \subset N$ and $T_N : N \to N$ is a bounded and nilpotent operator of degree d.

The necessary and sufficient conditions for a bounded operator to be upper (lower) semi-Browder are well-known (see [2, Theorems 2.62, 2.63], [17, Theorems 3, 4]), as well characterizations of upper (lower) semi-Browder spectrum of a bounded operator [14], [13, Corollary 19.20, Theorem 19.21], [1, Corollaries 3.45, 3.47], [2, Theorems 4.4, 4.5]. The purpose of this paper is to extend that results to a larger class, that is, the class of closed operators. Precisely we generalize Theorems 3 and 4 from [17] to the case of closed operators. The present paper is also motivated by a paper of T. Alvarez, F. Fakhfakh and M. Mnif, [3], in which the authors, continuing investigation started in [4, 5], gave one characterization of closed upper (lower) semi-Browder operators (Theorems 3.2, 3.3), as well a characterization of upper (lower) semi-Browder spectrum (Theorems 4.1, 4.2). In our paper we extend Theorems 3.2, 3.3, 4.1, 4.2, 4.3 and 4.4 from [3] to more general settings, and also give another equivalent characterizations of closed upper and lower semi-Browder operators, as well characterizations of corresponding spectra.

In this paper we use the definition of commutativity of linear operators in the same way as Goldman and Kračkovskii did in [8].

Definition 1.1. Let $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$, be a linear operator and $S \in \mathcal{L}(X)$. We say that S commutes with T if

(i) $Sx \in \mathcal{D}(T)$ for every $x \in \mathcal{D}(T)$,

(ii) $STx = TSx$ for every $x \in \mathcal{D}(T)$.

Notice that there is a slightly more general definition of commutativity by Kaashoek and Lay in [10]:

Definition 1.2. *Let X be a Banach space, $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$ and $K : \mathcal{D}(K) \to X$, $\mathcal{D}(K) \subset X$, two linear operators. We say that K commutes with T if*

(i) $\mathcal{D}(T) \subset \mathcal{D}(K)$,

(ii) $Kx \in \mathcal{D}(T)$ *whenever* $x \in \mathcal{D}(T)$,

(iii) $KTx = TKx$ *for* $x \in \mathcal{D}(T^2)$.

T. Alvarez et al., using the Kato decomposition, proved [3, Theorem 3.2] that if T is upper semi-Browder then there exists $A \in \mathcal{C}(X)$ and $B \in \mathcal{F}(X)$ such that $T = A + B$, $\mathcal{D}(A) = \mathcal{D}(T)$, A is bounded below and B commutes with T in the sense of Definition 1.2; and the converse assertion holds if

$$T(\mathcal{D}(T)) \subset \mathcal{D}(T). \tag{1.4}$$

We see that Definition 1.2 is more general then Definition 1.1. However, notice that the condition (1.4) implies $\mathcal{D}(T) = \mathcal{D}(T^2) = \mathcal{D}(T^3) = \cdots$, and therefore,

$$\text{Definition } 1.2 + (1.4) \textit{ is stronger then } \text{Definition } 1.1. \tag{1.5}$$

In this paper we use Definition 1.1 and also the Kato decomposition to get the previously mentioned result in a different way. To be precise, because of (1.5), our result (the equivalence (2.2.1)$\Longleftrightarrow$ (2.2.6) in Theorem 2.2): $T \in \mathcal{B}_+(X)$ if and only if there exist $A \in \mathcal{C}(X)$ and $B \in \mathcal{F}(X)$ such that $T = A + B$, A is a bounded below operator with $\mathcal{D}(A) = \mathcal{D}(T)$, $B \in \mathcal{F}(X)$ and B commutes with T in the sense of Definition 1.1, is an extension of Theorem 3.2 in [3].

T. Alvarez et al. also proved [3, Theorem 3.3] that if $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$, T is lower semi-Browder and $\rho_\Phi(T) \neq \emptyset$, then there exist $A \in \mathcal{C}(X)$ and $B \in \mathcal{F}(X)$ such that $T = A + B$, $\mathcal{D}(A) = \mathcal{D}(T)$, A is surjective and B commutes with T in the sense of Definition 1.2; and the converse assertion holds if $\rho(T) \neq \emptyset$ and $T'(\mathcal{D}(T')) \subset \mathcal{D}(T')$.

Now it is important to mention the following result [5, Lemma 3.3]:

Lemma 1.3. *Let $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$, $K \in \mathcal{BL}(X)$ and K commutes with T in the sense of Definition 1.2. If $\rho(T) \neq \emptyset$ or $\rho(T+K) \neq \emptyset$, then $KTx = TKx$ for all $x \in \mathcal{D}(T)$, that is, K commutes with T in the sense of Definition 1.1.*

We prove (the equivalence (3.1.1)$\Longleftrightarrow$ (3.1.6) in Theorem 3.1): If $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$ and $\rho_\Phi(T) \neq \emptyset$, then $T \in \mathcal{B}_-(X)$ if and only if there exist $A \in \mathcal{C}(X)$ and $B \in \mathcal{F}(X)$ such that $T = A + B$, A is a surjective operator with $\mathcal{D}(A) = \mathcal{D}(T)$ and B commutes with T in the sense of Definition 1.1. According to Lemma 1.3 we remark that this assertion improves Theorem 3.3 in [3].

In the following section we investigate properties of upper semi-Browder operators. Lower semi-Browder operators are considered in the third section.

2 Upper semi-Browder operators and upper semi-Browder spectrum

First we prove the following result useful for the proof of the main result of this section, which is otherwise more elementary for bounded operators.

Theorem 2.1. Let $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$, be a linear operator and $S \in \mathcal{L}(X)$ such that S is bijective, $S\left(\mathcal{D}(T)\right) = \mathcal{D}(T)$, and S commutes with T. Then

$$\mathcal{N}(T - S) \subset \mathcal{R}^\infty(T).$$

Proof. Let $x \in \mathcal{D}(T)$. Then there exists $u \in \mathcal{D}(T)$ such that $Su = x$ and

$$TSu = STu \Longrightarrow Tx = STS^{-1}x \Longrightarrow S^{-1}Tx = TS^{-1}x.$$

Therefore, S^{-1} commutes with T.

Let $x \in \mathcal{N}(T-S)$. Then $Tx = Sx \in \mathcal{D}(T)$ and $T^2x = T(Tx) = T(Sx) = S(Tx) = S^2x \in \mathcal{D}(T)$. By induction we conclude

$$T^n x = S^n x \in \mathcal{D}(T) \quad \text{for every} \quad n \in \mathbb{N}_0. \tag{2.6}$$

Observe that from $Tx = Sx$ it follows that

$$TS^{-1}x = S^{-1}Tx = x = SS^{-1}x$$

and therefore, $S^{-1}x \in \mathcal{N}(T-S)$. Consequently,

$$(S^{-1})^m x \in \mathcal{N}(T-S) \quad \text{for every } m \in \mathbb{N}_0. \tag{2.7}$$

As above, from (2.7) we conclude

$$T^n(S^{-1})^m x = S^n(S^{-1})^m x \in \mathcal{D}(T) \quad \text{for every} \quad n,\ m \in \mathbb{N}_0. \tag{2.8}$$

Fix $n_0 \in \mathbb{N}$. From $T^{n_0}x = S^{n_0}x = SS^{n_0-1}x$ it follows that $S^{-1}T^{n_0}x = S^{n_0-1}x$, and now from (2.6) and the fact that S^{-1} commutes with T we get $T^{n_0}S^{-1}x = S^{n_0-1}x$. Continuing this method and using (2.8) we obtain $T^{n_0}(S^{-1})^{n_0}x = x$. Hence $x \in \mathcal{R}(T^{n_0})$ and since n_0 was arbitrary, we get $x \in \mathcal{R}^\infty(T)$. $\square$

The following result was proved for $T \in \mathcal{L}(X)$ ([9, Lemma 38.1]), but using the same technic we can express it for a linear operator $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$.

Lemma 2.1. *Let $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$, be a linear operator with $\alpha(T) < \infty$. Then $T(\mathcal{D}(T) \cap \mathcal{R}^\infty(T)) = \mathcal{R}^\infty(T)$.*

Let $T : \mathcal{D}(T) \to X$, $\mathcal{D}(T) \subset X$ be a linear operator and $\epsilon > 0$. We write

$$\mathrm{comm}_\epsilon^{-1}(T) = \{S \in \mathcal{BL}(X)^{-1} : S \ \text{ commutes with } T,\ \|S\| < \epsilon\},$$

where $\mathcal{BL}(X)^{-1}$ is the group of invertible elements in $\mathcal{BL}(X)$.

For $T \in \mathcal{C}(X)$ we say that T is *almost bounded below (surjective)* if there exists an $\epsilon > 0$ such that $T - \lambda I$ is a bounded below (surjective) operator for $\lambda \in \mathbb{C}$, $0 < |\lambda| < \epsilon$.

Let $P \in \mathcal{BL}(X)$ be a projector which commutes with $T \in \mathcal{C}(X)$. Put $X_0 = \mathcal{R}(P)$ and $X_1 = \mathcal{N}(P)$. It is easy to check that

$$T\left(X_j \cap \mathcal{D}(T)\right) \subset X_j \quad \text{for} \quad j = 0, 1.$$

Now we define operators $T_j : X_j \to X_j,\ j = 0, 1$ as

$$T_j x = Tx, \quad x \in X_j \cap \mathcal{D}(T), \quad j = 0, 1.$$

The following theorem is our first main result where we give several necessary and sufficient conditions for a closed operator to be upper semi-Browder.

Theorem 2.2. *If X is a Banach space and $T \in \mathcal{C}(X)$, then the following conditions are equivalent:*

(2.2.1) *T is upper semi-Browder;*

(2.2.2) *T is upper semi-Fredholm, and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ with $S(\mathcal{D}(T)) = \mathcal{D}(T)$, it follows that $T - S$ is bounded below;*

(2.2.3) *T is upper semi-Fredholm and almost bounded below;*

(2.2.4) *There exists a projector $P \in \mathcal{F}(X)$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is nilpotent bounded operator, and T_1 is bounded below;*

(2.2.5) *There exists a projector $P \in \mathcal{F}(X)$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, TP is nilpotent bounded operator, and $T + P$ is bounded below;*

(2.2.6) *There exists $B \in \mathcal{F}(X)$ which commutes with T such that $T - B$ is bounded below;*

(2.2.7) *There exists $B \in \mathcal{K}(X)$ which commutes with T such that $T - B$ is bounded below.*

Proof. (2.2.1) $\Longrightarrow$ (2.2.2): Suppose that $T \in \mathcal{C}(X)$ is upper semi-Browder. Then $T \in \Phi_+(X)$, so by [11, Lemma 543], $\mathcal{R}(T^n)$ is closed for every $n \in \mathbb{N}$. According to [11, Theorem 1] there exists some $\epsilon_1 > 0$ such that if $B \in \mathcal{BL}(X)$ and $\|B\| < \epsilon_1$, then $T - B \in \Phi_+(X)$. Let $X_1 = \mathcal{R}^\infty(T)$. X_1 is a Banach space and $T(\mathcal{D}(T) \cap X_1) = X_1$ by Lemma 2.1. The operator $T_1 : X_1 \to X_1$ induced by T is closed with $\alpha(T_1) < \infty$ and $\beta(T_1) = 0$. From $T_1 \in \Phi(X_1)$, again by [11, Theorem 1], it follows that there exists some $\epsilon_2 > 0$, such that for $B \in \mathcal{BL}(X_1)$, $\|B\| < \epsilon_2$ implies $T_1 - B \in \Phi(X_1)$, $\alpha(T_1 - B) \le \alpha(T_1)$, $\beta(T_1 - B) \le \beta(T_1)$, $i(T_1 - B) = i(T_1)$. Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, and let $S \in \mathrm{comm}_\epsilon^{-1}(T)$ such that $S(\mathcal{D}(T)) = \mathcal{D}(T)$. Since S commutes with T, it follows that $S(\mathcal{R}(T^n)) \subset \mathcal{R}(T^n)$ for every $n \in \mathbb{N}$, and therefore, $S(X_1) = S(\bigcap_{n=1}^\infty \mathcal{R}(T^n)) = \bigcap_{n=1}^\infty S(\mathcal{R}(T^n)) \subset \bigcap_{n=1}^\infty \mathcal{R}(T^n) = X_1$. Let $S_1 : X_1 \to X_1$ be the operator induced by operator S. Operator S_1 is bounded, $\|S_1\| < \epsilon$ and $\beta(T_1) = 0$, so $\beta(T_1 - S_1) = 0$. From Theorem 2.1 we have

$$\alpha(T - S) = \alpha(T_1 - S_1) = i(T_1 - S_1) = i(T_1) = \alpha(T_1).$$

From [3, Lemma 2.1(iii)] it follows that $\mathcal{N}(T) \cap \mathcal{R}^\infty(T) = \{0\}$ and hence, $\alpha(T_1) = 0$. Therefore, $\alpha(T - S) = 0$. Since $T - S$ has closed range, we get that $T - S$ is bounded below.

(2.2.2) $\Longrightarrow$ (2.2.3): Obvious.

(2.2.3) $\Longrightarrow$ (2.2.4): Suppose that T is upper semi-Fredholm and there exists $\epsilon > 0$ such that $T - \lambda I$ is injective with closed range for $0 < |\lambda| < \epsilon$. From Theorem 1.1 it follows that there exist two closed subspaces M and N such that $X = M \oplus N$, $T(M \cap \mathcal{D}(T)) \subset M$, $T_M : M \cap \mathcal{D}(T) \to N$ is a closed and semi regular operator; $N \subset \mathcal{D}(T)$, $\dim N < \infty$, $T(N) \subset N$ and $T_N : N \to N$ is a bounded and nilpotent operator. Let P be a projector such that $\mathcal{R}(P) = N$ and $\mathcal{N}(P) = M$. Clearly, $P \in \mathcal{F}(X)$, $R(P) \subset \mathcal{D}(T)$ and $P(\mathcal{D}(T)) \subset \mathcal{D}(T)$. For $x \in \mathcal{D}(T)$ there exist $u \in \mathcal{N}(P) \cap \mathcal{D}(T)$ and $v \in \mathcal{R}(P)$ such that $x = u + v$. Since $TPx = Tv$ and $PTx = P(Tu + Tv) = Tv$, we conclude that P commutes with T. For $T_0 = T_N$ and $T_1 = T_M$, evidently T_0 is a nilpotent bounded operator and $T_1 - \lambda I$ is injective for $0 < |\lambda| < \epsilon$. Since T_1 is semi regular, from [11, Theorem 3, p. 297], we conclude that T_1 is injective. Thus, T_1 is bounded below.

(2.2.4) $\Longrightarrow$ (2.2.5): Suppose that there exists a projector $P \in \mathcal{F}(X)$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is a nilpotent bounded operator of degree p, and T_1 is bounded below. From $TP = T_0P$ it follows that TP is bounded. For $x \in X$ there exist $u \in \mathcal{N}(P)$ and $v \in \mathcal{R}(P)$ such that $x = u + v$. Then

$$\begin{aligned}
(TP)^p x &= \underbrace{(TP)(TP)\dots(TP)(TP)}_{p} x = \underbrace{(TP)(TP)\dots(TP)}_{p-1} Tv \\
&= \underbrace{(TP)(TP)\dots(TP)}_{p-2} TTv = \dots = T^p v = T_0^p v = 0,
\end{aligned}$$

and so TP is nilpotent. From $T \in \mathcal{C}(X)$ and $P \in \mathcal{BL}(X)$ it follows that $T + P \in \mathcal{C}(X)$. Since T_0 is a nilpotent bounded operator, we get that $T_0 + I$ is invertible, and hence $\mathcal{N}(T + P) = \mathcal{N}(T_1) \oplus \mathcal{N}(T_0 + I) = \{0\}$ and $\mathcal{R}(T + P) = \mathcal{R}(T_1) \oplus \mathcal{R}(T_0 + I) = \mathcal{R}(T_1) \oplus \mathcal{R}(P)$. Since $\mathcal{R}(T_1)$ is closed and $\dim \mathcal{R}(P) < \infty$ we get that $\mathcal{R}(T + P)$ is closed. Therefore, $T + P$ is bounded below.

(2.2.5) $\Longrightarrow$ (2.2.6): Let there exists a projector $P \in \mathcal{F}(X)$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, TP is a nilpotent bounded operator, and $T + P$ is bounded below. For $B = -P$ we have that B commutes with T, and $T - B$ is bounded below.

(2.2.6) $\Longrightarrow$ (2.2.7) Obvious.

(2.2.7) $\Longrightarrow$ (2.2.1) Let there exists $B \in \mathcal{K}(X)$ which commutes with T such that $T - B$ is bounded below. Put $A = T - B$. Then $\operatorname{asc} A < \infty$ and $A + \lambda B \in \Phi_+(X)$ for $\lambda \in [0, 1]$ according to [12, Chapter 4, Theorem 5.26]. Since B commutes with A, from [8, Theorem 3] it follows that the function $\lambda \to \overline{\mathcal{N}^\infty(A + \lambda B)} \cap \mathcal{R}^\infty(A + \lambda B)$ is locally constant on the set $[0, 1]$ and therefore this function is constant on $[0,1]$. As $\operatorname{asc}(A) < \infty$, from Lemma 1.2 (i) it follows that $\overline{\mathcal{N}^\infty(A)} \cap \mathcal{R}^\infty(A) = \mathcal{N}^\infty(A) \cap \mathcal{R}^\infty(A) = \{0\}$ and hence, $\overline{\mathcal{N}^\infty(A+B)} \cap \mathcal{R}^\infty(A+B) = \{0\}$. It implies $\mathcal{N}^\infty(A+B) \cap \mathcal{R}^\infty(A+B) = \{0\}$, and by Lemma 1.2 (ii), we get $\operatorname{asc}(A+B) < \infty$. Therefore, $T = A + B \in \mathcal{B}_+(X)$. $\square$

From the equivalence (2.2.1) $\Longleftrightarrow$ (2.2.6) it follows that $T \in \mathcal{B}_+(X)$ if and only if there exist $A \in \mathcal{C}(X)$ and $B \in \mathcal{F}(X)$ such that $T = A + B$, A is a bounded below operator with $\mathcal{D}(A) = \mathcal{D}(T)$, $B \in \mathcal{F}(X)$ and B commutes with T. We remark that this result is an improvement of Theorem 3.2 in [3].

Further, T. Alvarez et al. proved (Theorem 4.3 in [3]) that if $T \in \mathcal{B}_+(X)$ and $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$, then there exists $\eta > 0$ such that $T - \lambda \in \mathcal{B}_+(X)$ for $0 < |\lambda| < \eta$. Notice that the equivalence (2.2.1) $\Longleftrightarrow$ (2.2.3) is an extension of this result.

For $K \subset \mathbb{C}$, $\operatorname{acc} K$ denotes the set of all accumulation points of K.

Corollary 2.1. *Let $T \in \mathcal{C}(X)$. Then*

$$\sigma_{\mathcal{B}_+}(T) = \sigma_{\Phi_+}(T) \cup \operatorname{acc} \sigma_a(T). \tag{2.9}$$

Proof. Follows from the equivalence (2.2.1)$\Longleftrightarrow$(2.2.3). $\square$

Corollary 2.2. *Let $T \in \mathcal{C}(X)$. Then $\sigma_{\mathcal{B}_+}(T)$ is a closed set.*

Proof. From [11, Theorem 1] it follows that $\sigma_{\Phi_+}(T)$ is closed. Now from (2.9) we conclude that $\sigma_{\mathcal{B}_+}(T)$ is a closed set as the union of two closed sets. $\square$

For $T \in \mathcal{C}(X)$ set

$$\mathcal{F}_T(X) = \{F \in \mathcal{F}(X) : F \text{ commutes with } T\}$$

and

$$\mathcal{K}_T(X) = \{K \in \mathcal{K}(X) : K \text{ commutes with } T\}.$$

The following corollary improves Theorem 4.1 in [3].

Corollary 2.3. *Let $T \in \mathcal{C}(X)$. Then*

$$\sigma_{\mathcal{B}_+}(T) = \bigcap_{F \in \mathcal{F}_T(X)} \sigma_a(T+F) = \bigcap_{K \in \mathcal{K}_T(X)} \sigma_a(T+K).$$

Proof. Suppose that $\lambda \notin \bigcap_{K \in \mathcal{K}_T(X)} \sigma_a(T+K)$. Then there exists $K \in \mathcal{K}_T(X)$ such that $\lambda \notin \sigma_a(T+K)$, that is $T+K-\lambda$ is bounded below. Since $-K$ commutes with $T-\lambda$, from Theorem 2.2 (precisely, from the equivalence (2.2.1)$\Longleftrightarrow$(2.2.7)) it follows that $T-\lambda \in \mathcal{B}_+(X)$, i.e. $\lambda \notin \sigma_{\mathcal{B}_+}(T)$. Therefore, $\sigma_{\mathcal{B}_+}(T) \subset \bigcap_{K \in \mathcal{K}_T(X)} \sigma_a(T+K) \subset \bigcap_{F \in \mathcal{F}_T(X)} \sigma_a(T+F)$.

To prove the converse, suppose that $\lambda \notin \sigma_{\mathcal{B}_+}(T)$. Then $T-\lambda \in \mathcal{B}_+(X)$, and from Theorem 2.2 (from the equivalence (2.2.1)$\Longleftrightarrow$(2.2.6)) it follows that there exists $F \in \mathcal{F}(X)$ which commutes with $T-\lambda$ such that $T-\lambda-F$ is bounded below. Then $F_1 = -F \in \mathcal{F}(X)$ commutes with T and hence $F_1 \in \mathcal{F}_T(X)$. Moreover, $T+F_1-\lambda$ is bounded below and so $\lambda \notin \sigma_a(T+F_1)$. Thus, we have just proved the inclusion $\bigcap_{F \in \mathcal{F}_T(X)} \sigma_a(T+F) \subset \sigma_{\mathcal{B}_+}(T)$. $\square$

3 Lower semi-Browder operators and lower semi-Browder spectrum

We first prove the following simply lemma.

Lemma 3.1. *Let $T \in \mathcal{C}(X)$ be a densely defined operator, and let $S \in \mathcal{BL}(X)$. If S commutes with T, then S' commutes with T'.*

Proof. For $y' \in \mathcal{D}(T')$ it follows that

$$\|S'y'(Tx)\| = \|(y'S)(Tx)\| = \|(y'T)(Sx)\| \leq \|y'T\|\|S\|\|x\| \quad \text{for every} \quad x \in \mathcal{D}(T),$$

and hence $S'y' \in \mathcal{D}(T')$. Therefore, $S'(\mathcal{D}(T')) \subset \mathcal{D}(T')$. It remains to prove the commutativity relation. For $y \in \mathcal{D}(T')$ we find

$$(T'S')y' = \overline{(S'y')T},$$

$$(S'T')y' = \overline{y'T}S.$$

Since for $x \in \mathcal{D}(T)$ it holds

$$(S'y')Tx = (y'S)(Tx) = (y'T)(Sx),$$

it follows that $\overline{(S'y')T} = \overline{y'T}S$, and so $(T'S')y' = (S'T')y'$.

$\square$

The following theorem is our second main result where we characterize closed lower semi-Browder operators.

Theorem 3.1. *If $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$ and $\rho_\Phi(T) \neq \emptyset$, then the following conditions are equivalent:*

(3.1.1) *T is lower semi-Browder;*

(3.1.2) *T is lower semi-Fredholm, and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ with $S(\mathcal{D}(T)) = \mathcal{D}(T)$, it follows that $T - S$ is onto;*

(3.1.3) *T is lower semi-Fredholm and almost surjective;*

(3.1.4) *There exists a projector $P \in F(X)$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, T_0 is a nilpotent bounded operator and T_1 is surjective;*

(3.1.5) *There exists a projector $P \in F(X)$ which commutes with T such that $R(P) \subset \mathcal{D}(T)$, TP is a nilpotent bounded operator and $T + P$ is surjective;*

(3.1.6) *There exists $B \in F(X)$ which commutes with T such that $T - B$ is surjective;*

(3.1.7) *There exists $B \in K(X)$ which commutes with T such that $T - B$ is surjective.*

Proof. (3.1.1) $\Longrightarrow$ (3.1.2): Let $T \in \mathcal{B}_-(X)$. Then T' is a closed operator and from [3, Proposition 3.1 (iii)] it follows that $\mathrm{asc}\,(T') < \infty$. Further, since $\mathcal{R}(T)$ is closed, then $\mathcal{R}(T')$ is also closed by [7, Theorem IV.1.2] and from [7, Theorem IV.2.3, i.] it follows that $\alpha(T') = \beta(T) < \infty$. Therefore, $T' \in \mathcal{B}_+(X')$.

Let $S \in \mathcal{BL}(X)$ be an arbitrary bijection with $S(\mathcal{D}(T)) = \mathcal{D}(T)$ and let S commutes with T. Then $S' \in \mathcal{BL}(X')$, $\|S'\| = \|S\|$, S' is bijective, and by Lemma 3.1, $S'(\mathcal{D}(T')) \subset \mathcal{D}(T')$ and S' commutes with T' .

We shall show that $S'(\mathcal{D}(T')) = \mathcal{D}(T')$. Suppose that $y' \in \mathcal{D}(T')$. Then there exists the unique functional $z' \in X'$ such that $y' = S'z' = z'S$. It follows that $z' = y'S^{-1}$ and since S^{-1} commutes with T by Theorem 2.1, for $x \in \mathcal{D}(T)$ the following holds

$$\begin{aligned} \|(z'T)x\| &= \|y'\left(S^{-1}(Tx)\right)\| = \|y'\left(T(S^{-1}x)\right)\| = \\ &= \|(y'T)(S^{-1}x)\| \leq \|y'T\|\|S^{-1}\|\|x\|, \end{aligned}$$

which proves that $z' \in \mathcal{D}(T')$. Therefore, $\mathcal{D}(T') \subset S'(\mathcal{D}(T'))$, and so $S'(\mathcal{D}(T')) = \mathcal{D}(T')$.

According to Theorem 2.2 there exists some $\epsilon > 0$ such that $T' - A$ is bounded below for every operator $A \in \mathcal{BL}(X')$ such that $A \in \mathrm{comm}_\epsilon^{-1}(T')$ and $A(\mathcal{D}(T')) = \mathcal{D}(T')$. Using a previous analysis for $S \in \mathrm{comm}_\epsilon^{-1}(T)$ with $S(\mathcal{D}(T)) = \mathcal{D}(T)$, it follows that $S' \in \mathrm{comm}_\epsilon^{-1}(T')$ and $S'(\mathcal{D}(T')) = \mathcal{D}(T')$, and therefore, $T' - S'$ is bounded below. According to Lemma 1.2 and [7, Theorem IV.1.2] we get that $\mathcal{R}(T-S)$ is closed, and also by [7, Theorem IV. 2.3, i.] we conclude that

$$\beta(T-S)=\alpha((T-S)')=\alpha(T'-S')=0.$$

Therefore, $T-S$ is onto.

(3.1.2) $\Longrightarrow$ (3.1.3): Obvious.

(3.1.3) $\Longrightarrow$ (3.1.4): Suppose that $T\in\Phi_-(X)$ and T is almost surjective. From Theorem 1.1 it follows that there exist two closed subspaces M and N such that $X=M\oplus N$, $T(M\cap\mathcal{D}(T))\subset M$, $T_M: M\cap\mathcal{D}(T)\to M$ is a closed and semi regular operator; $N\subset\mathcal{D}(T)$, $\dim N<\infty$, $T(N)\subset N$ and $T_N: N\to N$ is a bounded and nilpotent operator of degree p. Let P be the projector on N parallel to M. Then $P\in\mathcal{F}(X)$, $\mathcal{R}(P)\subset\mathcal{D}(T)$, $P(\mathcal{D}(T))\subset\mathcal{D}(T)$ and $T_0=T_N$ is a bounded and nilpotent operator. In the same way as in the proof of the implication (2.2.3) $\Longrightarrow$ (2.2.4) we get that P commutes with T. Since there exists $\epsilon>0$ such that $T-\lambda I$ is surjective for $0<|\lambda|<\epsilon$, it follows that $T_M-\lambda I$ is surjective for $0<|\lambda|<\epsilon$. In the same way as T_M is semi regular, from [11, Theorem 3, p. 297], we conclude that $T_1=T_M$ is surjective.

(3.1.4) $\Longrightarrow$ (3.1.5): Suppose that there exists a projector $P\in\mathcal{F}(X)$ which commutes with T such that $R(P)\subset\mathcal{D}(T)$, T_0 is nilpotent bounded operator of degree p and T_1 is surjective. As in the proof of the implication (2.2.4) $\Longrightarrow$ (2.2.5) we get that TP is a nilpotent bounded operator. From $\mathcal{R}(T+P)=\mathcal{R}(T_1)\oplus\mathcal{R}(T_0+I)=\mathcal{N}(P)\oplus\mathcal{R}(P)=X$, we see that $T+P$ is a surjection.

(3.1.5) $\Longrightarrow$ (3.1.6) Put $B=-P$.

(3.1.6) $\Longrightarrow$ (3.1.7) Obvious.

(3.1.7) $\Longrightarrow$ (3.1.1) Let there exists $B\in\mathcal{K}(X)$ which commutes with T such that $T-B$ is surjective. Put $A=T-B$. The operator B' is compact and commutes with T'. The operator $T-B$ is surjective, so it has closed range. From [7, Theorem IV.1.2] and Lemma 1.2 we see that $\mathcal{R}(T'-B')$ is also closed and by [7, Theorem IV.2.3], we find that $\alpha(T'-B')=\beta(T-B)=0$. It follows that the operator $T'-B'$ is bounded below and from Theorem 2.2 it follows that $T'\in\mathcal{B}_+(X')$. Using again [7, Theorem IV.1.2] and [7, Theorem IV.2.3, i.] we get that $\mathcal{R}(T)$ is closed and $\beta(T)=\alpha(T')<\infty$, so $T\in\Phi_-(X)$. From [3, Proposition 3.1] we conclude that $\operatorname{dsc}(T)=\operatorname{asc}(T')<\infty$ and thus, $T\in\mathcal{B}_-(X)$. $\square$

If $T\in\mathcal{C}(X)$, such that $\overline{\mathcal{D}(T)}=X$ and $\rho_\Phi(T)\neq\emptyset$, from the equivalence (3.1.1) $\Longleftrightarrow$ (3.1.6) it follows that $T\in\mathcal{B}_-(X)$ if and only if there exist $A\in\mathcal{C}(X)$ and $B\in\mathcal{F}(X)$ such that $T=A+B$, A is a surjective operator with $\mathcal{D}(A) = \mathcal{D}(T)$, $B \in \mathcal{F}(X)$ and B commutes with T. Notice that this result improves Theorem 3.3 in [3].

T. Alvarez et al. proved [3, Theorem 4.4] that if $T \in \mathcal{B}_-(X)$, $\overline{\mathcal{D}(T)} = X$, $T'(\mathcal{D}(T')) \subset \mathcal{D}(T')$ and $\rho(T) \neq \emptyset$, then there exists $\eta > 0$ such that $T - \lambda \in \mathcal{B}_-(X)$ for $0 < |\lambda| < \eta$. Furthermore, F. Fakhfakh and M. Mnif in [6, Proposition 2.1 (ii)] got the same result without the assumption $T'(\mathcal{D}(T')) \subset \mathcal{D}(T')$. We remark that this result is extended by the equivalence (3.1.1) $\Longleftrightarrow$ (3.1.3). Also from this equivalence immediately follows the next characterization of the lower semi-Browder spectrum.

Corollary 3.1. *Let $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$ and $\rho_\Phi(T) \neq \emptyset$. Then*

$$\sigma_{\mathcal{B}_-}(T) = \sigma_{\Phi_-}(T) \cup \operatorname{acc} \sigma_d(T).$$

The following corollary extend Corollary 2.1 (ii) in [6].

Corollary 3.2. *Let $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$ and $\rho_\Phi(T) \neq \emptyset$. Then $\sigma_{\mathcal{B}_-}(T)$ is a closed set.*

Proof. Follows from Corollary 3.1, since $\sigma_{\Phi_-}(T)$ and $\operatorname{acc} \sigma_d(T)$ are closed sets. $\square$

As a consequence of the equivalences (3.1.1)$\Longleftrightarrow$(3.1.6)$\Longleftrightarrow$(3.1.7) we get one more characterization of the lower semi-Browder spectrum.

Corollary 3.3. *Let $T \in \mathcal{C}(X)$, $\overline{\mathcal{D}(T)} = X$ and $\rho_\Phi(T) \neq \emptyset$. Then*

$$\sigma_{\mathcal{B}_-}(T) = \bigcap_{F \in \mathcal{F}_T(X)} \sigma_d(T + F) = \bigcap_{K \in \mathcal{K}_T(X)} \sigma_d(T + K).$$

Proof. Suppose that $\lambda \notin \bigcap_{K \in \mathcal{K}_T(X)} \sigma_d(T+K)$. Then there exists $K \in \mathcal{K}_T(X)$ such that $\lambda \notin \sigma_d(T+K)$, that is $T+K-\lambda$ is surjective. Since $-K$ commutes with $T - \lambda$, $\overline{\mathcal{D}(T - \lambda)} = \overline{\mathcal{D}(T)} = X$, $\rho_\Phi(T - \lambda) \neq \emptyset$, from Theorem 3.1 (precisely, from the equivalence (3.1.1)$\Longleftrightarrow$(3.1.7)) it follows that $T - \lambda \in \mathcal{B}_-(X)$, i.e. $\lambda \notin \sigma_{\mathcal{B}_-}(T)$. Therefore, $\sigma_{\mathcal{B}_-}(T) \subset \bigcap_{K \in \mathcal{K}_T(X)} \sigma_d(T + K) \subset \bigcap_{F \in \mathcal{F}_T(X)} \sigma_d(T + F)$.

To prove the opposite inclusion, suppose that $\lambda \notin \sigma_{\mathcal{B}_-}(T)$. Then $T - \lambda \in \mathcal{B}_-(X)$, and since $\overline{\mathcal{D}(T - \lambda)} = X$ and $\rho_\Phi(T-\lambda) \neq \emptyset$, from Theorem 3.1 (from the equivalence (3.1.1)$\Longleftrightarrow$(3.1.6)) it follows that there exists $F \in \mathcal{F}(X)$ which commutes with $T - \lambda$ such that $T - \lambda - F$ is surjective. Then $F_1 =$

$-F \in \mathcal{F}(X)$ commutes with T and hence $F_1 \in \mathcal{F}_T(X)$. Moreover, $T+F_1-\lambda$ is surjective and so, $\lambda \notin \sigma_d(T+F_1)$. Consequently, $\bigcap_{F\in\mathcal{F}_T(X)} \sigma_d(T+F) \subset \sigma_{\mathcal{B}_+}(T)$. $\square$

Using Lemma 1.3 we can conclude that Corollary 3.3 is an improvement of Theorem 4.2 in [3].

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