The reverse order law $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution

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Abstract

Several equivalent conditions for the reverse order law $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution are presented. Also, we investigate necessary and sufficient conditions for $(ab)^\# = b^\dagger a^\dagger$ to hold.

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1 Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. In the theory of generalized inverses, one of fundamental procedures is to find generalized inverses of products. If $a, b \in \mathcal{R}$ are invertible, then ab is also invertible, and the inverse of the product ab satisfied $(ab)^{-1} = b^{-1}a^{-1}$. This equality is called the reverse order law and it cannot trivially be extended to various generalized inverse of the product ab. The reverse order laws for generalized inverses have been investigated in the literature since the 1960s [1, 2, 3, 4, 6, 7].

Let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$(1)\ aa^\#a = a, \quad (2)\ a^\#aa^\# = a^\#, \quad (5)\ aa^\# = a^\#a;$$

$a^\#$ is a group inverse of a and it is uniquely determined by these equations. The group inverse $a^\#$ double commutes with a, that is, $ax = xa$ implies $a^\#x = xa^\#$ [1]. Denote by $\mathcal{R}^\#$ the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

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An element $a \in \mathcal{R}$ is self-adjoint (or Hermitian) if $a^* = a$.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $a^\dagger \in \mathcal{R}$, if the following equations hold [10]:

$$(1)\ aa^\dagger a = a, \quad (2)\ a^\dagger aa^\dagger = a^\dagger, \quad (3)\ (aa^\dagger)^* = aa^\dagger, \quad (4)\ (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that above conditions hold. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

If $\delta \subset \{1, 2, 3, 4, 5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denoted by $a\{\delta\}$. Notice that $a\{1, 2, 5\} = \{a^\#\}$ and $a\{1, 2, 3, 4\} = \{a^\dagger\}$. If a is invertible, then $a^\#$ and $a^\dagger$ coincide with the ordinary inverse of a. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

An element $a \in \mathcal{R}$ is: left *-cancellable if $a^*ax = a^*ay$ implies $ax = ay$; it is right *-cancellable if $xaa^* = yaa^*$ implies $xa = ya$; and it is *-cancellable if it is both left and right *-cancellable. We observe that a is left *-cancellable if and only if a^* is right *-cancellable. In C^*-algebras all elements are *-cancellable. A ring $\mathcal{R}$ is called *-reducing if every element of $\mathcal{R}$ is *-cancellable. This is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$ for all $a \in \mathcal{R}$.

One of the basic topics in the theory of generalized inverses is to investigate various reverse order laws related to generalized inverses products. The reverse order law for the generalized inverse is an useful computational tool in applications (solving linear equations in linear algebra or numerical analysis), and it is also interesting from the theoretical point of view.

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [5]. A Hilbert space version of their result was given by Isumino [7]. The results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's paper [11].

In this paper we present some necessary and sufficient conditions for the reverse order law $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution. We also study the equivalent conditions involving $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ to ensure that $(ab)^\# = b^\dagger a^\dagger$ is satisfied. Some equivalent conditions to $(a^\dagger ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger$ and $(abb^\dagger)^\# = (a^\dagger abb^\dagger)^\dagger a^\dagger$ are given too. Similar results related to the reverse order laws $(ab)^\# = b^*(a^*abb^*)^\dagger a^*$,$(a^*ab)^\# = b^*(a^*abb^*)^\dagger$,$(abb^*)^\# = (a^*abb^*)^\dagger a^*$ are investigated.

In the end of this section, we state the following well-known results on the Moore-Penrose inverse, which be used later.

Lemma 1.1. [9] *If $a \in \mathcal{R}^\dagger$, then*

(i) $aa^{(1,3)} = aa^\dagger$, *for any* $a^{(1,3)} \in a\{1, 3\}$;

(ii) $a^{(1,4)}a = a^\dagger a$, *for any* $a^{(1,4)} \in a\{1,4\}$.

Lemma 1.2. [9] *Let* $a, b \in \mathcal{R}$.

(i) *If* $a, a^\dagger ab \in \mathcal{R}^\dagger$, *then* $(a^\dagger ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger a$.

(ii) *If* $b, abb^\dagger \in \mathcal{R}^\dagger$, *then* $(abb^\dagger)^\dagger = bb^\dagger(abb^\dagger)^\dagger$.

Lemma 1.3. *Let* $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$. *Then*

(i) $(a^\dagger abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger a$,

(ii) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger(a^\dagger abb^\dagger)^\dagger$.

Proof. From

$$\begin{aligned} (a^\dagger abb^\dagger)^\dagger a^\dagger a &= (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger a \\ &= (a^\dagger abb^\dagger)^\dagger(a^\dagger aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger)^* \\ &= (a^\dagger abb^\dagger)^\dagger(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger)^* = (a^\dagger abb^\dagger)^\dagger, \end{aligned}$$

we conclude that (i) holds. The statement (ii) can be proved in the similar way. □

By Remark after Theorem 2.4 in [8], [8, Theorem 2.1] can be formulated as follows.

Theorem 1.1. *Let* $\mathcal{R}$ *be a ring with involution, let* $a, b \in \mathcal{R}^\dagger$ *and let* $(1 - a^\dagger a)b$ *be left *-cancellable. Then the following conditions are equivalent:*

(a) $abb^\dagger a^\dagger ab = ab$;

(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;

(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;

(d) $a^\dagger abb^\dagger$ *is an idempotent;*

(e) $bb^\dagger a^\dagger a$ *is an idempotent;*

(f) $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;

(g) $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.

2 Reverse order laws

In the following theorem, the reverse order law $(ab)^{\#} = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$ in rings with involution is characterized.

Theorem 2.1. *If $a, b, a^{\dagger}abb^{\dagger} \in \mathcal{R}^{\dagger}$ and $ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$,

(ii) $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} \in (ab)\{5\}$,

(iii) $abaa^{\dagger} = ab$ *and* $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}ba = a(a^{\dagger}abb^{\dagger})^{\dagger}$,

(iv) $b^{\dagger}bab = ab$ *and* $ba(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} = (a^{\dagger}abb^{\dagger})^{\dagger}b$,

(v) $b\{1,3,4\} \cdot (a^{\dagger}abb^{\dagger})\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{5\}$,

(vi) $(a^{\dagger}abb^{\dagger})^{\dagger} = b(ab)^{\#}a$ *and* $abaa^{\dagger} = ab = b^{\dagger}bab$.

Proof. (i) $\Rightarrow$ (ii): It is trivial.

(ii) $\Rightarrow$ (iii): Notice that $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} \in (ab)\{1\}$, by

$$abb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}ab = a(a^{\dagger}abb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}abb^{\dagger})b = aa^{\dagger}abb^{\dagger}b = ab. \qquad (1)$$

Since $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} \in (ab)\{5\}$, we obtain

$$abaa^{\dagger} = ababb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}aa^{\dagger} = ababb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} = ab$$

and

$$b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}(abaa^{\dagger}) = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}ab = abb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}. \qquad (2)$$

Multiplying (2) by a from the right side and applying Lemma 1.3, we get $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}ba = a(a^{\dagger}abb^{\dagger})^{\dagger}$. So, the item (iii) holds.

(iii) $\Rightarrow$ (v): Suppose that $abaa^{\dagger} = ab$ and $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}ba = a(a^{\dagger}abb^{\dagger})^{\dagger}$. If $b^{(1,3,4)} \in b\{1,3,4\}$, $(a^{\dagger}abb^{\dagger})^{(1,3,4)} \in (a^{\dagger}abb^{\dagger})\{1,3,4\}$ and $a^{(1,3,4)} \in a\{1,3,4\}$, by Lemma 1.1 and Lemma 1.3, we have

$$\begin{aligned}
a(bb^{(1,3,4)})(a^\dagger abb^\dagger)^{(1,3,4)}a^{(1,3,4)} &= a(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3,4)})a^{(1,3,4)} \\
&= aa^\dagger a(bb^\dagger(a^\dagger abb^\dagger)^\dagger)a^{(1,3,4)} \\
&= (a(a^\dagger abb^\dagger)^\dagger)a^\dagger(aa^{(1,3,4)}) \\
&= b^\dagger(a^\dagger abb^\dagger)^\dagger baa^\dagger aa^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger baa^\dagger \\
&= b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger(abaa^\dagger) = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab \\
&= (b^\dagger b)b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab \\
&= b^{(1,3,4)}(bb^\dagger(a^\dagger abb^\dagger)^\dagger)a^\dagger ab \\
&= b^{(1,3,4)}((a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)b \\
&= b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)}(a^\dagger a)bb^\dagger b \\
&= b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)}a^{(1,3,4)}ab.
\end{aligned}$$

Therefore, for any $b^{(1,3,4)} \in b\{1,3,4\}$, $(a^\dagger abb^\dagger)^{(1,3,4)} \in (a^\dagger abb^\dagger)\{1,3,4\}$ and $a^{(1,3,4)} \in a\{1,3,4\}$, $b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)}a^{(1,3,4)} \in (ab)\{5\}$ and (v) holds.

(v) $\Rightarrow$ (i): Because $b^\dagger \in b\{1,3,4\}$, $(a^\dagger abb^\dagger)^\dagger \in (a^\dagger abb^\dagger)\{1,3,4\}$ and $a^\dagger \in a\{1,3,4\}$, the hypothesis $b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{5\}$ implies $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{5\}$. Since the equalities (1) hold and

$$b^\dagger((a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger)a^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger,$$

we conclude that $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,2\}$. Thus, the statement (i) is satisfied.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v): In the similar way as (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v), we can prove these implications.

(i) $\Rightarrow$ (vi): Since (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), then $abaa^\dagger = ab = b^\dagger bab$ and, by Lemma 1.3,

$$b(ab)^\# a = bb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger a = (a^\dagger abb^\dagger)^\dagger.$$

(vi) $\Rightarrow$ (i): Let $(a^\dagger abb^\dagger)^\dagger = b(ab)^\# a$ and $abaa^\dagger = ab = b^\dagger bab$. Now, we have

$$\begin{aligned}
b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger &= b^\dagger b(ab)^\# aa^\dagger = (b^\dagger bab)[(ab)^\#]^3(abaa^\dagger) \\
&= ab[(ab)^\#]^3 ab = (ab)^\#.
\end{aligned}$$

□

Analogously to Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *If $a, b, a^*abb^* \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^*(a^*abb^*)^{\dagger}a^*$,

(ii) $b^*(a^*abb^*)^{\dagger}a^* \in (ab)\{5\}$,

(iii) $abaa^{\dagger} = ab$ *and* $b^*(a^*abb^*)^{\dagger}a^*aba = abb^*(a^*abb^*)^{\dagger}a^*a$,

(iv) $b^{\dagger}bab = ab$ *and* $babb^*(a^*abb^*)^{\dagger}a^* = bb^*(a^*abb^*)^{\dagger}a^*ab$,

(v) $b^* \cdot (a^*abb^*)\{1,3,4\} \cdot a^* \subseteq (ab)\{5\}$.

Proof. This theorem can be proved similarly as Theorem 2.1, applying the equalities $a = (a^{\dagger})^*a^*a$ and $a^* = a^*aa^{\dagger}$. □

In the following result, we show that $(ab)\{5\} \subseteq b\{1,3,4\}\cdot(a^{\dagger}abb^{\dagger})\{1,3,4\}\cdot a\{1,3,4\}$ is equivalent to $(ab)\{5\} = b\{1,3,4\} \cdot (a^{\dagger}abb^{\dagger})\{1,3,4\} \cdot a\{1,3,4\}$.

Theorem 2.3. *If $a, b, a^{\dagger}abb^{\dagger} \in \mathcal{R}^{\dagger}$ and $ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)\{5\} \subseteq b\{1,3,4\} \cdot (a^{\dagger}abb^{\dagger})\{1,3,4\} \cdot a\{1,3,4\}$,

(ii) $(ab)\{5\} = b\{1,3,4\} \cdot (a^{\dagger}abb^{\dagger})\{1,3,4\} \cdot a\{1,3,4\}$.

Proof. (i) ⇒ (ii): Assume that $(ab)\{5\} \subseteq b\{1,3,4\}\cdot(a^{\dagger}abb^{\dagger})\{1,3,4\}\cdot a\{1,3,4\}$. Then there exist $b^{(1,3,4)} \in b\{1,3,4\}$, $(a^{\dagger}abb^{\dagger})^{(1,3,4)} \in (a^{\dagger}abb^{\dagger})\{1,3,4\}$ and $a^{(1,3,4)} \in a\{1,3,4\}$ such that $(ab)^{\#} = b^{(1,3,4)}(a^{\dagger}abb^{\dagger})^{(1,3,4)}a^{(1,3,4)}$. Notice that, by Lemma 1.1 and Lemma 1.3, we get

$$\begin{aligned} b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} &= b^{(1,3,4)}(bb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}a)a^{(1,3,4)} = b^{(1,3,4)}(a^{\dagger}abb^{\dagger})^{\dagger}a^{(1,3,4)} \\ &= b^{(1,3,4)}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}abb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{(1,3,4)} \\ &= b^{(1,3,4)}(a^{\dagger}abb^{\dagger})^{(1,3,4)}(a^{\dagger}a)(bb^{\dagger})(a^{\dagger}abb^{\dagger})^{(1,3,4)}a^{(1,3,4)} \\ &= (b^{(1,3,4)}(a^{\dagger}abb^{\dagger})^{(1,3,4)}a^{(1,3,4)})ab(b^{(1,3,4)}(a^{\dagger}abb^{\dagger})^{(1,3,4)}a^{(1,3,4)}) \\ &= (ab)^{\#}ab(ab)^{\#} = (ab)^{\#}. \end{aligned}$$

Using Theorem 2.1, we deduce that $b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger} = (ab)^{\#}$ implies $b\{1,3,4\} \cdot (a^{\dagger}abb^{\dagger})\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{5\}$. So, the equality (ii) holds.

(ii) ⇒ (i): Obviously. □

In the similar way as in the proof of Theorem 2.3, we obtain the next theorem.

Theorem 2.4. *If $a, b, a^*abb^* \in \mathcal{R}^{\dagger}$ and $ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)\{5\} \subseteq b^* \cdot (a^*abb^*)\{1,3,4\} \cdot a^*$,

(ii) $(ab)\{5\} = b^* \cdot (a^*abb^*)\{1,3,4\} \cdot a^*$.

In the following theorem, we prove a group of equivalent conditions for $(ab)^\# = b^\dagger a^\dagger$ to be satisfied.

Theorem 2.5. *Let $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, let $ab \in \mathcal{R}^\#$ and let $(1-a^\dagger a)b$ be left *-cancellable. Then $(ab)^\# = b^\dagger a^\dagger$ if and only if $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ and any one of the following equivalent conditions holds:*

(a) $abb^\dagger a^\dagger ab = ab$;

(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;

(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;

(d) $a^\dagger abb^\dagger$ *is an idempotent;*

(e) $bb^\dagger a^\dagger a$ *is an idempotent;*

(f) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;

(g) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.

Proof. $\Longrightarrow$: Since $(ab)^\# = b^\dagger a^\dagger$, then $abb^\dagger a^\dagger ab = ab$ which implies that, by Theorem 1.1, the conditions (a)-(g) are satisfied and $(ab)^\# = b^\dagger a^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$.

$\Longleftarrow$: Conversely, the conditions (a)-(g) imply $b^\dagger a^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$. From the assumption $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, we deduce that $(ab)^\# = b^\dagger a^\dagger$. $\square$

The condition $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in Theorem 2.5 can be replaced by some equivalent conditions from Theorem 2.1.

Remark. Theorem 2.5 holds in C^*-algebras and *-reducing rings without the hypothesis $(1-a^\dagger a)b$ is left *-cancellable, since this condition is automatically satisfied.

The relation between the reverse order laws $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$ and $(a^\dagger ab)^\dagger = (ab)^\# a$ is studied in the next theorem.

Theorem 2.6. *If $b \in \mathcal{R}$, $a, a^\dagger ab \in \mathcal{R}^\dagger$ and if $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$,

(ii) $(a^\dagger ab)^\dagger = (ab)^\# a$ *and* $abaa^\dagger = ab$.

Proof. (i) $\Rightarrow$ (ii): The condition $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$ implies

$$abaa^\dagger = (ab)^2(ab)^\# aa^\dagger = (ab)^2(a^\dagger ab)^\dagger a^\dagger aa^\dagger = (ab)^2(a^\dagger ab)^\dagger a^\dagger = ab.$$

Therefore, by Lemma 1.2, $(ab)^\# a = (a^\dagger ab)^\dagger a^\dagger a = (a^\dagger ab)^\dagger$.

(ii) $\Rightarrow$ (i): Using the equalities $(a^\dagger ab)^\dagger = (ab)^\# a$ and $abaa^\dagger = ab$, we obtain that (i) holds:

$$(a^\dagger ab)^\dagger a^\dagger = (ab)^\# aa^\dagger = [(ab)^\#]^2(abaa^\dagger) = [(ab)^\#]^2 ab = (ab)^\#.$$

$\square$

Similarly as Theorem 2.6, we can show the following theorem.

Theorem 2.7. *If* $a \in \mathcal{R}$, $b, abb^\dagger \in \mathcal{R}^\dagger$ *and if* $ab \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(ab)^\# = b^\dagger(abb^\dagger)^\dagger$,

(ii) $(abb^\dagger)^\dagger = b(ab)^\#$ *and* $ab = b^\dagger bab$.

The reverse order law $(a^\dagger ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger$ is characterized in the following result.

Theorem 2.8. *If* $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $a^\dagger ab \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(a^\dagger ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger$,

(ii) $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{5\}$,

(iii) $b^\dagger ba^\dagger ab = a^\dagger ab$ *and* $ba^\dagger a(a^\dagger abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger b$,

(iv) $b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\} \subseteq (a^\dagger ab)\{5\}$,

(v) $b^\dagger ba^\dagger ab = a^\dagger ab$ *and* $(a^\dagger abb^\dagger)^\dagger = b(a^\dagger ab)^\#$.

Proof. (i) $\Rightarrow$ (ii): This implication is trivial.

(ii) $\Rightarrow$ (iii): Observe that $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{1\}$, by

$$a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = (a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)b = a^\dagger abb^\dagger b = a^\dagger ab. \qquad (3)$$

Further, from $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{5\}$ and Lemma 1.3, we have

$$b^\dagger b(a^\dagger ab) = b^\dagger bb^\dagger(a^\dagger abb^\dagger)^\dagger(a^\dagger ab)^2 = b^\dagger(a^\dagger abb^\dagger)^\dagger(a^\dagger ab)^2 = a^\dagger ab \qquad (4)$$

and

$$ba^\dagger a(a^\dagger abb^\dagger)^\dagger = b(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger) = (bb^\dagger(a^\dagger abb^\dagger)^\dagger)a^\dagger a)b = (a^\dagger abb^\dagger)^\dagger b.$$

(iii) $\Rightarrow$ (i): Applying $b^\dagger ba^\dagger ab = a^\dagger ab$ and $ba^\dagger a(a^\dagger abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger b$, we obtain

$$\begin{aligned} b^\dagger((a^\dagger abb^\dagger)^\dagger a^\dagger a)b &= b^\dagger((a^\dagger abb^\dagger)^\dagger b) = b^\dagger ba^\dagger a(a^\dagger abb^\dagger)^\dagger \\ &= (b^\dagger ba^\dagger ab)b^\dagger(a^\dagger abb^\dagger)^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger. \end{aligned}$$

Hence, $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{5\}$. Since the equalities (3) hold and

$$b^\dagger((a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger) = b^\dagger(a^\dagger abb^\dagger)^\dagger,$$

we deduce that $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{1,2\}$. Thus, the condition (i) is satisfied.

(ii) $\Rightarrow$ (iv): Suppose that $b^\dagger(a^\dagger abb^\dagger)^\dagger \in (a^\dagger ab)\{5\}$. For $b^{(1,3,4)} \in b\{1,3,4\}$ and $(a^\dagger abb^\dagger)^{(1,3,4)} \in (a^\dagger abb^\dagger)\{1,3,4\}$, by Lemma 1.1 and Lemma 1.3, we obtain

$$\begin{aligned} b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)}a^\dagger ab &= b^{(1,3,4)}((a^\dagger abb^\dagger)^{(1,3,4)}a^\dagger abb^\dagger)b \\ &= b^{(1,3,4)}(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger b \\ &= (b^{(1,3,4)}b)b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = b^\dagger bb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab \\ &= a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3,4)} \\ &= a^\dagger abb^{(1,3,4)}(a^\dagger abb^\#)^{(1,3,4)}. \end{aligned}$$

So, for any $b^{(1,3,4)} \in b\{1,3,4\}$ and $(a^\dagger abb^\dagger)^{(1,3,4)} \in (a^\dagger abb^\dagger)\{1,3,4\}$, we get $b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)} \in (a^\dagger ab)\{5\}$ and (iv) is satisfied.

(iv) $\Rightarrow$ (ii): By $b^\dagger \in b\{1,3,4\}$ and $(a^\dagger abb^\dagger)^\dagger \in (a^\dagger abb^\dagger)\{1,3,4\}$.

(i) $\Rightarrow$ (v): Let $(a^\dagger ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger$. Since (i) $\Rightarrow$ (iii), then $b^\dagger ba^\dagger ab = a^\dagger ab$ and, by Lemma 1.3, $(a^\dagger abb^\dagger)^\dagger = b(b^\dagger(a^\dagger abb^\dagger)^\dagger) = b(a^\dagger ab)^\#$.

(v) $\Rightarrow$ (i): Assume that $b^\dagger ba^\dagger ab = a^\dagger ab$ and $(a^\dagger abb^\dagger)^\dagger = b(a^\dagger ab)^\#$. Now,

$$\begin{aligned} (a^\dagger ab)^\# &= (a^\dagger ab)[(a^\dagger ab)^\#]^2 = b^\dagger b(a^\dagger ab[(a^\dagger ab)^\#]^2) \\ &= b^\dagger(b(a^\dagger ab)^\#) = b^\dagger(a^\dagger abb^\dagger)^\dagger. \end{aligned}$$

□

In the same way as in Theorem 2.8, we obtain the following theorems.

Theorem 2.9. *If $a \in \mathcal{R}$, $b, a^*abb^* \in \mathcal{R}^\dagger$ and $a^*ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^*ab)^\# = b^*(a^*abb^*)^\dagger$,

(ii) $b^*(a^*abb^*)^\dagger \in (a^*ab)\{5\}$,

(iii) $b^\dagger ba^*ab = a^*ab$ *and* $ba^*abb^*(a^*abb^*)^\dagger = bb^*(a^*abb^*)^\dagger a^*ab$,

(iv) $b^* \cdot (a^*abb^*)\{1,3,4\} \subseteq (a^*ab)\{5\}$,

(v) $b^\dagger ba^*ab = a^*ab$ *and* $(a^*abb^*)^\dagger = (b^\dagger)^*(a^*ab)^\#$.

Theorem 2.10. *If* $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $abb^\dagger \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(abb^\dagger)^\# = (a^\dagger abb^\dagger)^\dagger a^\dagger$,

(ii) $(a^\dagger abb^\dagger)^\dagger a^\dagger \in (abb^\dagger)\{5\}$,

(iii) $abb^\dagger aa^\dagger = abb^\dagger$ *and* $(a^\dagger abb^\dagger)^\dagger bb^\dagger a = a(a^\dagger abb^\dagger)^\dagger$,

(iv) $(a^\dagger abb^\dagger)\{1,3,4\} \cdot a\{1,3,4\} \subseteq (abb^\dagger)\{5\}$,

(v) $abb^\dagger aa^\dagger = abb^\dagger$ *and* $(a^\dagger abb^\dagger)^\dagger = (abb^\dagger)^\# a$.

Theorem 2.11. *If* $b \in \mathcal{R}$, $a, a^*abb^* \in \mathcal{R}^\dagger$ *and* $abb^* \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(abb^*)^\# = (a^*abb^*)^\dagger a^*$,

(ii) $(a^*abb^*)^\dagger a^\dagger \in (abb^*)\{5\}$,

(iii) $abb^*aa^\dagger = abb^*$ *and* $(a^*abb^*)^\dagger a^*abb^*a = abb^*(a^*abb^*)^\dagger a^*a$,

(iv) $(a^*abb^*)\{1,3,4\} \cdot a^* \subseteq (abb^*)\{5\}$,

(v) $abb^*aa^\dagger = abb^*$ *and* $(a^*abb^*)^\dagger = (abb^*)^\#(a^\dagger)^*$.

Now, we prove that $(a^\dagger ab)\{5\} \subseteq b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\}$ is equivalent to $(a^\dagger ab)\{5\} = b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\}$.

Theorem 2.12. *If* $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ *and* $a^\dagger ab \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(a^\dagger ab)\{5\} \subseteq b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\}$,

(ii) $(a^\dagger ab)\{5\} = b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\}$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $(a^\dagger ab)\{5\} \subseteq b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\}$. Then there exist $b^{(1,3,4)} \in b\{1,3,4\}$ and $(a^\dagger abb^\dagger)^{(1,3,4)} \in (a^\dagger abb^\dagger)\{1,3,4\}$ such that $(a^\dagger ab)^\# = b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)}$. Using Lemma 1.1 and Lemma 1.3, we get

$$\begin{aligned} b^\dagger(a^\dagger abb^\dagger)^\dagger &= b^{(1,3,4)}(bb^\dagger(a^\dagger abb^\dagger)^\dagger) = b^{(1,3,4)}(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger \\ &= (b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)})a^\dagger ab(b^{(1,3,4)}(a^\dagger abb^\dagger)^{(1,3,4)}) \\ &= (a^\dagger ab)^\# a^\dagger ab(a^\dagger ab)^\# = (a^\dagger ab)^\#. \end{aligned}$$

Therefore, by Theorem 2.8, $b\{1,3,4\} \cdot (a^\dagger abb^\dagger)\{1,3,4\} \subseteq (a^\dagger ab)\{5\}$ and (ii) is satisfied.

(ii) $\Rightarrow$ (i): Obviously. $\square$

The next results can be checked similarly as Theorem 2.12

Theorem 2.13. *If $a \in \mathcal{R}$, $b, a^*abb^* \in \mathcal{R}^\dagger$ and $a^*ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^*ab)\{5\} \subseteq b^* \cdot (a^*abb^*)\{1,3,4\}$,

(ii) $(a^*ab)\{5\} = b^* \cdot (a^*abb^*)\{1,3,4\}$.

Theorem 2.14. *If $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $abb^\dagger \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abb^\dagger)\{5\} \subseteq (a^\dagger abb^\dagger)\{1,3,4\} \cdot a\{1,3,4\}$,

(ii) $(abb^\dagger)\{5\} = (a^\dagger abb^\dagger)\{1,3,4\} \cdot a\{1,3,4\}$.

Theorem 2.15. *If $b \in \mathcal{R}$, $a, a^*abb^* \in \mathcal{R}^\dagger$ and $abb^* \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abb^*)\{5\} \subseteq (a^*abb^*)\{1,3,4\} \cdot a^*$,

(ii) $(abb^*)\{5\} = (a^*abb^*)\{1,3,4\} \cdot a^*$.

Sufficient conditions for the reverse order law $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ are investigated in the following theorem.

Theorem 2.16. *Suppose that $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ and $ab, a^\dagger ab, abb^\dagger \in \mathcal{R}^\#$. Then each of the following conditions is sufficient for $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ to hold:*

(i) $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ *and* $(a^\dagger ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger$,

(ii) $(ab)^{\#} = b^\dagger (abb^\dagger)^{\#}$ *and* $(abb^\dagger)^{\#} = (a^\dagger abb^\dagger)^\dagger a^\dagger$.

Proof. (i) From $(ab)^{\#} = (a^\dagger ab)^{\#} a^\dagger$ and $(a^\dagger ab)^{\#} = b^\dagger (a^\dagger abb^\dagger)^\dagger$, we get

$$(ab)^{\#} = (a^\dagger ab)^{\#} a^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger.$$

(iii) It follows as item (ii). □

If we replace the conditions $(a^\dagger ab)^{\#} = b^\dagger (a^\dagger abb^\dagger)^\dagger$ and $(abb^\dagger)^{\#} = (a^\dagger abb^\dagger)^\dagger a^\dagger$ in Theorem 2.16 by some equivalent conditions from Theorem 2.8 and Theorem 2.10, we obtain list of sufficient conditions for $(ab)^{\#} = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ to be satisfied.

Similarly to Theorem 2.16, we get the next theorem.

Theorem 2.17. *Suppose that $a, b \in \mathcal{R}$, $a^*abb^* \in \mathcal{R}^\dagger$ and $ab, a^*ab, abb^* \in \mathcal{R}^{\#}$. Then each of the following conditions is sufficient for $(ab)^{\#} = b^*(a^*abb^*)^\dagger a^*$ to hold:*

(i) $(ab)^{\#} = (a^*ab)^{\#} a^*$ *and* $(a^*ab)^{\#} = b^*(a^*abb^*)^\dagger$,

(ii) $(ab)^{\#} = b^*(abb^*)^{\#}$ *and* $(abb^*)^{\#} = (a^*abb^*)^\dagger a^*$.

Finally, we give an example to illustrate our results.

Example 2.1. Consider a 2×2 block matrices $A = \begin{bmatrix} 1 & a \\ a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$, where $a, b \in \mathbf{R} \setminus \{0\}$. Notice that $A^\dagger = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{a} & -\frac{1}{a^2} \end{bmatrix}$, $B^\dagger = \begin{bmatrix} \frac{1}{b} & 0 \\ 0 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} b & a \\ ab & 0 \end{bmatrix}$. Since statements of Theorem 2.5 (or Theorem 2.6 or Theorem 2.7) are satisfied, we obtain $(AB)^{\#} = \begin{bmatrix} 0 & \frac{1}{ab} \\ \frac{1}{a} & -\frac{1}{a^2} \end{bmatrix} = B^\dagger A^\dagger$.

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