Abstract “Simple permanence” is one of several variants of “spectral permanence”, which are curiously interrelated.

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0. Introduction

This is a reworking of our previous note [DZH], in which we deployed “Drazin permanence” and quasipolar Banach algebra elements in the proof of a variant of the “spectral permanence” enjoyed by C* algebras. Here we use instead “simple permanence” and simply polar elements of semigroups and rings: we believe that the argument is now more transparent and more elementary.

1. Generalized permanence

If $T: A \rightarrow B$ is a “semigroup homomorphism” [DZH] then there is inclusion

1.1

$T(A^{-1}) \subseteq B^{-1} \subseteq B$ ,

where $A^{-1}$ is the invertible group of $A$, and hence also

1.2

$A^{-1} \subseteq T^{-1}B^{-1} \subseteq A$ ;

equality here is what is known as the “Gelfand property”, or spectral permanence, for the homomorphism

T:

1.3

$T^{-1}B^{-1} \subseteq A^{-1}$ .

More generally the “relatively regular” elements

1.4

$A^\cap = \{a \in A : a \in aAa\}$

satisfy

1.5

$A^{-1} = A^{-1}_{left} \cap A^{-1}_{right} \subseteq A^{-1}_{left} \cup A^{-1}_{right} \subseteq A^\cap$ ,

and if $T: A \rightarrow B$ is a semigroup homomorphism then

1.6

$T(A^\cap) \subseteq B^\cap \subseteq B$ ,

and hence

1.7

$A^\cap \subseteq T^{-1}B^\cap \subseteq A$ .

Equality in this case will be described as generalized permanence for $T$:

1.8

$T^{-1}B^\cap \subseteq A^\cap$ .

We recall [DZH] that spectral permanence does not in general imply generalized permanence:

Theorem 1 For ring homomorphisms $T: A \rightarrow B$ there is implication

spectral and generalized permanence together imply one-one .

Proof. Generally $T: A \rightarrow B$ has spectral permanence only if

1.9

$T^{-1}(0) \subseteq \text{Rad}(A)$ ,

has generalized permanence only if

1.10

$T^{-1}(0) \subseteq A^\cap$ ,

and evidently

1.11

$\text{Rad}(A) \cap A^\cap = O \equiv \{0\}$ ,

where

1.12

$\text{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\}$ .
2. Simple polarity

If \( a \in A \) has a commuting generalized inverse we shall call it “group invertible” or simply polar:

2.1 \( \text{SP}(A) = \{ a \in A : a \in \text{comm}(a) \} \).

If \( T : A \to B \) is a homomorphism then

2.2 \( \text{TSP}(A) \subseteq \text{SP}(B) \subseteq B \),
equivalently

2.3 \( \text{SP}(A) \subseteq T^{-1}\text{SP}(B) \subseteq A \).

When there is equality here we say that \( T \) has simple permanence. If we think of the counterimage \( T^{-1}B^{-1} \) as in some sense “Fredholm” elements of the semigroup \( A \), then the counterimage \( T^{-1}\text{SP}(B) \) abstracts what Caradus [C] and Schmoeger [S] have called generalized Fredholm operators.

Necessary and sufficient for \( a \in A \) to be simply polar is \([X],[HLu]\) that

2.4 \( a \in A a^2 = a^2 A \):
recall
\[
a^2 u = a = va^2 \implies aua = a = ava
\]
and take \( c = va\) for a “group inverse”. Also necessary and sufficient for \( a \in \text{SP}(A) \), in rings, is ([S];[KDH] Theorem 5) that there be a “semigroup inverse”, \( c \in A \) for which

2.5 \( a = aca ; 1 - ac - ca \in A^{-1} \).

Notice also

2.6 \( \text{SP}(A) \subseteq A^{-1} = \{ a \in A : a \in aA^{-1}a \} \):
observe that \( a + (1 - ac) \) and \( cac + (1 - ac) \) are mutually inverse. It follows

2.7 \( \text{SP}(A) \cap A^{-1}_{\text{left}} = A^{-1} = \text{SP}(A) \cap A^{-1}_{\text{right}} \).

**Theorem 2** If the semigroup \( A \) is commutative and the range

2.8 \( T(A) \cap B^{-1}_{\text{left}} \setminus B^{-1} \neq \emptyset \)
then \( T : A \to B \) does not have generalized permanence. It follows that spectral permanence and one one do not together imply generalized permanence.

**Proof.** If \( A \) is commutative then, using (2.7),

2.9 \( T(a) \in B^{\cap} \setminus \text{SP}(B) \implies a \notin A^{\cap} \),
v violating generalized permanence. In particular if

2.10 \( T = J : A = \text{comm}_A^2(a) \subseteq B \)
then \( T \) is one one and has spectral permanence, while

2.11 \( a \in B^{-1}_{\text{left}} \setminus B^{-1} \implies a \in B^{\cap} \setminus \text{SP}(B) \bullet \)

For a specific example ([DZH] Theorem 3.2) take \( a \in B = B(\ell_2) \) to be the (forward) unilateral shift.

Alternatively, replace the natural embedding \( J \) by the left regular representation \( L \). For another example look at the embedding, for a compact Hausdorff space \( X \),

2.12 \( C(X) \subseteq C^X \),
or alternatively, for a Banach space \( X \),

2.13 \( B(X) \subseteq L(X) \);
here of course spectral permanence follows from the open mapping theorem.

**Theorem 3** When \( T : A \to B \) is a ring homomorphism then

2.14 \( T \) one one with spectral permanence \( \implies T \) has simple permanence .

**Proof.** The last implication is the argument of Theorem 1; conversely observe

2.15 \( \text{SP}(A) \cap T^{-1}B^{-1}_{\text{left}} \subseteq A^{\cup} \cap T^{-1}B^{-1}_{\text{left}} \subseteq A^{-1} + T^{-1}(0) \bullet \)
In general (2.12) spectral permanence and one one do not guarantee simple permanence.
3. Simply polar operators

When \( a \in A = L(X) \) is in the ring of additive maps on an abelian group \( X \) then necessary and sufficient that \( a \in \text{SP}(A) \) is that it is both “of ascent 1”, in the sense that

\[
a^{-2}(0) \subseteq a^{-1}(0),
\]
equivalently

\[
a^{-1}(0) \cap a(X) = O,
\]
and also “of descent 1”, in the sense that

\[
a(X) \subseteq a^2(X),
\]
equivalently

\[
a^{-1}(0) + a(X) = X.
\]

The same conditions characterise simple polarity in the ring of linear mappings on a vector space, and also in the ring \( A = B(X) \) of bounded linear mappings on a Banach space: here however two or three applications of the open mapping theorem are necessary. For incomplete normed spaces however the conditions (3.1) and (3.3), even together with relative regularity \( a \in A^\circ \), are not in general sufficient:

**Theorem 4** If \( a \in A \) is arbitrary in the ring \( A \) then, with

\[
b = \begin{pmatrix} a & -1 \\ 0 & 0 \end{pmatrix} \in B = \begin{pmatrix} A & A \\ A & A \end{pmatrix}, \quad d = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in B,
\]
then automatically

\[
b = bdb \in B^\circ,
\]
while there is implication

\[
b \in Bb^2 \implies a \in A_{left}^{-1},
\]
and also implication

\[
b \in b^2B \implies a \in A_{right}^{-1}.
\]

Hence

\[
b \in \text{SP}(B) \implies a \in A^{-1}.
\]

**Proof.** Look at the top right hand corner element •

For example ([H] (7.3.6.8)) we may take \( A = B(X) \) with \( X = c_{00} \subseteq c_0 \) the space of “terminating sequences”, and \( a = w \in A \) the “standard weight”

\[
w(x)_n = (1/n)x_n.
\]

When \( A = B(X) \) for a normed space \( X \) and \( a \in A \) is of ascent and descent one then \([X]\) each of the following conditions is sufficient for simple polarity:

- \( X \) complete;
- \( a \in A \) Fredholm;
- \( a \in A \) finite rank;
- \( b \in X \) a normed algebra and \( a \in \{L_0, R_0\} \subseteq B(X) \).
4. Koliha-Drazin permanence

More generally if there is \( n \in \mathbb{N} \) for which \( a^n \) is simply polar we shall also say that \( a \in A \) is “polar”, or Drazin invertible. If \( a \in A \) is polar then there is \( c \in A \) for which \( ac = ca \) and \( a - ac \) is nilpotent. More generally still if we write, in a Banach algebra \( A \),

\[
QN(A) = \{ a \in A : 1 - Ca \subseteq A^{-1} \}
\]

for the quasinilpotents of \( A \), then \( a \in QN(A) \) if and only if \( \sigma_A(a) \subseteq \{0\} \), while with some complex analysis we can prove that if \( a \in QN(A) \) then

\[
\|a^n\|^{1/n} \to 0 \ (n \to \infty).
\]

Since (4.1) and (4.2) are equivalent it follows that also equivalent [H2],[K] is the condition

\[
QN(A) = \{ a \in A : 1 - \text{comm}(a)a \subseteq A^{-1} \}.
\]

In the ultimate generalization of “group invertibility”, we shall write \( QP(A) \) for the quasipolar elements \( a \in A \), those which have a spectral projection, \( q \in A \) for which

\[
q = q^2 ; \ aq = qa ; \ a + q \in A^{-1} ; \ aq \in QN(A).
\]

Now [K] the spectral projection and the Koliha-Drazin inverse

\[
a^\bullet = q , \ a^\times = (a + q)^{-1}(1 - q)
\]

are uniquely determined and lie in the double commutant of \( a \in A \). It is easy to see that if (4.4) is satisfied then

\[
0 \notin \text{acc} \ \sigma_A(a) : 
\]

the origin cannot be an accumulation point of the spectrum; conversely if (4.6) holds then we can display the spectral projection as a sort of “vector-valued winding number”

\[
a^\bullet = \frac{1}{2\pi i} \oint (z - a)^{-1}dz,
\]

where we integrate counter clockwise round a small circle \( \gamma \) centre the origin whose connected hull \( \eta \gamma \) is a disc whose intersection with the spectrum is at most the point \( \{0\} \). By the same technique we can display the Koliha-Drazin inverse in the form

\[
a^\times = \frac{1}{2\pi i} \oint_{\sigma'(a)} z^{-1}(z - a)^{-1}dz,
\]

where \( \sigma'(a) = \sigma(a) \setminus \{0\} \). Now generally for a homomorphism \( T : A \to B \) there is inclusion

\[
T \ QP(A) \subseteq QP(B),
\]

while if \( T : A \to B \) has spectral permanence in the sense (1.3) then it is clear from (4.6) that there is also “Drazin permanence” in the sense that

\[
QP(A) = T^{-1}QP(B) \subseteq A.
\]
Theorem 5 For Banach algebra homomorphisms $T : A \to B$ there is implication

4.11 $\text{spectral permanence} \implies \text{Drazin permanence}.$

Proof. Equality in (1.3), expressed [DZH] in terms of the spectrum, together with (4.6) •

We recall ([DZH] Theorem 2) that in (2.11) the shift $a \in B \setminus \text{QP}(B)$.

As a sort of converse to Theorem 5, and squaring the circle in Theorem 3,

Theorem 6 If $T : A \to B$ is a Banach algebra homomorphism then

4.12 $\text{QP}(A) \cap T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0)$

and if $T : A \to B$ is one one then

4.13 $\text{QP}(A) \cap T^{-1}\text{SP}(B) = \text{SP}(A).$

Hence if $a \in B$ and $T = J : A = \text{comm}^2(a) \subseteq B$ then

4.14 $A^\cap = T^{-1}\text{SP}(B).$

Hence if $T^{-1}(0) = \{0\}$ is one one then

4.15 $\text{Drazin} \implies \text{simple} \implies \text{spectral permanence}.$

Proof. Uniqueness guarantees that the spectral projection $T(a^*)$ of $Ta \in \text{SP}(B) \subseteq \text{QP}(B)$ commutes with $T(a) \in B$, and one-one-ness guarantees the same for $a \in A$ •
5. Moore-Penrose permanence

By a star semigroup we shall understand a semigroup $A$ with an involution, $*: A \to A$ satisfying, for arbitrary $a, c \in A$,

$$ (a^*)^* = a ; \quad (ca)^* = a^*c^* ; \quad 1^* = 1 . $$

In rings and algebras involutions are assumed to be additive, and “conjugate linear”. Obviously there is implication

$$ a \in H(A) \Rightarrow a^* \in H(A) $$

for each $H(A) \in \{ A^{-1}, A^\cap, \text{SP}(A) \}$. Elements $a \in A$ are said to be hermitian or “real” when they are the same as their adjoints:

$$ \text{Re}(A) = \{ a \in A : a^* = a \} . $$

A Moore-Penrose inverse for $a \in A$ is $c = a^\dagger \in A$ for which the induced idempotents are hermitian:

$$ a = aca ; \quad c = cac ; \quad (ca)^* = ca ; \quad (ac)^* = ac . $$

We write $A^\dagger \subseteq A^\cap$ for those $a \in A$ for which $a^\dagger$ exists. The argument ([HM] Theorem 5) for “C* algebras” works in semigroups [X2], and says that

$$ a^{\dagger} \in \text{comm}^2(a, a^*) $$
is unique and double commutes with $\{ a, a^* \}$ in $A$. The “B* condition”, in a Banach algebra $A$, says that

$$ \| a^*a \| = \| a \|^2 . $$

It follows

$$ a, x \in A \Rightarrow \| ax \|^2 \leq \| x^* \| \| a^* ax \| $$

and hence that * is cancellable in the sense that

$$ a \in A \Rightarrow L_{a^*a}(0) \subseteq L_{a^*a}(0) ; $$
in words ([HL] Definition 1) the pair $(L_{a^*}, L_a)$ is “left skew exact”. We need one more object: the “star polars”

$$ \text{SP}^*(A) = \{ a \in A : a^*a \in A^\cap \} . $$

Our main objective is to prove again the Harte/Mbekhta observation ([HM] Theorem 6) that in a C* algebra $A$

$$ A^\cap \subseteq A^\dagger , $$
relatively regular elements always have Moore-Penrose inverse, and that [HM2] isometric C* algebra homomorphisms have generalized permanence. We begin by collecting some elementary observations:

**Theorem 7** If the involution $*: A \to A$ is cancellable then there is inclusion

$$ A^\dagger \subseteq \text{SP}^*(A) \subseteq A^\cap , $$

**Proof.** With cancellation there is implication

$$ a \in \text{SP}^*(A) \Rightarrow a \in aAa^*a \subseteq Aa^*a^*aA , $$

and equality

$$ \text{Re} (A) \cap \text{SP}^*(A) = \text{Re} (A) \cap \text{SP}(A) . $$

If $a = aca \in A^\dagger$ with $a^\dagger = c$ then

$$ a^*a = a^*(ac)(ac)^*a = a^*acc^*a^*a \in a^*aAa^*a ; $$

conversely (5.7)

$$ a^*a = a^*ada^*a \Rightarrow a = ada^*a ; $$

hence also

$$ a \in Aa^*a , \quad \Leftrightarrow \quad a^* \in a^*aA . $$

Hence if $a^* = a$ then (2.4) follows $\bullet$
Now it is clear that isometric C* homomorphisms have “Moore-Penrose permanence”:

**Theorem 8** If $T : A \to B$ is a * homomorphism with simple permanence there is inclusion

$$T^{-1}B^\dagger \subseteq A^\dagger.$$  

*Proof.* We claim

$$A^\dagger = \{ a \in A : a^*a \in \text{SP}(A) \},$$

with implication

$$a^*a \in \text{SP}(A) \implies a^\dagger = (a^*a)^{\times}a^*.$$  

If $a \in A^\dagger$ with $a = cca$ and $(ca)^* = ca$ and $(ac)^* = ac$ then, with $d = cc^*$, we have

$$a^*ad = a^*acc^* = a^*c^*a^*c^* = ca$$

and

$$da^*a = cc^*a^*a = ca.$$  

Conversely if $a^*a = a^*ada^*a$ with $a^*ad = da^*a$ and (wlog: $d \mapsto \frac{1}{2}(d + d^*)$) $d = d^*$ then, with $c = da^*$,

$$aca = ada^*a = a$$

and $ca = da^*a = a^*ad = a^*c^*$. 

Now if $a \in A$ there is, using Theorem 3, implication

$$Ta \in B^\dagger \implies T(a^*a) \in \text{SP}(B) \implies a^*a \in \text{SP}(A) \implies a \in A^\dagger \bullet$$

Thanks to (5.9), this is of course “generalized permanence”. The Harte/Mbekhta result is derived by using the “poor man’s path” to convert the idempotents $ca$ and $ac$ into self adjoint idempotents. Alternatively, thanks to the Gelfand/Naimark/Segal representation, we can look first in the very special algebra $D = B(X)$ of bounded Hilbert space operators:

**Theorem 9** If $d \in D = B(X)$ for a Hilbert space $X$ then

5.12

$$(d^*d)^{-1}(0) \subseteq d^{-1}(0)$$

and

5.13

$$\text{cl } d(X) + d^{-1}(0) = X ;$$

hence

5.14

$$\text{cl } d(X) = d(X) \implies d^*(X) = d^*d(X) , \implies \text{cl } d^*d(X) = d^*d(X) .$$

There is inclusion

5.15

$$\text{Re}(D)_{\cap}D^\cap \subseteq \text{SP}(D) ;$$

hence

5.16

$$d \in D^\cap \implies d \in \text{SP}^*(D) \implies d^*d \in \text{SP}(D) \implies d \in D^\dagger .$$

*Proof.* For arbitrary $\xi \in X$ there is [DZH] inequality

$$\|d\xi\|^2 \leq \|\xi\| \|d^*d\xi\| ,$$

and also

$$\text{cl } d(X) = d^*^{-1}(0)^\dagger \bullet$$

Both of the Harte/Mbekhta observations now follow:

**Theorem 10** If $T : A \to B$ is isometric then

5.17

$$T^{-1}(B^\cap) \subseteq A^\dagger .$$

*Proof.* With $S : B \to D = B(X)$ a GNS mapping we argue, using again Theorem 3,

$$Ta \in B^\cap \implies ST(a^*a) \in \text{SP}(D) \implies a^*a \in \text{SP}(A) \implies a \in A^\dagger \bullet$$

In Theorem 4.1 of [DZH] we established this using the more esoteric QP($A$) rather than SP($A$). It would be entertaining to be able to replace the GNS representation in Theorem 9 with the much more elementary left regular representation $L : A \to B(A)$. 

7
6. Polar decomposition

We conclude with a discussion of the “polar decomposition” of C* algebra elements. In the algebra of operators $A = B(X)$ it is familiar that an arbitrary element $a \in A$ can be written as the product of a “partial isometry” and a positive operator. It is not clear that this can be done in a general C* algebra: for example if $A = C[0, 1]$ there are only two idempotents in $A$ and hence only two possible partial isometries.

We want here to observe that [H3] at least the Moore-Penrose invertibles have polar decomposition. By a generalized polar decomposition for an element $a \in A$ of a C* algebra we shall understand a pair $(u, c) \in A^2$ for which

6.1 \[ u = uu^*u ; \]
6.2 \[ c = c^* ; \]
6.3 \[ L_u^{-1}(0) \subseteq L_c^{-1}(0). \]

If in addition

6.4 \[ 0 \leq c \text{ and } L_c^{-1}(0) \subseteq L_u^{-1}(0) \]

then we shall say that $(u, c)$ a polar decomposition of $a \in A$. We claim ([H3] Theorem 4)

**Theorem 11** If $(u, c) \in A^2$ is a generalized polar decomposition of $a \in A$ then

6.5 \[ a^*a = c^2 \text{ and } u^*a = c . \]

If $(u, c)$ is a polar decomposition of $a$ then each of $u$ and $c$ are uniquely determined and lie in the double commutant of $(a, a^*)$. Also

6.6 \[ aa^*u = ua^*a . \]

**Proof.** For the first part of (6.5) observe that

\[ u^*uc - c \in L_u^{-1}(0) \subseteq L_c^{-1}(0) ; \]

now

\[ (u^*a - c)^*(u^*a - c) = c(u^*u - 1)^2c = 0 , \]

and the second part of (6.5) follows by cancellation. When $(u, c)$ is a polar decomposition then the positivity gives the uniqueness of $c$:

6.7 \[ c = |a| = (a^*a)^{1/2} . \]

The uniqueness of $u^*u$ and $uu^*$ follows from their status as “support” and “cosupport” projections for $a$; for the uniqueness of $u$ suppose $a = uc = vc$ satisfying (6.1)-(6.4): then

\[ (1 - v^*u)c = 0 \implies c(1 - u^*v) = 0 , \implies u(1 - u^*v) = 0 . \]

Now

\[ u^*u = uu^*u , \implies u^*(u - v) = 0 , \]

similarly $v^*(u - v) = 0$, and hence $v = u$ by cancellation.

It is clear from (6.7) that $c$ is in the double commutant of $(a, a^*)$, as are also the support and cosupport $u^*u$ and $uu^*$. Finally if $d \in \operatorname{comm}(a, a^*)$ then it also commutes with each of $c$, $u^*u$ and $uu^*$ and hence

\[ cu^*d = dcu^* = cd^*u^* \implies uu^*d = ud^* = duu^* = uu^*d = u^*d . \]

and hence

\[ du = uu^*udu^*u = uu^*ud = ud . \]

Finally, for (6.6),

\[ aa^*u = uc^2u^*u = ua^*au^*u = ua^*a . \]
We shall write

\[ (u, c) = (\text{sgn}(a), |a|) . \]

Evidently, taking limits of polynomials in \( a^*a \),

\[ |a^*|u = u|a| ; \]

It follows

\[ (\text{sgn}(a^*), |a^*|) = (\text{sgn}(a)^*, \text{sgn}(a)|a|\text{sgn}(a)^*) . \]

We can characterise ([H3] Theorem 5) relative regularity in terms of the polar decomposition:

**Theorem 12** If \( a \in A^1 \subseteq A \) has a Moore-Penrose inverse then it has a polar decomposition, with

\[ \text{sgn}(a) = (a^*)^*|a| . \]

If \( a \in A \) has polar decomposition \((u, c)\) then

\[ d = c + 1 - u^*u \implies L_d^{-1}(0) = \{0\} , \]

and

\[ a \in A^1 \implies d \in A^{-1} \implies a \in A^\cap . \]

**Proof.** We argue, with \( c = a^\dagger \) and \( u = c^*|a| \), that

\[ uu^*u = c^*|a|^2cc^*a = c^*a^*acc^*|a| = (ac)^*(ac)c^*|a| = c^*a^*c^*|a| = c^*|a| \]

and

\[ u|a| = c^*|a|^2 = c^*a^*a = (ac)a = a . \]

If \( x \in A \) is arbitrary there is implication

\[ dx = 0 \implies ucx = 0 \implies cx = 0 = u^*ueu = 0 \implies u^8ux = 0 = (1 - u^*u)x = 0 . \]

Also

\[ d \in A^{-1} \implies ad^{-1}u^*a = udd^{-1}a^*a = ua^*a = a . \]

Conversely if \( a \in A^1 \) then \( a^\dagger a = u^*u \) and \( aa^\dagger = uu^* \) and hence

\[ d' = (a^\dagger a + 1 - u^*u) \implies dd' = 1 = d'd \bullet . \]
References


[DR] DS Djordjević and V Rakocević, Lectures on generalized inverses, University of Nis 2008.


