Weighted generalized Drazin inverse in rings

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Abstract

In this paper we introduce and investigate the weighted generalized Drazin inverse for elements in rings. We also introduce and investigate the weighted EP elements

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1 Introduction

The Drazin inverse of an element in a semigroup was introduced in [2]. Let S be a multiplicative semigroup and let $a \in S$. The element $b \in S$ is a Drazin inverse of a, if the following hold:

$$bab = b,\ ab = ba, a^{n+1}b = a^n$$

for some non-negative integer n, and the least such n (if it exists) is the Drazin index of a. If the Drazin inverse of a exists, then it is unique and denoted by a^D. Such a is then called Drazin invertible, and the set of all Drazin invertible elements in S is denoted by S^D.

The most interesting semigroup is the set of complex square matrices, or, more generally, the set of all linear bounded operators on a Banach space. By now, many papers appeared dealing with the Drazin inverse and its generalizations, including linear bounded operators on Banach or Hilbert spaces, elements in Banach algebras or rings.

The generalization that we are particularly interested in, deals with the weighted Drazin inverse, it is introduced in [7], and also investigated in [6]. Although both papers consider only linear bounded operators between two

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Banach spaces, with small changes it is not difficult to see that these results are valid in Banach algebras also.

In this paper we introduce and investigate the weighted generalized Drazin inverse and the weighted EP elements in rings. In the rest of this section we recall some properties of the Drazin inverse and the Moore-Penrose inverse. The paper is organized as follows. In Section 2 we prove several results concerning the weighed generalized Drazin inverse. In Section 3 we prove the main characterization of EP elements in rings.

2 Background

Let $\mathcal{R}$ be an arbitrary ring with the unit 1. The set of all invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$, and the set of all idempotents in $\mathcal{R}$ is denoted by $\mathcal{R}^{\bullet}$. We use $\mathcal{R}^{nil}$ to denote the set of all nilpotent elements of $\mathcal{R}$.

Let $a \in \mathcal{R}$. Then $a^D = x \in \mathcal{R}$ is the Drazin inverse of a, if the following hold:

$$xax = x, \ ax = xa, \ \text{and} \ a(ax - 1) \in \mathcal{R}^{nil}.$$

For any element $b \in \mathcal{R}$ the commutant and the double commutant of b, respectively, are defined by

$$\mathrm{comm}(b) = \{x \in \mathcal{R} : bx = xb\},$$

$$\mathrm{comm}^2(b) = \{x \in \mathcal{R} : xy = yx \text{ for all } y \in \mathrm{comm}(b)\}.$$

In [3] quasinilpotent elements of a ring $\mathcal{R}$ are introduced as follows:

$q \in \mathcal{R}$ is quasinilpotent, if $1 + xq \in \mathcal{R}^{-1}$ for all $x \in \mathrm{comm}(q)$.

We use $\mathcal{R}^{qnil}$ to denote the set of all quasinilpotent elements of $\mathcal{R}$.

Thus, the generalized Drazin inverse of $a \in \mathcal{R}$ is defined as (see [4]) the element $a^d = x$ satisfying:

$$x \in \mathrm{comm}^2(a), \ xax = x, \ a(1 - ax) \in \mathcal{R}^{qnil}.$$

If a^d exists, then it is unique [4]. In Banach algebras it is enough to assume $x \in \mathrm{comm}(a)$ instead of $x \in \mathrm{comm}^2(a)$. We use $\mathcal{R}^d$ to denote the set of all generalized Drazin invertible elements of $\mathcal{R}$.

Definition 2.1. [5] *An element $a \in \mathcal{R}$ is quasipolar if there exists $p \in \mathcal{R}$ such that*

$$p^2 = p, \qquad p \in \mathrm{comm}^2(a), \qquad ap \in \mathcal{R}^{qnil}, \qquad a + p \in \mathcal{R}^{-1}.$$

The following result holds.

Theorem 2.1. [5] *An element $a \in \mathcal{R}$ is generalized Drazin invertible if and only if the element a is quasipolar. In this case $a \in \mathcal{R}$ has a unique generalized Drazin inverse a^d given by the equation*

$$a^d = (a + a^\pi)^{-1}(1 - a^\pi).$$

From now on, a^π will be the unique spectral idempotent of a, and $a^\pi = 1 - aa^d$.

Operator matrices seem to be the useful tool for investing Drazin invertibility of linear bounded operators on Banach spaces. In rings we have to use idempotents. Let $a \in \mathcal{R}$, and let $p, q \in \mathcal{R}^\bullet$. Then we write

$$a = paq + pa(1-q) + (1-p)aq + (1-p)a(1-q)$$

and use the notations

$$a_{11} = paq, \quad a_{12} = pa(1-q), \quad a_{21} = (1-p)aq, \quad a_{22} = (1-p)a(1-q).$$

Hence, the elements $p, q \in \mathcal{R}^\bullet$ induce a representation of an arbitrary element $a \in \mathcal{R}$, which is given by the following matrix form:

$$a = \begin{bmatrix} paq & pa(1-q) \\ (1-p)aq & (1-p)a(1-q) \end{bmatrix}_{p,q} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{p,q}.$$

It is well-known that $a \in \mathcal{R}^d$ can be represented in the following matrix form

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{p,p},$$

relative to $p = aa^d = 1 - a^\pi$, a_1 is invertible in the algebra $p\mathcal{R}p$ and a_2 is quasinilpotent in the algebra $(1-p)\mathcal{R}(1-p)$. Then the generalized Drazin inverse of a is given by

$$a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p,p}.$$

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

We say that $b = a^\dagger$ is the Moore-Penrose inverse (or MP-inverse) of a, if the following hold:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [1]). The set of all Moore-Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

Definition 2.2. *Let $\mathcal{R}$ be a ring with involution and e, f two invertible elements in $\mathcal{R}$. We say that the element $a \in \mathcal{R}$ has the weighted MP-inverse with weights e, f if there exists $b \in \mathcal{R}$ such that*

$$aba = a, \quad bab = b, \quad (eba)^* = eba, \quad (fab)^* = fab.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^\dagger_{e,f}$ if it exists [1]. The set of all weighted MP-invertible elements of $\mathcal{R}$ with weights e, f, will be denoted by $\mathcal{R}^\dagger_{e,f}$.

3 Weighted generalized Drazin inverse

Let $\mathcal{R}$ be a ring with the unit 1, and let $w \in \mathcal{R}$. Let $\mathcal{R}_w$ be the ring $\mathcal{R}$ equipped with the w-product: $a * b = awb$ for $a, b \in \mathcal{R}$. If w is invertible, then $1_w = w^{-1}$ is the unit of the ring $\mathcal{R}_w$. In the rest of the paper we assume that $w \in \mathcal{R}^{-1}$. It is not difficult to see that, with this assumption, the equality $\mathcal{R}^{-1} = \mathcal{R}_w^{-1}$ holds.

Now, we introduce the weighted and the weighted generalized Drazin inverse in rings.

Definition 3.1. *Let $w \in \mathcal{R}^{-1}$. An element $a \in \mathcal{R}$ is called:*

(a) *weighted Drazin invertible, or w–Drazin invertible, if a is Drazin invertible in the ring $\mathcal{R}_w$. The w–Drazin inverse $a^{D,w}$ of a is then defined as the Drazin inverse of a in the ring $\mathcal{R}_w$, i.e. $a^{D,w} = a^D_{\mathcal{R}_w}$.*

(b) *weighted generalized Drazin invertible, or wg–Drazin invertible, if a is generalized Drazin invertible in the ring $\mathcal{R}_w$. The wg–Drazin inverse $a^{d,w}$ of a is then defined as the generalized Drazin inverse of a in the ring $\mathcal{R}_w$, i.e. $a^{d,w} = a^d_{\mathcal{R}_w}$.*

The set of all wg–Drazin invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{d,w}$, and the set of all w–Drazin invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{D,w}$. By Theorem 2.1, it follows that the wg–Drazin inverse is unique if it exists.

We prove the following result concerning quasinilpotent elements in $\mathcal{R}$ and $\mathcal{R}_w$.

Theorem 3.1. *Let $\mathcal{R}$ be a ring with the unit 1, and let $w \in \mathcal{R}^{-1}$. For $a \in \mathcal{R}$ the following statements are equivalent:*

(a) *$a \in \mathcal{R}_w^{qnil}$;*

(b) *$aw \in \mathcal{R}^{qnil}$;*

(c) *$wa \in \mathcal{R}^{qnil}$.*

Proof. (a) $\Longrightarrow$ (b): Let $a \in \mathcal{R}_w^{qnil}$. Suppose that $y \in \mathcal{R}$ such that $awy = yaw$. We take $x = yw^{-1}$. Then

$$a * x = awx = awyw^{-1} = yaww^{-1} = ya,$$

and

$$x * a = xwa = yw^{-1}wa = ya.$$

Hence, x commutes with a in $\mathcal{R}_w$. Then $w^{-1} + a * x \in \mathcal{R}_w^{-1}$, so there exists some $b \in \mathcal{R}_w^{-1}$ such that $b * (w^{-1} + a * x) = w^{-1}$, which is equivalent to $bw(w^{-1} + awx) = w^{-1}$, or $bw(1 + awy) = 1$. We know that $b \in \mathcal{R}^{-1}$ also, so it follows that $1 + awy \in \mathcal{R}^{-1}$. We have just proved that $aw \in \mathcal{R}^{qnil}$, so (b) is satisfied.

(b) $\Longrightarrow$ (a): Let $aw \in \mathcal{R}^{qnil}$. Assume that $x \in \mathcal{R}$ such that $a*x = x*a$. It follows that $awx = xwa$, and $awxw = xwaw$. Let $y = xw$. We get that aw commutes with y. Consequently, $1 + yaw \in \mathcal{R}^{-1}$, so there exists some $c \in \mathcal{R}^{-1}$ satisfying $c(1+yaw) = 1$, which is equivalent to $c(w^{-1}+x*a)w = 1$. Hence, $w^{-1} + x * a \in \mathcal{R}^{-1} = \mathcal{R}_w^{-1}$, so (a) is satisfied.

The equivalence (a) $\Longleftrightarrow$ (c) can be proved similarly. $\square$

Now, we prove the main characterization of the wg–Drazin invertible elements in rings.

Theorem 3.2. *Let $\mathcal{R}$ be a ring with the unit 1, and let $w \in \mathcal{R}^{-1}$. For $a \in \mathcal{R}$ the following statements are equivalent:*

(a) *a is wg-Drazin invertible with the wg-Drazin inverse $a^{d,w} = b \in \mathcal{R}$.*

(b) *aw is generalized Drazin invertible in $\mathcal{R}$ with $(aw)^d = bw$.*

(c) *wa is generalized Drazin invertible in $\mathcal{R}$ with $(wa)^d = wb$.*

The wg–Drazin inverse $a^{d,w}$ of a then satisfies

$$a^{d,w} = ((aw)^d)^2 a = a((wa)^d)^2. \tag{1}$$

Proof. (a) $\Longrightarrow$ (b): Suppose that a has the wg-Drazin inverse, which is denoted by b. Then

$$b \in \mathrm{comm}_w^2(a), \qquad b * a * b = b, \qquad a * b * a - a \in \mathcal{R}_w^{qnil}.$$

Suppose that $y \in \mathcal{R}$ such that $awy = yaw$ holds. We take $x = yw^{-1}$. Then

$$a * x = awx = awyw^{-1} = yaww^{-1} = ya,$$

and

$$x * a = xwa = yw^{-1}wa = ya.$$

Hence, $a * x = x * a$. Since $b \in \text{comm}_w^2(a)$, $b * x = x * b$, i.e.

$$bwyw^{-1} = yw^{-1}wb.$$

So, $bwy = ybw$. Then $c = bw \in \text{comm}^2(aw)$.

Since $b * a * b = b$, we have $(bw)^2 a = b$ and

$$c^2(aw) = (bw)^2 aw = bw = c.$$

From $a*b*a-a \in \mathcal{R}_w^{qnil}$, we obtain $(aw)^2 b - a \in \mathcal{R}_w^{qnil}$. Then $(aw)^2 bw - aw \in \mathcal{R}^{qnil}$ by Theorem 3.1. Thus, $(aw)^2 c - aw$ is quasinilpotent in $\mathcal{R}$, and (b) is proved, with

$$(aw)^d = c = bw.$$

(b) $\Longrightarrow$ (a): Assume that $aw \in \mathcal{R}$ has the g-Drazin inverse c. Then

$$c \in \text{comm}^2(aw), \qquad c^2(aw) = c, \qquad c(aw)^2 - aw \in \mathcal{R}^{qnil}.$$

Let $b = c^2 a$. Suppose that $x \in \mathcal{R}$ such that $a * x = x * a$. Set $y = xw$. Now,

$$awy = awxw = xwaw = yaw.$$

Since $c \in \text{comm}^2(aw)$, we get $cy = yc$. From

$$bwy = c^2 awy = c^2 yaw = yc^2 aw = ybw,$$

i.e

$$bwxw = xwbw,$$

we get

$$b * x = bwx = bwxww^{-1} = xwbww^{-1} = xwb = x * b.$$

Therefore, $b \in \text{comm}_w^2(a)$. The equation $c^2(aw) = c$ imply

$$b * a * b = (c^2 aw)(awc^2)a = c^2 a = b.$$

Since $c(aw)^2 - aw$ is quasinilpotent in $\mathcal{R}$, we obtain that

$$a * b * a - a = (awc^2)awa - a = cawa - a$$

is quasinilpotent in $\mathcal{R}_w$, by Theorem 3.1. Hence, a is wg-Drazin invertible with $a^{d,w} = c^2 a$.

The equivalence (a) $\Longleftrightarrow$ (c) can be proved similarly. $\square$

Finally, we prove the following result concerning the matrix form of $a \in \mathcal{R}^{d,w}$.

Theorem 3.3. *Let $\mathcal{R}$ be a ring with the unit 1, and let $w \in \mathcal{R}^{-1}$. Then $a \in \mathcal{R}$ is wg-Drazin invertible if and only if there exist $p, q \in \mathcal{R}^\bullet$ such that $p \in \mathrm{comm}^2(aw)$, $q \in \mathrm{comm}^2(wa)$, and*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{p,q}, \quad w = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}_{q,p},$$

where $a_1w_1 \in (p\mathcal{R}p)^{-1}$, $w_1a_1 \in (q\mathcal{R}q)^{-1}$, $a_2w_2 \in ((1-p)\mathcal{R}(1-p))^{qnil}$ and $w_2a_2 \in ((1-q)\mathcal{R}(1-q))^{qnil}$. The wg–Drazin inverse of a is given by

$$a^{d,w} = \begin{bmatrix} a_1((w_1a_1)^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix}_{p,q} = \begin{bmatrix} ((a_1w_1)^{-1})^2a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p,q}. \tag{2}$$

Proof. If $a \in \mathcal{R}$ is wg-Drazin invertible, then aw and wa are generalized Drazin invertible. We have the following matrix representations of aw and wa relative to $p = aw(aw)^d$ and $q = wa(wa)^d$:

$$aw = \begin{bmatrix} (aw)_1 & 0 \\ 0 & (aw)_2 \end{bmatrix}_{p,p}, \quad wa = \begin{bmatrix} (wa)_1 & 0 \\ 0 & (wa)_2 \end{bmatrix}_{q,q},$$

where $(aw)_1 = (aw)^2(aw)^d \in (p\mathcal{R}p)^{-1}$, $(aw)_2 \in ((1-p)\mathcal{R}(1-p))^{qnil}$, $(wa)_1 = (wa)^2(wa)^d \in (q\mathcal{R}q)^{-1}$, $(wa)_2 \in ((1-q)\mathcal{R}(1-q))^{qnil}$.

The idempotents $p = aw(aw)^d$, $q = wa(wa)^d$ induce a representation of a given by

$$a = \begin{bmatrix} paq & pa(1-q) \\ (1-p)aq & (1-p)a(1-q) \end{bmatrix}_{p,q} = \begin{bmatrix} a_1 & a_{12} \\ a_{21} & a_2 \end{bmatrix}_{p,q}.$$

Thus, we obtain $a_{12} = pa(1-q) = 0$ and $a_{21} = (1-p)aq = 0$. Hence,

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{p,q}.$$

Similarly, the idempotents q, p induce the following representation of w

$$w = \begin{bmatrix} qwp & qw(1-p) \\ (1-q)wp & (1-q)w(1-p) \end{bmatrix}_{q,p} = \begin{bmatrix} w_1 & w_{12} \\ w_{21} & w_2 \end{bmatrix}_{q,p}.$$

From $w_{12} = qw(1-p) = 0$ and $w_{21} = (1-q)wp = 0$, we get

$$w = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}_{q,p}.$$

Now,

$$aw = \begin{bmatrix} a_1w_1 & 0 \\ 0 & a_2w_2 \end{bmatrix}_{p,p}, \quad wa = \begin{bmatrix} w_1a_1 & 0 \\ 0 & w_2a_2 \end{bmatrix}_{q,q},$$

where $a_1w_1 = (aw)_1 = (aw)^2(aw)^d \in (p\mathcal{R}p)^{-1}$, $a_2w_2 = (aw)_2 \in ((1-p)\mathcal{R}(1-p))^{qnil}$, $w_1a_1 = (wa)_1 = (wa)^2(wa)^d \in (q\mathcal{R}q)^{-1}$, $w_2a_2 = (wa)_2 \in ((1-q)\mathcal{R}(1-q))^{qnil}$.

The wg–Drazin inverse of a is equal, by (1), to

$$\begin{aligned} a^{d,w} &= a((wa)^d)^2 \\ &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{p,q} \begin{bmatrix} (w_1a_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}^2_{q,q} \\ &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{p,q} \begin{bmatrix} ((w_1a_1)^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix}_{q,q} \\ &= \begin{bmatrix} a_1((w_1a_1)^{-1})^2 & 0 \\ 0 & 0 \end{bmatrix}_{p,q}. \end{aligned}$$

The second equation in (2) can be obtained from $a^{d,w} = ((aw)^d)^2a$.

Conversely, if the decompositions with the specified properties exist, then

$$aw = \begin{bmatrix} a_1w_1 & 0 \\ 0 & a_2w_2 \end{bmatrix}_{p,p}.$$

Since $a_1w_1 \in (p\mathcal{R}p)^{-1}$ and $a_2w_2 \in ((1-p)\mathcal{R}(1-p))^{qnil}$, it follows that aw is generalized Drazin invertible. Then a is wg–Drazin invertible. □

We can also consider rings with involution. The following result is proved in [5].

Theorem 3.4. *Let $\mathcal{R}$ be a ring with involution. Then a is generalized Drazin invertible if and only if a^* is generalized Drazin invertible. In this case $(a^*)^d = (a^d)^*$.*

We prove the following result.

Theorem 3.5. *Let $\mathcal{R}$ be a ring with involution. Then $a \in \mathcal{R}$ is wg-Drazin invertible if and only if a^* is w^*g-Drazin invertible. In this case $(a^*)^{d,w^*} = (a^{d,w})^*$.*

Proof. From the preceding theorem aw is generalized Drazin invertible if and only if $(aw)^* = w^*a^*$ is generalized Drazin invertible. Then a is wg–Drazin invertible if and only if a^* is w^*g–Drazin invertible, by Theorem 3.2. □

Finally, if $\mathcal{R}$ is a complex Banach algebra, then we can assume more general conditions. Precisely, we do not need the unit in $\mathcal{R}$, and consequently w is not necessarily invertible. We use the norm $\|\cdot\|_w$ in $\mathcal{R}_w$ as follows: if $a \in \mathcal{R}$, then $\|a\|_w = \|a\|\|w\|$. Thus, $\mathcal{R}_w$ becomes a complex Banach algebra. We adjoin the unit 1 to $\mathcal{R}_w$. Hence, the concept of the spectrum and the spectral radius is available. The most important technical statement, concerning the weighted Drazin invertibility, is the following result [6].

Lemma 3.1. *Let $\mathcal{R}$ be a complex Banach algebra, and let $a, w \in \mathcal{R}$. Then $r_w(a) = r(aw) = r(wa)$, where $r(\cdot)$ denotes the spectral radius in $\mathcal{R}$, and $r_w(\cdot)$ denotes the spectral radius in $\mathcal{R}_w$. Consequently, $a \in \mathcal{R}_w^{qnil}$, if and only if $aw \in \mathcal{R}^{qnil}$, if and only if $wa \in \mathcal{R}^{qnil}$.*

For the sake of completeness, we state and give a short proof of the following result in Banach algebras. See also [6].

Theorem 3.6. *Let $\mathcal{R}$ be a complex Banach algebra, and let $w \in \mathcal{R}$ be a nonzero element. Then for $a \in \mathcal{R}$ the following statements are equivalent:*

(a) *$a \in \mathcal{R}_w^d$ and $a^{d,w} = b \in \mathcal{R}$.*

(b) *$aw \in \mathcal{R}^d$ and $(aw)^d = bw$.*

(c) *$wa \in \mathcal{R}^d$ and $(wa)^d = wb$.*

The wg-Drazin inverse $a^{d,w}$ of a then satisfies

$$a^{d,w} = ((aw)^d)^2 a = a((wa)^d)^2. \tag{3}$$

Proof. (a) $\Longrightarrow$ (b): Suppose that $a^{d,w} = b$. The conditions $a * b = b * a$, $b*a*b = b$, and $a*b*a-a \in \mathcal{R}_w^{qnil}$, translate to $awb = bwa$, $(bw)^2a = b$, and $t = (aw)^2b - a \in \mathcal{R}_w^{qnil}$. Let $c = bw$. Then $(aw)c = c(aw)$ and $c^2(aw) = c$. By Lemma 3.1, we have $r(tw) = r_w(t) = 0$. Hence, $(aw)^2c - aw = tw$ is quasinilpotent in $\mathcal{R}$ and (b) is proved, with $(aw)^d = c = bw$.

(b) $\Longrightarrow$ (a): Assume that $(aw)^d = c$. Let $b = c^2a$. The equations $(aw)c = c(aw)$ and $c^2(aw) = c$ imply $a * b = awc^2a = c^2awa = b * a$ and $b * a * b = (c^2aw)(awc^2)a = c^2a = b$. Write $a * b * a - a = (awc^2)awa - a = cawa - a = s$. Since $sw = c(aw)^2 - aw$ is quasinilpotent in $\mathcal{R}$, we get $r_w(s) = r(sw) = 0$ and s is quasinilpotent in $\mathcal{R}_w$. Therefore, a is wg-Drazin invertible with $a^{d,w} = c^2a$. □

Finaly, we mention that Theorem 2.3 is also valid if we suppose that $\mathcal{R}$ is a Banach algebra, without the assumption $w \in \mathcal{R}^{-1}$.

4 Weighted EP elements in rings

We recall the following definition of EP element [5].

Definition 4.1. *An element a of a ring $\mathcal{R}$ with involution is said to be EP if $a \in \mathcal{R}^d \cap \mathcal{R}^\dagger$ and $a^d = a^\dagger$. An element a is generalized EP (or gEP for short) if there exists $k \in N$ such that a^k is EP.*

Now, we introduce the weighted EP and the weighted generalized EP elements in rings.

Definition 4.2. *An element a of a ring $\mathcal{R}$ with involution is said to be weighted EP if $a \in \mathcal{R}^d_w \cap \mathcal{R}^\dagger_w$ and $a^{d,w} = a^\dagger_{\mathcal{R}_w}$. An element a is weighted generalized EP (or wgEP for short) if there exists $k \in N$ such that a^k is weighted EP.*

In the following theorem we prove the main characterization of EP elements in rings.

Theorem 4.1. *Let $\mathcal{R}$ be a ring with involution and with the unit 1, and let $w \in \mathcal{R}^{-1}$. For $a \in \mathcal{R}$ the following statements are equivalent:*

(a) *a is weighted EP.*

(b) *$aw \in \mathcal{R}^d \cap \mathcal{R}^\dagger_{w^*,w^*}$ and $(aw)^d = (aw)^\dagger_{w^*,w^*}$.*

(c) *$wa \in \mathcal{R}^d \cap \mathcal{R}^\dagger_{w^{-1},w^{-1}}$ and $(wa)^d = (wa)^\dagger_{w^{-1},w^{-1}}$.*

Proof. (a) $\Longrightarrow$ (b): Suppose that a is weighted EP, i.e. $a \in \mathcal{R}^d_w \cap \mathcal{R}^\dagger_w$ and $a^{d,w} = a^\dagger_{\mathcal{R}_w} \equiv b$. From $a \in \mathcal{R}^d_w$ and Theorem 3.2, we get $aw \in \mathcal{R}^d$ with $(aw)^d = bw$. Since $a \in \mathcal{R}^\dagger_w$ and $a^\dagger_{\mathcal{R}_w} = b$, by definition, we have

$$a * b * a = a, \quad b * a * b = b, \quad (a * b)^* = a * b, \quad (b * a)^* = b * a,$$

i.e.

$$awbwa = a, \quad bwawb = b, \quad (awb)^* = awb, \quad (bwa)^* = bwa.$$

Then, we get

$$awbwaw = aw,$$

$$bwawbw = bw,$$

$$(w^*awbw)^* = w^*(awb)^*w = w^*awbw,$$

$$(w^*bwaw)^* = w^*(bwa)^*w = w^*bwaw.$$

Hence, $aw \in \mathcal{R}^{\dagger}_{w^*,w^*}$ and $(aw)^{\dagger}_{w^*,w^*} = bw = (aw)^d$.

(b) $\Longrightarrow$ (a): Let $aw \in \mathcal{R}^d \cap \mathcal{R}^{\dagger}_{w^*,w^*}$ and $(aw)^d = (aw)^{\dagger}_{w^*,w^*}$. By Theorem 3.2 and $aw \in \mathcal{R}^d$, we have $a \in \mathcal{R}^d_w$ and $b = a^{d,w}$. Since $bw = (aw)^d = (aw)^{\dagger}_{w^*,w^*}$, by definition of the weighted MP-inverse, we obtain

$$awbwaw = aw, \quad bwawbw = bw,$$

$$(w^*awbw)^* = w^*awbw, \quad (w^*bwaw)^* = w^*bwaw.$$

Now, we get

$$a * b * a = awbwaww^{-1} = aww^{-1} = a,$$

$$b * a * b = bwawbww^{-1} = bww^{-1} = b,$$

$$(a * b)^* = ((w^*)^{-1}w^*awbww^{-1})^* = (w^*)^{-1}w^*awbww^{-1} = a * b,$$

$$(b * a)^* = ((w^*)^{-1}w^*bwaww^{-1})^* = (w^*)^{-1}w^*bwaww^{-1} = b * a.$$

Thus, $a \in \mathcal{R}^d_w$ and $a^{\dagger}_{\mathcal{R}_w} = b = a^{d,w}$. So, a is weighted EP element.

(a) $\Longrightarrow$ (c): Suppose that a is weighted EP, i.e. $a \in \mathcal{R}^d_w \cap \mathcal{R}^{\dagger}_w$ and $a^{d,w} = a^{\dagger}_{\mathcal{R}_w} \equiv b$. From $a \in \mathcal{R}^d_w$ and Theorem 3.2, we get $wa \in \mathcal{R}^d$ with $(wa)^d = wb$. By $a \in \mathcal{R}^{\dagger}_w$, we have

$$a * b * a = a, \quad b * a * b = b, \quad (a * b)^* = a * b, \quad (b * a)^* = b * a,$$

i.e.

$$awbwa = a, \quad bwawb = b, \quad (awb)^* = awb, \quad (bwa)^* = bwa.$$

Now, we obtain

$$wawbwa = wa,$$

$$wbwawb = wb,$$

$$(w^{-1}wawb)^* = (awb)^* = awb = w^{-1}wawb,$$

$$(w^{-1}wbwa)^* = (bwa)^* = bwa = w^{-1}wbwa.$$

Therefore, $wa \in \mathcal{R}^{\dagger}_{w^{-1},w^{-1}}$ and $(wa)^{\dagger}_{w^{-1},w^{-1}} = wb = (wa)^d$.

(c) $\Longrightarrow$ (a): Assume that $wa \in \mathcal{R}^d \cap \mathcal{R}^{\dagger}_{w^{-1},w^{-1}}$ and $(wa)^d = (wa)^{\dagger}_{w^{-1},w^{-1}}$. By Theorem 3.2 and $wa \in \mathcal{R}^d$, we have $a \in \mathcal{R}^d_w$ and $b = a^{d,w}$. Since $wb = (wa)^d = (wa)^{\dagger}_{w^{-1},w^{-1}}$, we obtain

$$wawbwa = wa, \quad wbwawb = wb,$$

$$(w^{-1}wawb)^* = w^{-1}wawb, \quad (w^{-1}wbwa)^* = w^{-1}wbwa.$$

Then, we get

$$a * b * a = w^{-1}wawbwa = w^{-1}wa = a,$$

$$b * a * b = w^{-1}wbwawb = w^{-1}wb = b,$$

$$(a * b)^* = (w^{-1}wawb)^* = w^{-1}wawb = a * b,$$

$$(b * a)^* = (w^{-1}wbwa)^* = w^{-1}wbwa = b * a.$$

Thus, $a \in \mathcal{R}_w^\dagger$ and $a_{\mathcal{R}_w}^\dagger = b = a^{d,w}$. So, a is weighted EP element. $\square$

References

[1] D. Cvetković, D. S. Djordjević, J. J. Koliha, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl. 426 (2007), 371-381.

[2] M. P. Drazin, *Pseudo-inverses in associate rings and semigroups*, Amer. Math. Monthly 65 (1958), 506-514.

[3] R. E.Harte, *On quasinilpotents in rings*, Panamer. Math. J. 1 (1991), 10-16.

[4] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. 38 (1996), 367–381.

[5] J. J. Koliha, P. Patricio *Elements of rings with equal spectral idempotents*, J. Australian Math. Soc. 72 (2002), 137–152.

[6] A. Dajić, J. J. Koliha, *The weighted g–Drazin inverse for operators*, J. Australian Math. Soc. 82 (2007), 163–181.

[7] V. Rakočević, Y. Wei, *A weighted Drazin inverse and applications*, Linear Algebra Appl. 350 (2002), 25–39.

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