On simple permanence

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Abstract *"Simple permanence" is one of several variants of "spectral permanence", which are curiously interrelated.*

2010 *Mathematics Subject Classification.* 46H05, 47A05, 47A53.

Keywords and Phrases. Simply polar elements, semigroup homomorphisms.

This research is supported by the Ministry of Science and Technological Development of Serbia, grant no. 174007

0. Introduction

This is a reworking of our previous note [DZH], in which we deployed "Drazin permanence" and *quasipolar* Banach algebra elements in the proof of a variant of the "spectral permanence" enjoyed by C* algebras. Here we use instead "simple permanence" and *simply polar* elements of semigroups and rings: we believe that the argument is now more transparent and more elementary.

1. Generalized permanence

If $T : A \to B$ is a "semigroup homomorphism" [DZH] then there is inclusion

1.1 $$T(A^{-1}) \subseteq B^{-1} \subseteq B \ ,$$

where A^{-1} is the invertible group of A, and hence also

1.2 $$A^{-1} \subseteq T^{-1}B^{-1} \subseteq A \ ;$$

equality here is what is known as the "Gelfand property", or *spectral permanence*, for the homomorphism T:

1.3 $$T^{-1}B^{-1} \subseteq A^{-1} \ .$$

More generally the "relatively regular" elements

1.4 $$A^{\cap} = \{a \in A : a \in aAa\}$$

satisfy

1.5 $$A^{-1} = A^{-1}_{left} \cap A^{-1}_{right} \subseteq A^{-1}_{left} \cup A^{-1}_{right} \subseteq A^{\cap} \ ,$$

and if $T : A \to B$ is a semigroup homomorphism then

1.6 $$T(A^{\cap}) \subseteq B^{\cap} \subseteq B \ ,$$

and hence

1.7 $$A^{\cap} \subseteq T^{-1}B^{\cap} \subseteq A \ .$$

Equality in this case will be described as *generalized permanence* for T:

1.8 $$T^{-1}B^{\cap} \subseteq A^{\cap} \ .$$

We recall [DZH] that spectral permanence does not in general imply generalized permanence:

Theorem 1 *For ring homomorphisms $T : A \to B$ there is implication*

spectral and generalized permanence together imply one-one .

Proof. Generally $T : A \to B$ has spectral permanence only if

1.9 $$T^{-1}(0) \subseteq \mathrm{Rad}(A) \ ,$$

has generalized permanence only if

1.10 $$T^{-1}(0) \subseteq A^{\cap} \ ,$$

and evidently

1.11 $$\mathrm{Rad}(A)_{\cap}A^{\cap} = O \equiv \{0\} \ ,$$

where

1.12 $$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} \bullet$$

2. Simple polarity

If $a \in A$ has a commuting generalized inverse we shall call it "group invertible" or *simply polar*:

2.1 $$\mathrm{SP}(A) = \{a \in A : a \in a\ \mathrm{comm}(a)a\} \ .$$

If $T : A \to B$ is a homomorphism then

2.2 $$T\mathrm{SP}(A) \subseteq \mathrm{SP}(B) \subseteq B \ ,$$

equivalently

2.3 $$\mathrm{SP}(A) \subseteq T^{-1}\mathrm{SP}(B) \subseteq A \ .$$

When there is equality here we say that T has *simple permanence*. If we think of the counterimage $T^{-1}B^{-1}$ as in some sense "Fredholm" elements of the semigroup A, then the counterimage $T^{-1}\mathrm{SP}(B)$ abstracts what Caradus [C] and Schmoeger [S] have called *generalized Fredholm* operators.

Necessary and sufficient for $a \in A$ to be simply polar is [X],[HLu] that

2.4 $$a \in Aa^2{}_\cap a^2A \ :$$

recall

$$a^2u = a = va^2 \Longrightarrow aua = a = ava$$

and take $c = vau$ for a "group inverse". Also necessary and sufficient for $a \in \mathrm{SP}(A)$, in rings, is ([S];[KDH] Theorem 5) that there be a "semigroup inverse", $c \in A$ for which

2.5 $$a = aca \ ; \ 1 - ac - ca \in A^{-1} \ .$$

Notice also

2.6 $$\mathrm{SP}(A) \subseteq A^{\cup} = \{a \in A : a \in aA^{-1}a\} \ :$$

observe that $a + (1 - ac)$ and $cac + (1 - ac)$ are mutually inverse. It follows

2.7 $$\mathrm{SP}(A)_\cap A^{-1}_{left} = A^{-1} = \mathrm{SP}(A)_\cap A^{-1}_{right} \ .$$

Theorem 2 *If the semigroup A is commutative and the range*

2.8 $$T(A)_\cap B^{-1}_{left} \setminus B^{-1} \neq \emptyset$$

then $T : A \to B$ does not have generalized permanence. It follows that spectral permanence and one one do not together imply generalized permanence.

Proof. If A is commutative then, using (2.7),

2.9 $$T(a) \in B^{\cap} \setminus \mathrm{SP}(B) \Longrightarrow a \notin A^{\cap} \ ,$$

violating generalized permanence. In particular if

2.10 $$T = J : A = \mathrm{comm}^2_B(a) \subseteq B$$

then T is one one and has spectral permanence, while

2.11 $$a \in B^{-1}_{left} \setminus B^{-1} \Longrightarrow a \in B^{\cap} \setminus \mathrm{SP}(B) \ \bullet$$

For a specific example ([DZH] Theorem 3.2) take $a \in B = B(\ell_2)$ to be the (forward) unilateral shift.

Alternatively, replace the natural embedding J by the left regular representation L. For another example look at the embedding, for a compact Hausdorff space X,

2.12 $$C(X) \subseteq \mathbf{C}^X \ ,$$

or alternatively, for a Banach space X,

2.13 $$B(X) \subseteq L(X) \ ;$$

here of course spectral permanence follows from the *open mapping theorem*.

Theorem 3 *When $T : A \to B$ is a ring homomorphism then*

2.14 $$T \text{ one one with spectral permanence } \Longrightarrow T \text{ has simple permanence} \ .$$

Proof. The last implication is the argument of Theorem 1; conversely observe

2.15 $$\mathrm{SP}(A)_\cap T^{-1}B^{-1}_{left} \subseteq A^{\cup}{}_\cap T^{-1}B^{-1}_{left} \subseteq A^{-1} + T^{-1}(0) \ \bullet$$

In general (2.12) spectral permanence and one one do not guarantee simple permanence.

3. Simply polar operators

When $a \in A = L(X)$ is in the ring of additive maps on an abelian group X then necessary and sufficient that $a \in \mathrm{SP}(A)$ is that it is both "of ascent 1", in the sense that

3.1 $$a^{-2}(0) \subseteq a^{-1}(0) \ ,$$

equivalently

3.2 $$a^{-1}(0) \cap a(X) = O \ ,$$

and also "of descent 1", in the sense that

3.3 $$a(X) \subseteq a^2(X) \ ,$$

equivalently

3.4 $$a^{-1}(0) + a(X) = X \ .$$

The same conditions characterise simple polarity in the ring of linear mappings on a vector space, and also in the ring $A = B(X)$ of bounded linear mappings on a Banach space: here however two or three applications of the open mapping theorem are necessary. For incomplete normed spaces however the conditions (3.1) and (3.3), even together with relative regularity $a \in A^\cap$, are not in general sufficient:

Theorem 4 *If $a \in A$ is arbitrary in the ring A then, with*

3.5 $$b = \begin{pmatrix} a & -1 \\ 0 & 0 \end{pmatrix} \in B = \begin{pmatrix} A & A \\ A & A \end{pmatrix} \ , \ d = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in B \ ,$$

then automatically

$$b = bdb \in B^\cap \ ,$$

while there is implication

$$b \in Bb^2 \Longrightarrow a \in A^{-1}_{left} \ ,$$

and also implication

$$b \in b^2 B \Longrightarrow a \in A^{-1}_{right} \ .$$

Hence

$$b \in \mathrm{SP}(B) \Longrightarrow a \in A^{-1} \ .$$

Proof. Look at the top right hand corner element •

For example ([H] (7.3.6.8)) we may take $A = B(X)$ with $X = c_{00} \subseteq c_0$ the space of "terminating sequences", and $a = w \in A$ the "standard weight"

$$w(x)_n = (1/n)x_n \ .$$

When $A = B(X)$ for a normed space X and $a \in A$ is of ascent and descent one then [X] each of the following conditions is sufficient for simple polarity:

$$X \ \text{complete} \ ;$$

$$a \in A \ \text{Fredholm} \ ;$$

$$a \in A \ \text{finite rank} \ ;$$

$$b \in X \ \text{a normed algebra and} \ a \in \{L_b, R_b\} \subseteq B(X) \ .$$

4. Koliha-Drazin permanence

More generally if there is $n \in \mathbf{N}$ for which a^n is simply polar we shall also say that $a \in A$ is "polar", or *Drazin invertible*. If $a \in A$ is polar then there is $c \in A$ for which $ac = ca$ and $a - aca$ is *nilpotent*. More generally still if we write, in a Banach algebra A,

4.1 $$\mathrm{QN}(A) = \{a \in A : 1 - \mathbf{C}a \subseteq A^{-1}\}$$

for the *quasinilpotents* of A, then $a \in \mathrm{QN}(A)$ if and only if $\sigma_A(a) \subseteq \{0\}$, while with some complex analysis we can prove that if $a \in \mathrm{QN}(A)$ then

4.2 $$\|a^n\|^{1/n} \to 0 \ (n \to \infty) \ .$$

Since (4.1) and (4.2) are equivalent it follows that also equivalent [H2],[K] is the condition

4.3 $$\mathrm{QN}(A) = \{a \in A : 1 - \mathrm{comm}(a)a \subseteq A^{-1}\} \ .$$

In the ultimate generalization of "group invertibility", we shall write $\mathrm{QP}(A)$ for the *quasipolar* elements $a \in A$, those which have a *spectral projection*, $q \in A$ for which

4.4 $$q = q^2 \ ; \ aq = qa \ ; \ a + q \in A^{-1} \ ; \ aq \in \mathrm{QN}(A) \ .$$

Now [K] the spectral projection and the *Koliha-Drazin inverse*

4.5 $$a^\bullet = q \ , \ a^\times = (a + q)^{-1}(1 - q)$$

are uniquely determined and lie in the double commutant of $a \in A$. It is easy to see that if (4.4) is satisfied then

4.6 $$0 \notin \mathrm{acc}\ \sigma_A(a) \ :$$

the origin cannot be an accumulation point of the spectrum; conversely if (4.6) holds then we can display the spectral projection as a sort of "vector-valued winding number"

4.7 $$a^\bullet = \frac{1}{2\pi i} \oint_0 (z - a)^{-1} dz \ ,$$

where we integrate counter clockwise round a small circle γ centre the origin whose *connected hull* $\eta\gamma$ is a disc whose intersection with the spectrum is at most the point $\{0\}$. By the same technique we can display the Koliha-Drazin inverse in the form

4.8 $$a^\times = \frac{1}{2\pi i} \oint_{\sigma'(a)} z^{-1}(z - a)^{-1} dz \ ,$$

where $\sigma'(a) = \sigma(a) \setminus \{0\}$. Now generally for a homomorphism $T : A \to B$ there is inclusion

4.9 $$T\ \mathrm{QP}(A) \subseteq \mathrm{QP}(B) \ ,$$

while if $T : A \to B$ has spectral permanence in the sense (1.3) then it is clear from (4.6) that there is also "Drazin permanence" in the sense that

4.10 $$\mathrm{QP}(A) = T^{-1}\mathrm{QP}(B) \subseteq A \ :$$

Theorem 5 *For Banach algebra homomorphisms $T : A \to B$ there is implication*

4.11 $$\text{spectral permanence} \Longrightarrow \text{Drazin permanence.}$$

Proof. Equality in (1.3), expressed [DZH] in terms of the spectrum, together with (4.6) •

We recall ([DZH] Theorem 2) that in (2.11) the shift $a \in B^{\cap} \setminus \mathrm{QP}(B)$.

As a sort of converse to Theorem 5, and squaring the circle in Theorem 3,

Theorem 6 *If $T : A \to B$ is a Banach algebra homomorphism then*

4.12 $$\mathrm{QP}(A)_{\cap} T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0)$$

and if $T : A \to B$ is one one then

4.13 $$\mathrm{QP}(A)_{\cap} T^{-1}\mathrm{SP}(B) = \mathrm{SP}(A)\ .$$

Hence if $a \in B$ and $T = J : A = \mathrm{comm}^2(a) \subseteq B$ then

4.14 $$A^{\cap} = T^{-1}\mathrm{SP}(B)\ .$$

Hence if $T^{-1}(0) = \{0\}$ is one one then

4.15 $$\text{Drazin} \Longrightarrow \text{simple} \Longrightarrow \text{spectral permanence}\ .$$

Proof. Uniqueness guarantees that the spectral projection $T(a)^{\bullet}$ of $Ta \in \mathrm{SP}(B) \subseteq \mathrm{QP}(B)$ commutes with $T(a) \in B$, and one-one-ness guarantees the same for $a \in A$ •

5. Moore-Penrose permanence

By a *star semigroup* we shall understand a semigroup A with an *involution*, $*: A \to A$ satisfying, for arbitrary $a, c \in A$,

5.1 $$(a^*)^* = a\ ;\ (ca)^* = a^* c^*\ ;\ 1^* = 1\ .$$

In rings and algebras involutions are assumed to be additive, and "conjugate linear". Obviously there is implication

5.2 $$a \in H(A) \Longrightarrow a^* \in H(A)$$

for each $H(A) \in \{A^{-1}, A^\cap, \mathrm{SP}(A)\}$. Elements $a \in A$ are said to be *hermitian* or "real" when they are the same as their adjoints:

5.3 $$\mathrm{Re}(A) = \{a \in A : a^* = a\}\ .$$

A *Moore-Penrose* inverse for $a \in A$ is $c = a^\dagger \in A$ for which the induced idempotents are hermitian:

5.4 $$a = aca\ ;\ c = cac\ ;\ (ca)^* = ca\ ;\ (ac)^* = ac\ .$$

We write $A^\dagger \subseteq A^\cap$ for those $a \in A$ for which $a^\dagger$ exists. The argument ([HM] Theorem 5) for "C* algebras" works in semigroups [X2], and says that

5.5 $$a^\dagger \in \mathrm{comm}^2(a, a^*)$$

is unique and double commutes with $\{a, a^*\}$ in A. The "B* condition", in a Banach algebra A, says that

5.6 $$\|a^* a\| = \|a\|^2\ .$$

It follows

$$a, x \in A \Longrightarrow \|ax\|^2 \le \|x^*\|\ \|a^* ax\|$$

and hence that * is *cancellable* in the sense that

5.7 $$a \in A \Longrightarrow L_{a^*a}^{-1}(0) \subseteq L_a^{-1}(0)\ ;$$

in words ([HLa] Definition 1) the pair (L_{a^*}, L_a) is "left skew exact". We need one more object: the "star polars"

5.8 $$\mathrm{SP}^*(A) = \{a \in A : a^* a \in A^\cap\}\ .$$

Our main objective is to prove again the Harte/Mbekhta observation ([HM[Theorem 6) that in a C* algebra A

5.9 $$A^\cap \subseteq A^\dagger\ ,$$

relatively regular elements always have Moore-Penrose inverse, and that [HM2] isometric C* algebra homomorphisms have generalized permanence. We begin by collecting some elementary observations:

Theorem 7 *If the involution $*: A \to A$ is cancellable then there is inclusion*

5.10 $$A^\dagger \subseteq \mathrm{SP}^*(A) \subseteq A^\cap\ ,$$

Proof. With cancellation there is implication

$$a \in \mathrm{SP}^*(A) \Longrightarrow a \in aAa^*a \subseteq Aa^*a_\cap aAa\ ,$$

and equality

$$\mathrm{Re}(A)_\cap \mathrm{SP}^*(A) = \mathrm{Re}(A)_\cap \mathrm{SP}(A)\ .$$

If $a = aca \in A^\dagger$ with $a^\dagger = c$ then

$$a^* a = a^*(ac)(ac)^* a = a^* acc^* a^* a \in a^* aAa^* a\ ;$$

conversely (5.7)

$$a^* a = a^* ada^* a \Longrightarrow a = ada^* a\ ;$$

hence also

$$a \in Aa^* a\ ,\ \Longleftrightarrow a^* \in a^* aA\ .$$

Hence if $a^* = a$ then (2.4) follows $\bullet$

Now it is clear that isometric C* homomorphisms have "Moore-Penrose permanence":

Theorem 8 *If $T : A \to B$ is a * homomorphism with simple permanence there is inclusion*

5.11 $$T^{-1}B^\dagger \subseteq A^\dagger \ .$$

Proof. We claim
$$A^\dagger = \{a \in A : a^*a \in \mathrm{SP}(A)\} \ ,$$
with implication
$$a^*a \in \mathrm{SP}(A) \Longrightarrow a^\dagger = (a^*a)^\times a^* \ .$$
If $a \in A^\dagger$ with $a = aca$ and $(ca)^* = ca$ and $(ac)^* = ac$ then, with $d = cc^*$, we have
$$a^*ad = a^*acc^* = a^*c^*a^*c^* = ca$$
and
$$da^*a = cc^*a^*a = ca \ .$$
Conversely if $a^*a = a^*ada^*a$ with $a^*ad = da^*a$ and (wlog: $d \mapsto \frac{1}{2}(d + d^*)$) $d = d^*$ then, with $c = da^*$,
$$aca = ada^*a = a \text{ and } ca = da^*a = a^*ad = a^*c^* \ .$$
Now if $a \in A$ there is, using Theorem 3, implication
$$Ta \in B^\dagger \Longrightarrow T(a^*a) \in \mathrm{SP}(B) \Longrightarrow a^*a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger \bullet$$

Thanks to (5.9), this is of course "generalized permanence". The Harte/Mbekhta result is derived by using the "poor man's path" to convert the idempotents ca and ac into self adjoint idempotents. Alternatively, thanks to the Gelfand/Naimark/Segal representation, we can look first in the very special algebra $D = B(X)$ of bounded Hilbert space operators:

Theorem 9 *If $d \in D = B(X)$ for a Hilbert space X then*

5.12 $$(d^*d)^{-1}(0) \subseteq d^{-1}(0)$$

and

5.13 $$\mathrm{cl}\ d(X) + d^{*-1}(0) = X \ ;$$

hence

5.14 $$\mathrm{cl}\ d(X) = d(X) \Longrightarrow d^*(X) = d^*d(X) \ , \Longrightarrow \mathrm{cl}\ d^*d(X) = d^*d(X) \ .$$

There is inclusion

5.15 $$\mathrm{Re}(D)_\cap D^\cap \subseteq \mathrm{SP}(D) \ ;$$

hence

5.16 $$d \in D^\cap \Longrightarrow d \in \mathrm{SP}^*(D) \Longrightarrow d^*d \in \mathrm{SP}(D) \Longrightarrow d \in D^\dagger \ .$$

Proof. For arbitrary $\xi \in X$ there is [DZH] inequality
$$\|d\xi\|^2 \le \|\xi\| \ \|d^*d\xi\| \ ,$$
and also
$$\mathrm{cl}\ d(X) = d^{*-1}(0)^\perp \bullet$$

Both of the Harte/Mbekhta observations now follow:

Theorem 10 *If $T : A \to B$ is isometric then*

5.17 $$T^{-1}(B^\cap) \subseteq A^\dagger \ .$$

Proof. With $S : B \to D = B(X)$ a GNS mapping we argue, using again Theorem 3,
$$Ta \in B^\cap \Longrightarrow ST(a^*a) \in \mathrm{SP}(D) \Longrightarrow a^*a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger \bullet$$

In Theorem 4.1 of [DZH] we established this using the more esoteric $\mathrm{QP}(A)$ rather than $\mathrm{SP}(A)$. It would be entertaining to be able to replace the GNS representation in Theorem 9 with the much more elementary left regular representation $L : A \to B(A)$.

6. Polar decomposition

We conclude with a discussion of the "polar decomposition" of C* algebra elements. In the algebra of operators $A = B(X)$ it is familiar that an arbitrary element $a \in A$ can be written as the product of a "partial isometry" and a positive operator. It is not clear that this can be done in a general C* algebra: for example if $A = C[0,1]$ there are only two idempotents in A and hence only two possible partial isometries. We want here to observe that [H3] at least the Moore-Penrose invertibles have polar decomposition. By a *generalized polar decomposition* for an element $a \in A$ of a C* algebra we shall understand a pair $(u, c) \in A^2$ for which $a = uc$ with

6.1 $$u = uu^*u \ ;$$

6.2 $$c = c^* \ ;$$

6.3 $$L_u^{-1}(0) \subseteq L_c^{-1}(0) \ .$$

If in addition

6.4 $$0 \le c \text{ and } L_c^{-1}(0) \subseteq L_u^{-1}(0)$$

then we shall say that (u, c) a *polar decomposition* of $a \in A$. We claim ([H3] Theorem 4)

Theorem 11 *If $(u, c) \in A^2$ is a generalized polar decomposition of $a \in A$ then*

6.5 $$a^*a = c^2 \text{ and } u^*a = c \ .$$

If (u, c) is a polar decomposition of a then each of u and c are uniquely determined and lie in the double commutant of (a, a^*). Also

6.6 $$aa^*u = ua^*a \ .$$

Proof. For the first part of (6.5) observe that

$$u^*uc - c \in L_u^{-1}(0) \subseteq L_c^{-1}(0) \ ;$$

now

$$(u^*a - c)^*(u^*a - c) = c(u^*u - 1)^2c = 0 \ ,$$

and the second part of (6.5) follows by cancellation. When (u, c) is a polar decomposition then the positivity gives the uniqueness of c:

6.7 $$c = |a| = (a^*a)^{1/2} \ .$$

The uniqueness of u^*u and uu^* follows from their status as "support" and "cosupport" projections for a; for the uniqueness of u suppose $a = uc = vc$ satisfying (6.1)-(6.4): then

$$(1 - v^*u)c = 0 \implies c(1 - u^*v) = 0 \ , \implies u(1 - u^*v) = 0 \ .$$

Now

$$u^*u = u^*uuv \ , \implies u^*(u - v) = 0 \ ,$$

similarly $v^*(u - v) = 0$, and hence $v = u$ by cancellation.

It is clear from (6.7) that c is in the double commutant of (a, a^*), as are also the support and cosupport u^*u and uu^*. Finally if $d \in \mathrm{comm}(a, a^*)$ then it also commutes with each of c, u^*u and uu^* and hence

$$cu^*d = dcu^* = cdu^* \implies uu^*d = udu^* \implies duu^* = uu^*d = udu^*$$

and hence

$$du = duu^*uudu^*u = uu^*ud = ud \ .$$

Finally, for (6.6),

$$aa^*u = uc^2u^*u = ua^*au^*u = ua^*a \bullet$$

We shall write

6.8 $$(u, c) = (\mathrm{sgn}(a), |a|)\ .$$

Evidently, taking limits of polynomials in a^*a,

6.9 $$|a^*|u = u|a|\ ;$$

It follows

6.10 $$(\mathrm{sgn}(a^*), |a^*|) = (\mathrm{sgn}(a)^*, \mathrm{sgn}(a)|a|\mathrm{sgn}(a)^*)\ .$$

We can characterise ([H3] Theorem 5) relative regularity in terms of the polar decomposition:
Theorem 12 *If $a \in A^\dagger \subseteq A$ has a Moore-Penrose inverse then it has a polar decomposition, with*

6.11 $$\mathrm{sgn}(a) = (a^\dagger)^*|a|\ .$$

If $a \in A$ has polar decomposition (u, c) then

6.12 $$d = c + 1 - u^*u \Longrightarrow L_d^{-1}(0) = \{0\}\ ,$$

and

6.13 $$a \in A^\dagger \Longrightarrow d \in A^{-1} \Longrightarrow a \in A^\cap\ .$$

Proof. We argue, with $c = a^\dagger$ and $u = c^*|a|$, that

$$uu^*u = c^*|a|^2cc^*a = c^*a^*acc^*|a| = (ac)^*(ac)c^*|a| = c^*a^*c^*|a| = c^*|a|$$

and

$$u|a| = c^*|a|^2 = c^*a^*a = (ac)a = a\ .$$

If $x \in A$ is arbitrary there is implication

$$dx = 0 \Longrightarrow ucx = 0 \Longrightarrow cx = 0 = u^*ucx = 0 \Longrightarrow u^8ux = 0 = (1 - u^*u)x = 0\ .$$

Also

$$d \in A^{-1} \Longrightarrow ad^{-1}u^*a = udd^{-1}a^*a = ua^*a = a\ .$$

Conversely if $a \in A^\dagger$ then $a^\dagger a = u^*u$ and $aa^\dagger = uu^*$ and hence

$$d' = (a^\dagger a + 1 - u^*u) \Longrightarrow dd' = 1 = d'd\ \bullet$$

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